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Research Article

On Asymptotic Behavior for Reaction Diffusion Equation with Small Time Delay

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We investigate the asymptotic behavior of scalar diffusion equation with small time delay $u_t - \Delta u = f(u(t), u(t-\tau))$. Roughly speaking, any bounded solution will enter and stay in the neighborhood of one equilibrium when the equilibria are discrete.

1. Introduction

With delay systems appearing frequently in science, engineering, physics, biology, economics, and so forth, many authors have recently devoted their interests to the effect of small delays on the dynamics of some system. This problem is relatively well understood for linear systems, including both finite-dimensional and infinite-dimensional situations, see [1–5]. However, for nonlinear systems, the problem is much more complicated, but there are some very nice results in [6–10].

In this paper, we consider the following scalar reaction-diffusion equation with a time delay

$$u_t - \Delta u = f(u(t), u(t-\tau)) \quad \left(x \in \Omega \subset R^N\right). \tag{1.1}$$

It is proved in [11–13] that for such diffusion equation without delay,

$$u_t - \Delta u = f(u), \tag{1.2}$$

subject to homogeneous boundary conditions, all globally defined bounded solutions must approach the set of equilibria as t tends to infinity. This depends heavily on the fact that (1.2) is a gradient system with the Lyapunov function

$$V(u) := \frac{1}{2}\int_{\Omega} |\nabla u|^2 - \int_{\Omega} F(u), \tag{1.3}$$

where F is a primitive of f. It is well known that solutions of (1.1) will typically oscillate in t as $t \to \infty$ if the delay is not sufficiently small. However, we will point out such interesting result that oscillations do not happen for sufficiently small delay. Specifically we obtain the conclusion that for given $R, \varepsilon > 0$ there exists a sufficiently small $\tau > 0$ such that any solution of (1.1) satisfying $\limsup_{t\to\infty} \|u(x,t)\|_{H_0^1(\Omega)} \le R$ will ultimately enter and stay in the ε–neighborhood of some equilibrium.

As a matter of fact, for the finite-dimensional situation, in [6] Li and Wang considered the general nonlinear gradient system with multiple small time delays

$$x'(t) = f(x(t - r_1(t)), \ldots, x(t - r_n(t))). \tag{1.4}$$

Making use of the Morse structure of invariant sets of gradient systems, he obtained a similar result. Following this idea, we investigate (1.1) in the infinite-dimensional situation. The difference between the two situations is very great. For example, under the finite-dimensional situation there must exist convergent subsequence for any bounded sequence. This is not correct in the infinite-dimensional situation. We only have weak compactness. In other words, bounded sequences in a reflexive Banach space are weakly precompact. In order to overcome this difficulty, we apply the famous Aubin-Lions lemma [14].

2. Preliminaries

In this paper, we assume Ω to be an open, bounded subset of R^N and τ to be a positive parameter (the delay). Consider the following scalar delayed initial boundary value problem:

$$\begin{gathered} u_t - \Delta u = f(u(t), u(t-\tau)) \quad \text{in } \Omega \times (0,T], \\ u = u_0 \quad \text{on } \Omega \times [-\tau, 0], \\ u = 0 \quad \text{on } \partial\Omega \times (0,T], \end{gathered} \tag{2.1}$$

where the nonlinear $f : R^2 \to R$ is assumed to be continuous and to satisfy

$$|f(u,v)| \le C\left[1 + |u|^{\rho} + |v|^{\rho}\right], \tag{2.2}$$

$$f(u,v)u \le C_1\left(u^2 + uv\right) + C_2. \tag{2.3}$$

Here $C, C_1,$ and C_2 are all constants, $\rho = 1 + 2/N$. Firstly we will give the definition of weak solution for (2.1).

Definition 2.1. A function $u(x,t)$ is called a weak solution of (2.1) if and only if

(i) $u \in L^2(0,T;H_0^1(\Omega))$, with $u' \in L^2(0,T;H^{-1}(\Omega))$,

(ii) $u|_{[-\tau,0]} = u_0 \in L^2(\Omega)$,

(iii) $\int_0^T [\langle u_t, \varphi\rangle + (Du, D\varphi)]dt = \int_0^T (f(u(t), u(t-\tau)), \varphi)dt$,

for each $\varphi \in L^2(0,T;H_0^1(\Omega))$. Here $\langle , \rangle$ and $(,)$ denote the pair of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, the inner product in $L^2(\Omega)$, respectively. Next we will give two very important lemmas many times used in the proof of two theorems.

Lemma 2.2. *If $\{u_n\}$ is bounded in $L^2(-\tau,T;H_0^1(\Omega)) \cap L^\infty(-\tau,T;L^2(\Omega))$, then $\{f(u_n(t), u_n(t-\tau))\}$ is bounded in $L^2(0,T;L^2(\Omega))$.*

Proof. Let $a = (N-2)/N \in (0,1)$, and because $\rho = 1 + 2/N$, $2\rho = a(2N/(N-2)) + 2(1-a)$. Before testing the boundedness of $\|f\|_{L^2(0,T;L^2(\Omega))}$, we firstly estimate $\|u_n\|_{L^{2\rho}(\Omega)}$

$$\begin{aligned}\|u_n\|_{L^{2\rho}(\Omega)}^{2\rho} &= \int_\Omega |u_n|^{a(2N/(N-2))} \cdot |u_n|^{2(1-a)} dx \\ &\leq \left[\int_\Omega |u_n|^{2N/(N-2)} dx\right]^a \left[\int_\Omega |u_n|^2 dx\right]^{1-a} \\ &= \|u_n\|_{L^{2N/(N-2)}(\Omega)}^{2Na/(N-2)} \cdot \|u_n\|_{L^2(\Omega)}^{2(1-a)} \\ &\leq C_1 \|u_n\|_{H_0^1(\Omega)}^{2Na/(N-2)} = C_1 \|u_n\|_{H_0^1(\Omega)}^2.\end{aligned} \tag{2.4}$$

Here we utilize the Hölder inequality, the fact of $H_0^1(\Omega) \subset L^{2N/(N-2)}$ continuously and $\{u_n\}$ is bounded in $L^\infty(-\tau,T;L^2(\Omega))$. So

$$\begin{aligned}&\int_0^T \int_\Omega |u_n(x,t)|^{2\rho} dx\, dt \leq C_1 \|u_n(x,t)\|_{L^2(-\tau,T;H_0^1(\Omega))}^2, \\ &\int_0^T \int_\Omega |u_n(x,t-\tau)|^{2\rho} dx\, dt \overset{t-\tau=s}{=} \int_{-\tau}^{T-\tau} \int_\Omega |u_n(x,s)|^{2\rho} dx\, ds \\ &\quad \leq \int_{-\tau}^T \int_\Omega |u_n(x,t)|^{2\rho} dx\, dt \leq C_1 \|u_n(x,t)\|_{L^2(-\tau,T;H_0^1(\Omega))}^2.\end{aligned} \tag{2.5}$$

In view of (2.2), we can easily see

$$\left|f(u_n(t), u_n(t-\tau))\right|^2 \leq C\left[1 + |u_n(t)|^{2\rho} + |u_n(t-\tau)|^{2\rho}\right]. \tag{2.6}$$

Integrating the above inequality with t and x, we complete the proof. □

Remark 2.3. If $\{u_n\}$ is bounded in $L^\infty(-\tau,T;H_0^1(\Omega))$, we can also get the same conclusion. The underlying lemma is the famous Aubin-Lions lemma. We only give the statement of the lemma.

Lemma 2.4. *Let X_0, X, and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. Suppose also that X_0 and X_1 are reflexive spaces. For $1 < p, q < \infty$, let $W = \{u \in L^p([0,T;X_0) | u' \in L^q([0,T];X_1)\}$. Then the embedding of W into $L^p[0,T;X)$ is also compact.*

Finally we give the definition of equilibrium solution of (1.2) and omega limit set $\omega(u)$, where $u(x,t)$ is a bounded solution of (1.1). Selecting $H_0^1(\Omega)$ as our phase space, we denote by $\omega(u)$ the limit set

$$\omega(u) = \left\{ v \mid \text{there exists } t_n \longrightarrow \infty \text{ such that } \|u(\cdot, t_n) - v\|_{H_0^1(\Omega)} \longrightarrow 0 \right\}. \tag{2.7}$$

As usual, an equilibrium solution of (1.2) is defined as a solution which does not depend on t; the equilibrium states are thus the functions $u \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying the elliptic boundary value problem

$$\begin{aligned} -\Delta u &= f(u,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Omega \end{aligned} \tag{2.8}$$

in the weak sense.

Let each equilibrium be isolated and let $u(\cdot, t)$ be the bounded complete solution of (1.2). Then we have

$$\lim_{t \to -\infty} u(\cdot, t) = E_1, \qquad \lim_{t \to +\infty} u(\cdot, t) = E_2 \tag{2.9}$$

for some equilibrium E_1 and E_2 with $V(E_1) > V(E_2)$, where V is the Lyapunov function (1.3). A complete solution of (1.2) means a solution $u(\cdot, t)$ defined on $(-\infty, +\infty)$. Now we will introduce our main results.

3. Main Results

In this section, we will prove two theorems. One is the existence of global solution. The other is our core, Theorem 3.2.

Theorem 3.1. *For given $\tau > 0$, $u_0 \in L^2(\Omega)$, problem* (2.1) *has a global weak solution.*

Proof. We will use classical Galerkin's method to build a weak solution of (2.1). Consider the approximate solution $u_m(t)$ of the form

$$u_m(t) = \sum_{k=1}^{m} u_k^m(t) \omega_k, \tag{3.1}$$

where $\{\omega_k\}_{k=1}^{\infty}$ is an orthogonal basis of $H_0^1(\Omega)$ and $\{\omega_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$. We get u_m from solving the following ODES:

$$\begin{gathered}\left(u_m', \omega_k\right) + \left(Du_m, D\omega_k\right) = \left(f\left(u_m(t), u_m(t-\tau)\right), \omega_k\right) \quad (0 < t \leq T,\ k = 1, 2, \ldots m), \\ u_k^m(t) = \left(u_0, \omega_k\right) \quad (-\tau \leq t \leq 0,\ k = 1, 2, \ldots m).\end{gathered} \tag{3.2}$$

According to standard existence theory of ODES, we can obtain the local existence of u_m.

Next we will establish some priori estimates for u_m. Multiplying (2.1) by u_m and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_{L^2(\Omega)}^2 + \|u_m\|_{H_0^1(\Omega)}^2 = \int_\Omega f\left[u_m(t), u_m(t-\tau)\right]u_m\, dx. \tag{3.3}$$

Because of (2.3) and the Cauchy inequality, we can get

$$\frac{d}{dt}\|u_m\|_{L^2(\Omega)}^2 + 2\|u_m\|_{H_0^1(\Omega)}^2 \leq C_1'\|u_m\|_{L^2(\Omega)}^2 + C_2'. \tag{3.4}$$

Getting rid of the term $2\|u_m\|_{H_0^1(\Omega)}^2$, from the differential form of Gronwall's inequality, we yield the estimate

$$\max_{0\leq t\leq T}\|u_m\|_{L^2(\Omega)}^2 \leq C_1\|u_0\|_{L^2(\Omega)}^2 + C_2. \tag{3.5}$$

Returning once more to inequality (3.4), we integrate from 0 to T and employ the inequality above to find

$$\|u_m\|_{L^2\left(0,T;H_0^1(\Omega)\right)}^2 \leq C_1\|u_0\|_{L^2(\Omega)}^2 + C_2. \tag{3.6}$$

Multiplying (2.1) by u_m' and then integrating over Ω, we have

$$\|u_m'\|_{L^2(\Omega)}^2 + \int_\Omega Du_m \cdot Du_m'\, dx = \int_\Omega f\left[u_m(t), u_m(t-\tau)\right]u_m'\, dx. \tag{3.7}$$

Using the Cauchy inequality and Lemma 2.2, we get

$$\|u_m'\|_{L^2(\Omega)}^2 + \frac{d}{dt}\|u_m\|_{H_0^1(\Omega)}^2 \leq C. \tag{3.8}$$

Again from the differential form of Gronwall's inequality, we integrate from 0 to T

$$\|u_m'\|_{L^2\left(0,T;L^2(\Omega)\right)}^2 \leq C_1\|u_0\|_{L^2(\Omega)}^2 + C_2. \tag{3.9}$$

Since $L^2(\Omega) \subset H^{-1}(\Omega)$, so

$$\|u'_m\|^2_{L^2(0,T;H^{-1}(\Omega))} \le C_1\|u_0\|^2_{L^2(\Omega)} + C_2. \tag{3.10}$$

According to estimates (3.6), (3.10), Lemma 2.2, and weak compactness, we see that

$$\begin{gathered} u_m \rightharpoonup u \text{ weakly in } L^2\Big(0,T;H_0^1(\Omega)\Big), \\ u'_m \rightharpoonup u' \text{ weakly in } L^2\Big(0,T;H^{-1}(\Omega)\Big), \\ f[u_m(t),u_m(t-\tau)] \rightharpoonup \eta \text{ weakly in } L^2\Big(0,T;L^2(\Omega)\Big). \end{gathered} \tag{3.11}$$

Here a subsequence of $\{u_m\}_{m=1}^{\infty}$ is still denoted by $\{u_m\}_{m=1}^{\infty}$. Applying Lemma 2.2, we can conclude that $u_m \to u$ strongly in $L^2(0,T;L^2(\Omega))$. Hence $u_m \to u$ A.E. in $\Omega\times(0,T)$. Since f is continuous, it follows that $f[u_m(t),u_m(t-\tau)] \to f[u(t),u(t-\tau)]$ A.E. in $\Omega\times(0,T)$. Thanks to (3.11) and Lemma 1.3 in [14], one has

$$f[u_m(t),u_m(t-\tau)] \rightharpoonup f[u(t),u(t-\tau)] \text{ weakly in } L^2\Big(0,T;L^2(\Omega)\Big). \tag{3.12}$$

Next fix an integer N_0 and choose a function $v \in C^1([0,T];H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N_0} d^k(t)\omega_k, \tag{3.13}$$

where $\{d^k\}_{k=1}^{N_0}$ are given smooth functions. Choosing $m \ge N_0$ and multiplying (3.2) by $d^k(t)$ sum $k = 1,2,\ldots,N_0$, and then integrating with respect to t, we can find

$$\int_0^T \langle u'_m, v\rangle + (Du_m, Dv)dt = \int_0^T \big(f[u_m(t),u_m(t-\tau)],v\big)dt. \tag{3.14}$$

Recalling (3.11) and (3.12) and passing to weak limits, we get

$$\int_0^T \langle u', v\rangle + (Du, Dv)dt = \int_0^T \big(f[u(t),u(t-\tau)],v\big)dt. \tag{3.15}$$

Because functions of the form $v(t)$ are dense in $L^2(0,T;H_0^1(\Omega))$, so the above equality holds for all functions $v \in L^2(0,T;H_0^1(\Omega))$.

Lastly we will show $u|_{[-\tau,0]} = u_0 \in L^2(\Omega)$. Notice that for each $v \in C^1([0,T];H_0^1(\Omega))$ with $v(T) = 0$ we get the following from (3.15):

$$\int_0^T -\langle v', u\rangle + (Du, Dv)dt = \int_0^T \big(f[u(t),u(t-\tau)],v\big)dt + (u(0),v(0)). \tag{3.16}$$

Similarly, from (3.14), we deduce

$$\int_0^T -\langle v', u_m\rangle + (Du_m, Dv)dt = \int_0^T (f[u_m(t), u_m(t-\tau)], v)dt + (u_m(0), v(0)). \tag{3.17}$$

In view of (3.2), $u_m(0) \to u_0$ in $L^2(\Omega)$; once again employ (3.11) and (3.12) to find

$$\int_0^T -\langle v', u\rangle + (Du, Dv)dt = \int_0^T (f[u(t), u(t-\tau)], v)dt + (u_0, v(0)). \tag{3.18}$$

As $v(0)$ is arbitrary, so we get the result $u(0) = u_0$. Since for $t \in [-\tau, 0], u_m(t) \to u_0$ in $L^2(\Omega)$, we can obtain the result. As for T being arbitrary, we see the global existence of (2.1). □

Theorem 3.2. *Assume that each equilibrium of* (1.2) *is isolated. Let $R, \varepsilon > 0$ be given arbitrarily. Then there exists a sufficiently small $\tau > 0$ such that any solution of* (1.1) *with* $\limsup_{t\to+\infty}\|u(\cdot,t)\|_{H_0^1(\Omega)} \le R$ *will eventually enter and stay in the $\varepsilon-$ neighborhood of some equilibrium.*

Proof. Here we select $H_0^1(\Omega)$ as our phase space. For simplicity, we will verify the correctness of the conclusion for such bounded solutions $u(x,t)$ of (1.1) as $\|u(\cdot,t)\|_{H_0^1(\Omega)} \le R$ for all $t \in [0,\infty)$. That is to say they are in $\overline{\mathcal{B}}_R$.

Assume there are n equilibria of (1.2) $\{E_1, \dots, E_n\}$, ordered by $V(E_n) \ge V(E_{n-1}) \ge \cdots V(E_1)$, where V is the Lyapunov function (1.3). We will follow two steps to prove our result.

Step 1. We firstly verify that for any $\delta > 0$, there exists a sufficiently small $\tau > 0$ such that

$$\omega(u) \bigcap \left(\bigcup_{1\le j\le n} \mathcal{B}_\delta(E_j) \right) \ne \emptyset \tag{3.19}$$

for any solution $u(x,t)$ of (1.1) in $\overline{\mathcal{B}}_R$.

In order to prove (3.19), we proceed by contradiction, which is used repeatedly in the following proof. Assume that there was a decreasing sequence $\tau_k \to 0$ and a corresponding solution sequence u_k of (1.1) in $\overline{\mathcal{B}}_R$ satisfying

$$d(E_j, \omega(u_k)) \ge 2\delta \tag{3.20}$$

for all $1 \le j \le n$ and $k \in N$. According to the definition of $\omega(u)$, for each k we can take a $t_k > 0$ such that for $t \ge t_k, 1 \le j \le n$

$$\left\| u_k(\cdot,t) - E_j \right\|_{H_0^1(\Omega)} \ge \delta. \tag{3.21}$$

Let $\widetilde{u}_k(t) = u_k(t + t_k)$ for $t \ge 0$. It is easy to see $\widetilde{u}_k$ is the weak solution of

$$\partial_t \widetilde{u}_k - \Delta \widetilde{u}_k = f(\widetilde{u}_k(t), \widetilde{u}_k(t - \tau_k)). \tag{3.22}$$

Next we will show there is a strong convergent subsequence of $\{\tilde{u}_k\}_{k=1}^{\infty}$ in $L^2(t,t+1;H_0^1(\Omega))$ for $t \geq 0$. Still denoting $\tilde{u}_k$, we can also prove the limit $\tilde{u}$ is in fact the weak solution of (1.2). From the elliptic equation regularity theorem, we can multiply (3.22) by $-\Delta\tilde{u}_k$ and integrate over Ω

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}_k\|^2_{H_0^1(\Omega)} + \|\tilde{u}_k\|^2_{H^2(\Omega)} \leq \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|\tilde{u}_k\|^2_{H^2(\Omega)}\right). \tag{3.23}$$

Because of the remark in Section 2, we can get

$$\frac{d}{dt}\|\tilde{u}_k\|^2_{H_0^1(\Omega)} + \|\tilde{u}_k\|^2_{H^2(\Omega)} \leq C. \tag{3.24}$$

Integrating from t to $t+1$, from the boundedness of $\|\tilde{u}_k(\cdot,t)\|^2_{H_0^1(\Omega)}$ and $\|\tilde{u}_k(\cdot,t+1)\|^2_{H_0^1(\Omega)}$, we conclude that $\{\tilde{u}_k\}$ is bounded in $L^2(t,t+1;H^2(\Omega))$. Multiplying (3.22) by $\partial_t\tilde{u}_k$ and utilizing the same method above, we can also conclude that $\{\partial_t\tilde{u}_k\}$ is bounded in $L^2(t,t+1;L^2(\Omega))$. Applying the Aubin-Lions lemma, we can conclude that there is a strong convergent subsequence of $\{\tilde{u}_k\}_{k=1}^{\infty}$ in $L^2(t,t+1;H_0^1(\Omega))$ for $t \geq 0$. We may set $\tilde{u}_k \to \tilde{u}$ strongly in $L^2(t,t+1;H_0^1(\Omega))$. Of course $\tilde{u}_k \to \tilde{u}$ strongly in $L^2(t,t+1;L^2(\Omega))$. Hence $\tilde{u}_k \to \tilde{u}$ a.e. in $\Omega\times(t,t+1)$. Since f is continuous, it follows that $f[\tilde{u}_k(t),\tilde{u}_k(t-\tau_k)] \to f(\tilde{u},\tilde{u})$ a.e. in $\Omega\times(0,T)$. Thanks to the weak convergence of $f[\tilde{u}_k(t),\tilde{u}_k(t-\tau_k)]$ in $L^2(t,t+1;L^2(\Omega))$ and lemma 1.3 in [14], one has

$$f[\tilde{u}_k(t),\tilde{u}_k(t-\tau_k)] \rightharpoonup f(\tilde{u},\tilde{u}) \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right). \tag{3.25}$$

So we prove that $\tilde{u}$ is the weak solution of (1.2). Considering (3.21), we have the following estimate for $\tilde{u}$:

$$\begin{aligned}\int_t^{t+1}\left\|\tilde{u}-E_j\right\|_{H_0^1(\Omega)}ds &= \int_t^{t+1}\left\|\tilde{u}_k-E_j+\tilde{u}-\tilde{u}_k\right\|_{H_0^1(\Omega)}ds\\ &\geq \int_t^{t+1}\left\|\tilde{u}_k-E_j\right\|_{H_0^1(\Omega)}ds - \int_t^{t+1}\|\tilde{u}_k-\tilde{u}\|_{H_0^1(\Omega)}ds\\ &\geq \delta - \int_t^{t+1}\|\tilde{u}_k-\tilde{u}\|_{H_0^1(\Omega)}ds \geq \frac{\delta}{2}.\end{aligned} \tag{3.26}$$

From the above inequality, we can surely know $\lim_{t\to\infty}\|\tilde{u}-E_j\|_{H_0^1(\Omega)} \neq 0$ for all $1 \leq j \leq n$. However, because (1.2) is a gradient system, this contradicts the fact that $\lim_{t\to\infty}u(\cdot,t) = E_j$ for some E_j. We obtain the correctness of (3.19).

Step 2. We will complete the proof of the theorem that if τ is sufficiently small, then for any bounded solution $u(\cdot,t)$ of (1.1) there must exist a E_j and sufficiently large T such that for $t > T$

$$\left\|u(\cdot,t)-E_j\right\|_{H_0^1(\Omega)} < \varepsilon. \tag{3.27}$$

Here we also adopt contradiction method to prove the result. If the desired conclusion was not correct, there would be a decreasing sequence $\tau_k \to 0$ and a corresponding solution sequence u_k of (1.1) in $\overline{\mathcal{B}}_R$ which does not satisfy (3.27).

In view of $\tau_k \to 0$, it is easy to infer that

$$\lim_{k\to\infty} \min_{1\le j\le n} d\left(E_j, \omega(u_k)\right) = 0. \tag{3.28}$$

Without loss of generality, we can assume that for all $k \ge 1$

$$\omega(u_k) \bigcap \left(\bigcup_{1\le j\le n} \overline{\mathcal{B}}_\varepsilon(E_j) \right) \ne \emptyset. \tag{3.29}$$

Denote by j^k the smallest j satisfying

$$\omega(u_k) \bigcap \overline{\mathcal{B}}_\varepsilon(E_j) \ne \emptyset. \tag{3.30}$$

It is easy to see that there exists a subsequence $\{k_i^1\}_{i=1}^\infty$ of $\{k\}_{k=1}^\infty$ such that for some $j_1 \in [1, n]$, we have $j^{k_i^1} = j_1$.

We will claim if $j_1 < n$, then there exists a $\delta_1 \in (0, \varepsilon)$ and k_1^* such that for $k_i^1 > k_1^*$

$$d\left(E_{j_1}, \omega\left(u_{k_i^1}\right)\right) \ge \delta_1. \tag{3.31}$$

Indeed, if the fact did not hold, there would be a subsequence of $\{k_i^1\}_{i=1}^\infty$ (for simplicity still denoted by $\{k_i^1\}_{i=1}^\infty$) such that

$$\lim_{i\to\infty} d\left(E_{j_1}, \omega\left(u_{k_i^1}\right)\right) = 0. \tag{3.32}$$

According to the definition of j^k and (3.32), we can choose a sequence $t_i > 0$ satisfying

$$\begin{gathered} u_{k_i^1}(t_i) \in \mathcal{B}_\varepsilon(E_{j_1}), \qquad \lim_{i\to\infty} \left\| u_{k_i^1}(t_i) - E_{j_1} \right\|_{H_0^1(\Omega)} = 0, \\ \left\| u_{k_i^1}(t) - E_j \right\|_{H_0^1(\Omega)} > \varepsilon, \quad \text{for } t > t_i,\ j < j_1. \end{gathered} \tag{3.33}$$

Now we define

$$\eta_i = \sup\left\{ t \ge t_i \mid u_{k_i^1}([t_i, t]) \subset \overline{\mathcal{B}}_\varepsilon(E_{j_1}) \right\}. \tag{3.34}$$

Obviously $u_{k_i^1}(\eta_i) \in \partial\mathcal{B}_\varepsilon(E_{j_1})$. Let

$$v_i(\cdot, t) = u_{k_i^1}(\cdot, t + \eta_i), \quad t \in \left[-(\eta_i - t_i), +\infty\right). \tag{3.35}$$

From (3.33) and the definition of η_i, it is clear to see

$$v_i(0) \in \partial\mathcal{B}_\varepsilon(E_{j_1}), \qquad v_i(t) \in \overline{\mathcal{B}}_\varepsilon(E_{j_1}) \quad \text{for } -(\eta_i - t_i) \le t \le 0, \tag{3.36}$$

$$\left\| v_i(t) - E_j \right\|_{H_0^1(\Omega)} > \varepsilon, \quad \text{for } t > 0,\ j < j_1. \tag{3.37}$$

Obviously $v_i(\cdot, t)$ is the weak solution of

$$\partial_t v_i - \Delta v_i = f\left(v_i(t), v_i\left(t - \tau_{k_i^1}\right)\right). \tag{3.38}$$

Following the method above, we can also prove there is a strong convergent subsequence of $\{v_i\}_{i=1}^{\infty}$ in $L^2(t, t+1; H_0^1(\Omega))$ for $t \ge 0$. Still denoting $\{v_i\}_{i=1}^{\infty}$ and letting $T = \limsup(\eta_i - t_i)$, the limit v defined on $(-T, \infty)$ is indeed the weak solution of (1.2).

Next we will show that $T = +\infty$. In fact, if $T < +\infty$, then $v(t)$ can be well defined at $t = -T$. In view of (3.32), we see $v(-T) = E_{j_1}$. Hence $v(t) \equiv E_{j_1}$ for $t \ge -T$. That is to say, $v_i(t) \to E_{j_1}$ strongly in $L^2(-T, 0; H_0^1(\Omega))$. Because $v_i(\cdot, t) \in L^2(-(\eta_i - t_i), 0; H_0^1(\Omega))$ and $v_i'(\cdot, t) \in L^2(-(\eta_i - t_i), 0; L^2(\Omega))$, it follows from Theorem 4 in 5.9.2 of [15] that $v_i(\cdot, t) \in C([-(\eta_i - t_i), 0]; H_0^1(\Omega))$. That is to say

$$\lim_{t \to 0-} \|v_i(\cdot, t) - v_i(\cdot, 0)\|_{H_0^1(\Omega)} = 0. \tag{3.39}$$

By the definition of continuity, there exists $t_1 < 0$ such that

$$\|v_i(\cdot, t) - v_i(\cdot, 0)\|_{H_0^1(\Omega)} < \varepsilon_0 < \varepsilon \quad \text{for } t_1 < t < 0. \tag{3.40}$$

So

$$\begin{aligned} \varepsilon_0 &> \left\| v_i(\cdot, t) - E_{j_1} + E_{j_1} - v_i(\cdot, 0) \right\|_{H_0^1(\Omega)} \\ &\ge \left\| v_i(\cdot, 0) - E_{j_1} \right\|_{H_0^1(\Omega)} - \left\| v_i(\cdot, t) - E_{j_1} \right\|_{H_0^1(\Omega)}. \end{aligned} \tag{3.41}$$

Hence

$$\left\| v_i(\cdot, t) - E_{j_1} \right\|_{H_0^1(\Omega)} > \left\| v_i(\cdot, 0) - E_{j_1} \right\|_{H_0^1(\Omega)} - \varepsilon_0 = \varepsilon - \varepsilon_0 > 0. \tag{3.42}$$

Thus

$$\int_{-(\eta_i - t_i)}^{0} \left\| v_i(\cdot, t) - E_{j_1} \right\|_{H_0^1(\Omega)} dt \ge \int_{t_1}^{0} \left\| v_i(\cdot, t) - E_{j_1} \right\|_{H_0^1(\Omega)} dt \ge -t_1(\varepsilon - \varepsilon_0). \tag{3.43}$$

Obviously

$$\lim_{i \to \infty} \int_{-(\eta_i - t_i)}^{0} \left\| v_i(\cdot, t) - E_{j_1} \right\|_{H_0^1(\Omega)} dt \ne 0. \tag{3.44}$$

This contradicts the fact $v_i(t) \to E_{j_1}$ strongly in $L^2(-(\eta_i - t_i), 0; H_0^1(\Omega))$. So it must be $T = +\infty$.

Let $\lim_{t\to+\infty} v(t) = E_j$. Then there must be $j \geq j_1$. Otherwise for $j < j_1$

$$\begin{aligned}\int_t^{t+1} \left\| v(s) - E_j \right\|_{H_0^1(\Omega)} &= \int_t^{t+1} \left\| v(s) - v_i(s) + v_i(s) - E_j \right\|_{H_0^1(\Omega)} ds \\ &\geq \int_t^{t+1} \left\| v_i(s) - E_j \right\|_{H_0^1(\Omega)} - \int_t^{t+1} \| v_i(s) - v(s) \|_{H_0^1(\Omega)} ds \\ &\geq \frac{\varepsilon}{2} > 0,\end{aligned} \tag{3.45}$$

where we use (3.37) and the fact $v_i(s) \to v(s)$ strongly in $L^2(t, t+1; H_0^1(\Omega))$. So it is impossible that for $j < j_1$, $\lim_{t\to+\infty} v(t) = E_j$.

Lastly we need to verify $\lim_{t\to-\infty} v(t) = E_{j_1}$. Considering (3.36), for any $\varepsilon' > 0$ sufficiently small such that $-(\eta_i - t_i) \leq t \leq -(\eta_i - t_i) + \varepsilon' \leq 0$, we have

$$\begin{aligned}&\int_{-(\eta_i - t_i)}^{-(\eta_i - t_i)+\varepsilon'} \left\| v(t) - E_{j_1} \right\|_{H_0^1(\Omega)} dt \\ &\quad = \int_{-(\eta_i - t_i)}^{-(\eta_i - t_i)+\varepsilon'} \left\| v(t) - v_i(t) + v_i(t) - E_{j_1} \right\|_{H_0^1(\Omega)} dt \\ &\quad \leq \int_{-(\eta_i - t_i)}^{-(\eta_i - t_i)+\varepsilon'} \| v_i(t) - v(t) \|_{H_0^1(\Omega)} dt + \int_{-(\eta_i - t_i)}^{-(\eta_i - t_i)+\varepsilon'} \left\| v_i(t) - E_{j_1} \right\|_{H_0^1(\Omega)} dt \\ &\quad \leq \int_{-(\eta_i - t_i)}^{-(\eta_i - t_i)+\varepsilon'} \| v_i(t) - v(t) \|_{H_0^1(\Omega)} dt + \varepsilon' \varepsilon.\end{aligned} \tag{3.46}$$

Because $v_i \to v$ strongly in $L^2(-(\eta_i - t_i), -(\eta_i - t_i) + \varepsilon'; H_0^1(\Omega))$, we easily get the result. In a word we conclude that

$$\lim_{t\to-\infty} v(\cdot, t) = E_{j_1}, \qquad \lim_{t\to+\infty} v(\cdot, t) = E_j \left(j \geq j_1 \right). \tag{3.47}$$

This obviously contradicts (2.9). So we get the correctness of (3.31).

According the definition of $j^{k_i^1}$, we can conclude that

$$d\left(E_j, \omega\left(u_{k_i^1} \right) \right) \geq \delta_1 \quad \text{for all } k_i^1 > k_1^*,\ 1 \leq j \leq j_1. \tag{3.48}$$

For convenience we may assume that (3.48) holds for all k_i^1.

Fix a $0 < \delta_2' < \delta_1$, and denote by $j^{k_i^1}$ the smallest j satisfying

$$\omega\left(u_{k_i^1} \right) \bigcap \overline{\mathcal{B}}_{\delta_2'}(E_j) \neq \emptyset. \tag{3.49}$$

From (3.48) we know $j_{k_i^1} > j_1$ for all k_i^1. Similarly there are a subsequence $\{k_i^2\}_{i=1}^{\infty}$ of $\{k_i^1\}_{i=1}^{\infty}$ and a $j_2 \in (j_1, n]$ such that $j^{k_i^2} = j_2$ for all k_i^2. Following the same process above, we can prove that if $j_2 < n$, then there exists a $\delta_2 \in (0, \delta_2')$ and $k_2^* > k_1^*$ such that for $k_i^2 > k_2^*$

$$d\left(E_{j_2}, \omega\left(u_{k_i^2}\right)\right) \geq \delta_2. \tag{3.50}$$

By the choice of $\{k_i^2\}_{i=1}^{\infty}$, it is easy to see that

$$d\left(E_j, \omega\left(u_{k_i^2}\right)\right) \geq \delta_2 \quad \text{for all } k_i^2 > k_2^*,\ 1 \leq j \leq j_2. \tag{3.51}$$

Repeating the same argument again and again, we finally get sequences

$$j_1 < j_2 < \cdots < j_m = n \qquad \varepsilon > \delta_1 > \delta_2 > \cdots > \delta_m > 0, \qquad k_1^* < k_2^* < \cdots < k_m^*, \tag{3.52}$$

and $\{k_i^p\}_{i=1}^{\infty} (1 \leq p \leq m)$ such that

$$d\left(E_j, \omega\left(u_{k_i^p}\right)\right) \geq \delta_p, \quad \text{for all } k_i^p > k_p^*,\ 1 \leq j \leq j_p. \tag{3.53}$$

In particular, of course we have

$$d\left(E_j, \omega\left(u_{k_i^m}\right)\right) \geq \delta_m \quad \text{for all } k_i^m > k_m^*,\ 1 \leq j \leq j_m = n. \tag{3.54}$$

This clearly contradicts (3.28). And the proof is completed. □

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