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Research Article

Some Results on the Best Proximity Pair

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We give some new conditions for existence and uniqueness of best proximity point. We also introduce the concept of strongly proximity pair and give some interesting results.

1. Introduction

Let X be a metric space and A and B nonempty subsets of X. If there is a pair $(x_0, y_0) \in A \times B$ for which $d(x_0, y_0) = d(A, B)$, that $d(A, B)$ is distance of A and B, then the pair (x_0, y_0) is called a best proximity pair for A and B. Best proximity pair evolves as a generalization of the concept of best approximation, and reader can find some important results of it in [1–4].

Now, as in [5] (see also [6–14]), we can find the best proximity points of the sets A and B by considering a map $T : A \cup B \to A \cup B$ such that $T(A) \subset B$ and $T(B) \subset A$. We say that the point $x \in A \cup B$ is a best proximity point of the pair (A, B), if $d(x, Tx) = d(A, B)$, and we denote the set of all best proximity points of (A, B) by $P_T(A, B)$, that is,

$$P_T(A, B) := \{x \in A \cup B : d(x, Tx) = d(A, B)\}. \tag{1.1}$$

Best proximity pair also evolves as a generalization of the concept of fixed point of mappings, because if $A \cap B \neq \emptyset$, every best proximity point is a fixed point of T.

The concept of *approximate best proximity pair* on metric space was introduced in [10, Definition 1.1], but it is clear that $P_T^a(A, B) = A \times B$. Now, in section two of this paper, we give some conditions that guarantee the existence, uniqueness, or compactness of the set $P_T(A, B)$. Then, in section three, by introducing the concepts of T-approximatively compact pair and

T-strongly compact pair, we give some characterizations of a subclass of the best proximity points, namely, the strongly proximity pairs of sets.

2. Some Existence Theorems

In this section, we will consider the existence of the best proximity points, by considering some sequences which converge to that best proximity point. At first, we generalize some result of Eldred and Veeramani [6].

Theorem 2.1. *Let A and B be nonempty closed subsets of a complete metric space X. Suppose that the mapping $T : A \cup B \longrightarrow A \cup B$ satisfying $T(A) \subset B$, $T(B) \subset A$, and*

$$d(Tx, Ty) \le \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma d(A, B), \tag{2.1}$$

for all $x, y \in A \cup B$, where $\alpha, \beta, \gamma \ge 0$ and $\alpha + 2\beta + \gamma < 1$. If A (or B) is boundedly compact, then there exists $x \in A \cup B$ with $d(x, Tx) = d(A, B)$.

Proof. Suppose x_0 is an arbitrary point of $A \cup B$ and define $x_{n+1} = Tx_n$. Now,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\le \alpha d(x_n, x_{n+1}) + \beta[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] + \gamma d(A, B). \end{aligned} \tag{2.2}$$

So

$$d(x_{n+1}, x_{n+2}) \le \frac{\alpha + \beta}{1 - \beta} d(x_n, x_{n+1}) + \frac{\gamma}{1 - \beta} d(A, B), \tag{2.3}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \le k d(x_n, x_{n+1}) + (1 - k) d(A, B), \tag{2.4}$$

where $k = ((\alpha + \beta)/(1 - \beta)) < 1$. Hence, inductively, we have

$$d(x_{n+1}, x_n) \le k^n d(x_1, x_0) + (1 - k^n) d(A, B), \tag{2.5}$$

and so

$$d(x_{n+1}, x_n) \longrightarrow d(A, B). \tag{2.6}$$

Therefore, by Proposition 3.3 of [6], both sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded. Now, since A (or B) is boundedly compact then $\{x_{2n}\}$ has a convergent subsequence, and so, by Proposition 3.2 of [6], there exists $x \in A$ such that $d(x, Tx) = d(A, B)$. □

Now, we show that the mapping T, which satisfies (2.1), has a unique best proximity point in the uniformly convex Banach space X.

Theorem 2.2. *Let A and B be two nonempty closed and convex subsets of a uniformly convex Banach space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B$, $T(B) \subset A$ and the condition* (2.1). *Then, there exists a unique element $x \in A$ such that $\|x - Tx\| = d(A,B)$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the above unique element.*

Proof. One can prove this theorem by the method of the Proposition 3.10 of [6]. □

Note that the uniform convexity of the Banach space X is necessary for uniqueness of $P_T(A,B)$; for instance, let $X = \mathbb{R}^2$ with $\|\cdot\|_\infty$, $A = x_0 + B_X$, and $B = -x_0 + B_X$ where $x_0 = (2,0)$. If $T : A \cup B \to A \cup B$ by $T(a,b) = (-a,b)$, then one can easily see that $P_T(A,B)$ is an infinite set.

Corollary 2.3. *Let A and B be two nonempty closed and convex subsets of a uniformly convex Banach space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B$, $T(B) \subset A$, and*

$$\|Tx - Ty\| \leq a_1\|x - y\| + a_2\|x - Tx\| + a_3\|y - Ty\| + a_4 d(A,B) \tag{2.7}$$

for all $x, y \in A \cup B$, where $a_i \geq 0$, $i = 1,2,3,4$ and $\sum_{i=1}^{4} a_i < 1$. Then, there exists a unique element $x \in A \cup B$ with $\|x - Tx\| = d(A,B)$.

Proof. Interchange the roles of x and y in (2.7); then add the new inequality with (2.7). □

In the following, we present some new conditions on the mapping T, such as weak closedness, such that it has a best proximity point in the uniformly convex Banach space X. We remember that the mapping $T : A \cup B \to A \cup B$ is said to be weakly closed if $x_n \rightharpoonup x$ weakly in $A \cup B$ and $Tx_n \rightharpoonup y$ weakly, then $Tx = y$.

Theorem 2.4. *Let A and B be two nonempty closed and convex subsets of a uniformly convex Banach space X such that A is bounded. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B$, $T(B) \subset A$, and*

$$\|Tx - Ty\| \leq \alpha\|x - y\| + \beta[\|x - Tx\| + \|y - Ty\|], \tag{2.8}$$

for all $x, y \in A \cup B$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta \leq 1$. If one of the following conditions:

(i) *T is weakly closed and $T(A)$ is bounded,*

(ii) *T is weakly sequentially continuous,*

satisfies, then there exists $x \in A$ with $\|x - Tx\| = d(A,B)$.

Proof. Let

$$A_0 := \{x \in A : \|x - y\| = d(A,B),\ \text{for some } y \in B\}. \tag{2.9}$$

By Lemma 3.2 of [5], A_0 is nonempty; hence, there are $x_0 \in A$ and $y_0 \in B$ such that $\|x_0 - y_0\| = d(A,B)$. For every positive integer $n \in \mathbb{N}$, define

$$T_n(x) = \begin{cases} \dfrac{1}{n} y_0 + \left(1 - \dfrac{1}{n}\right) Tx, & x \in A, \\[2ex] \dfrac{1}{n} x_0 + \left(1 - \dfrac{1}{n}\right) Tx, & x \in B. \end{cases} \tag{2.10}$$

Then, for every $x, y \in A \cup B$,

$$\begin{aligned} \|T_n x - T_n y\| &\leq \left(1 - \frac{1}{n}\right) \|Tx - Ty\| + \frac{1}{n} d(A,B) \\ &\leq \left(1 - \frac{1}{n}\right) \alpha \|x - y\| + \left(1 - \frac{1}{n}\right) \beta (\|x - Tx\| + \|y - Ty\|) \\ &\quad + \frac{1}{n} d(A,B). \end{aligned} \tag{2.11}$$

Therefore, by Theorem 2.2, for every $n \in \mathbb{N}$, there exists $x_n \in A$ such that

$$\|x_n - T_n x_n\| = d(A,B). \tag{2.12}$$

Since A is bounded and closed, there exist $x \in A$ such that $x_n \rightharpoonup x$ (by passing to a subsequence, if necessary). If (i) holds, then the sequence $\{Tx_n\}$ has a weakly convergent subsequence $Tx_{n_k} \rightharpoonup Tx$, thanks to the weak closedness of T. So $x_{n_k} - Tx_{n_k} \rightharpoonup x - Tx$. On the other hands, since $\|T_n x_n - Tx_n\| = (1/n)\|y_0 - Tx_n\| \to 0$, we have

$$\|x_{n_k} - Tx_{n_k}\| \leq \|x_{n_k} - T_{n_k} x_{n_k}\| + \|T_{n_k} x_{n_k} - Tx_{n_k}\| \longrightarrow d(A,B). \tag{2.13}$$

Therefore,

$$\|x - Tx\| \leq \liminf_{k \to \infty} \|x_{n_k} - Tx_{n_k}\| \leq d(A,B). \tag{2.14}$$

The proof of the statement in the case (ii) is even simpler and is a part of the above proof. □

Theorem 2.5. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B$, $T(B) \subset A$, and*

$$d\left(Tx, T^2 x\right) \leq k d(x, Tx) + (1 - k) d(A,B), \tag{2.15}$$

for all $x \in A \cup B$, where $0 \leq k < 1$. If there are $u \in A \cup B$ and $n \in \mathbb{N}$ such that $T^n u = u$, then $d(u, Tu) = d(A,B)$.

Proof. Suppose there are $u \in A \cup B$ and $n \in \mathbb{N}$ such that $T^n u = u$. If $d(A,B) < d(u,Tu)$, since T satisfies (2.15), we have

$$\begin{aligned} d(u,Tu) &= d\left(T\left(T^{n-1}u\right), T^2\left(T^{n-1}u\right)\right) \\ &\leq kd\left(T^{n-1}u, T\left(T^{n-1}u\right)\right) + (1-k)d(A,B) \\ &\leq k^2 d\left(T^{n-2}u, T^{n-1}u\right) + \left(1-k^2\right)d(A,B) \\ &\ \ \vdots \\ &\leq k^n d(u,Tu) + (1-k^n)d(A,B) \\ &< d(u,Tu), \end{aligned} \tag{2.16}$$

which is a contradiction, so $d(u,Tu) = d(A,B)$. □

Corollary 2.6. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B$, $T(B) \subset A$, and* (2.1)*. If there are $u \in A \cup B$ and $n \in N$ such that $T^n u = u$, then $d(u,Tu) = d(A,B)$.*

Proof. If $y = Tx$ in (2.1), then

$$d\left(Tx, T^2x\right) \leq \alpha d(x,Tx) + \beta\left[d(x,Tx) + d\left(Tx, T^2x\right)\right] + \gamma d(A,B). \tag{2.17}$$

So

$$d\left(Tx, T^2x\right) \leq \frac{\alpha+\beta}{1-\beta} d(x,Tx) + \frac{\gamma}{1-\beta} d(A,B). \tag{2.18}$$

Hence, by Theorem 2.5, we have $d(u,Tu) = d(A,B)$. □

3. Strongly Proximity Pairs

Let A and B be nonempty subsets of a metric space X, $\delta > 0$, and $T : A \cup B \to A \cup B$ such that $T(A) \subset B$ and $T(B) \subset A$. Put

$$P_T^{\delta}(A,B) := \{x \in A \cup B : d(x,Tx) < d(A,B) + \delta\}. \tag{3.1}$$

We say that the pair (A,B) is a strongly proximity pair, if it is proximity pair, and, for any neighborhood V of 0 in X there exists $\delta > 0$ such that $P_T^{\delta}(A,B) \subseteq P_T(A,B) + V$.

For example, if

$$A = \left\{(x,y) : (x-2)^2 + y^2 \leq 1\right\}, \qquad B = \left\{(x,y) : (x+2)^2 + y^2 \leq 1\right\}, \tag{3.2}$$

and $T(x,y)=(-x,y)$, then for every $\epsilon>0$, $P_T^{\delta}(A,B)\subseteq P_T(A,B)+V$, where $\delta=\sqrt{4+\epsilon^2}-2$ and V is the sphere with radius ϵ and center of zero. Hence the pair (A,B) is a strongly proximity pair.

Also, if

$$A=\{(0,y):-1\leq y\leq 1\},\qquad B=\{(x,0):1\leq x\leq 2\ \text{or}\ -2\leq x<-1\} \tag{3.3}$$

and $T:A\cup B\to A\cup B$ such that

$$T(x,y)=\begin{cases}(y+1,x) & y\geq 0,(x,y)\in A,\\ (y-1,x) & y<0,(x,y)\in A,\\ (y,x-1) & x\geq 1,(x,y)\in B,\\ (y,x+1) & x<-1,(x,y)\in B,\end{cases} \tag{3.4}$$

Therefore $T(A)\subset B$, and $T(B)\subset A$ and $P_T(A,B)=\{(0,0),(1,0)\}$, but the pair (A,B) is not a strongly proximity pair, while it is a proximity pair.

Here, by introducing the concepts of T-approximatively compact pair and T-strongly compact pair, we give some characterizations of the strongly proximity pairs of sets.

Definition 3.1. Let A and B be nonempty subsets of a metric space X and $T:A\cup B\to A\cup B$ such that $T(A)\subset B$ and $T(B)\subset A$. We say the following.

(i) The sequence $\{z_n\}\subseteq A\cup B$ is T-minimizing if

$$\lim_{n\to\infty}d(z_n,Tz_n)=d(A,B). \tag{3.5}$$

(ii) The pair (A,B) is T-approximatively compact pair (T-a.c.p.) if every T-minimizing sequence $\{z_n\}\subseteq A\cup B$ has convergent subsequence.

(iii) The pair (A,B) is T-strongly compact pair (T-s.c.p.) if every T-minimizing sequence $\{z_n\}\subseteq A\cup B$ is convergent.

In the last section, we find some conditions on T such that $P_T(A,B)\neq\emptyset$, and so in this section, we can always suppose that $P_T(A,B)\neq\emptyset$. At the first, we state an elementary lemma, which can be used in the proof of the main theorems that follow.

Lemma 3.2. *Let A and B be nonempty subsets of a metric space X, $T:A\cup B\to A\cup B$ such that $T(A)\subset B$ and $T(B)\subset A$, and the pair (A,B) is T-s.c.p. Then, $P_T(A,B)$ is singleton.*

Proof. Let $x,y\in P_T(A,B)$, hence,

$$d(y,Ty)=d(x,Tx)=d(A,B). \tag{3.6}$$

Now, define

$$z_n = \begin{cases} x, & n \text{ odd}, \\ y, & n \text{ even}. \end{cases} \tag{3.7}$$

Then, the sequence $\{z_n\}$ is T-minimizing but is not convergent provided that $x \neq y$ and so a contradiction. □

Now, we can prove the main theorems of this section.

Theorem 3.3. *Let A and B be nonempty closed subsets of a normed space X, and $T : A \cup B \to A \cup B$ is a continuous function, such that $T(A) \subset B$ and $T(B) \subset A$ and $P_T(A,B) \neq \emptyset$. Then, the pair (A,B) is T-a.c.p. if and only if the pair (A,B) is strongly proximity pair and $P_T(A,B)$ compact.*

Proof. Let the pair (A,B) be T-a.c.p., and $\{x_n\} \subseteq P_T(A,B)$ is an arbitrary sequence. Then for each n, $d(x_n, Tx_n) = d(A,B)$, and, by hypothesis, the sequence $\{x_n\}$ has a convergent subsequence to an element of $P_T(A,B)$. Thus, $P_T(A,B)$ is compact.

Also, if (A,B) is not strongly proximity pair, then there exist a neighborhood V of 0 and a T-minimizing sequence $\{z_n\} \subseteq A \cup B$ with z_n not belonging to $P_T(A,B)+V$ for all $n \geq 1$. Since (A,B) is T-a.c.p., there is a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z_0$. Then, $z_0 \in P_T(A,B)$, and so $z_n \in z_0 + V \subseteq P_T(A,B) + V$ for sufficiently large n, that is a contradiction.

Conversely, suppose that (A,B) is a strongly proximity pair and $P_T(A,B)$ compact, but (A,B) is not T-a.c.p. Then, there is a T-minimizing sequence $\{z_n\} \subseteq A \cup B$ without any convergent subsequence. It follows that, for any $x \in P_T(A,B)$, there is a neighborhood U_x of x such that, for sufficiently large n, z_n does not belong to U_x. Since $P_T(A,B)$ is compact, one can cover $P_T(A,B)$ by finitely many $U_{x_i}, i = 1,2,\dots,n$. So there is a neighborhood V of 0 and $n_0 \in N$ such that for all $n \geq n_0$, z_n does not belong to $P_T(A,B) + V$. Since $P_T(A,B)$ is strongly proximity pair, there exists $\delta > 0$ such that $P_T^{\delta}(A,B) \subseteq P_T(A,B) + V$. Since $\{z_n\}$ is a T-minimizing sequence, $z_n \in P_T^{\delta}(A,B)$ for sufficiently large $n \in N$ and this is a contradiction. □

Corollary 3.4. *Let A and B be nonempty subsets of normed space X such that $A \cup B$ is compact and $T : A \cup B \to A \cup B$ is continuous such that $T(A) \subset B$ and $T(B) \subset A$. Then, the pair (A,B) is strongly proximity pair and $P_T(A,B)$ is compact.*

Proof. Since $A \cup B$ is compact, it is obvious that the pair (A,B) is T-a.c.p. Now, apply Theorem 3.3. □

Theorem 3.5. *Let A and B be nonempty closed subsets of a normed space X, and $T : A \cup B \to A \cup B$ is continuous such that $T(A) \subset B$ and $T(B) \subset A$ and $P_T(A,B) \neq \emptyset$. Then, the pair (A,B) is a T-s.c.p. if and only if the pair (A,B) is strongly proximity pair and $P_T(A,B)$ singleton.*

Proof. Suppose that (A,B) is T-s.c.p. By Theorem 3.3, (A,B) is strongly proximity pair, and, by Lemma 3.2, $P_T(A,B)$ is singleton.

Conversely, suppose (A,B) is strongly proximity pair and $P_T(A,B) = \{z_0\}$. Let V be a neighborhood of 0. Since (A,B) is strongly proximity pair, there exists $\delta > 0$ such that $P_T^{\delta}(A,B) \subseteq z_0 + V$. Thus, for any T-minimizing sequence $\{z_n\} \subseteq A \cup B$, $z_n \in P_T^{\delta}(A,B) \subseteq z_0 + V$ for sufficiently large n. Hence, $z_n \to z_0$. □

Theorem 3.6. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B$, $T(B) \subset A$, and, for every $x, y \in A$*

$$\|Tx - Ty\| \leq \alpha \|x - y\| \quad (0 < \alpha < 1). \tag{3.8}$$

Then, the pair (A, B) is a T-s.c.p. if and only if $P_T(A, B)$ is singleton.

Proof. The necessary condition follows from Theorem 3.5.

For the proof of sufficient condition, suppose that $P_T(A, B) = \{z_0\}$ but (A, B) is not a T-s.c.p. Then, there is a T-minimizing sequence $\{z_n\} \subseteq A \cup B$ that is not convergent. It follows that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and a scaler $d > 0$ such that for all integer k,

$$\|z_{n_k} - z\| \geq d. \tag{3.9}$$

By uniform convexity of X, there exists $\epsilon > 0$ such that

$$(d(A, B) + \epsilon)\left[1 - \delta_X\left(\frac{d(1 - \alpha)}{d(A, B) + \epsilon}\right)\right] < d(A, B). \tag{3.10}$$

Since $\lim_{n \to \infty} \|z_n - Tz_n\| = d(A, B)$, there exists k such that

$$\|z_{n_k} - Tz_{n_k}\| < d(A, B) + \epsilon. \tag{3.11}$$

Also,

$$\|z_0 - Tz_0\| = d(A, B) < d(A, B) + \epsilon. \tag{3.12}$$

But

$$\begin{aligned} \|z_0 - Tz_0 - (z_{n_k} - Tz_{n_k})\| &\geq \|z_0 - z_{n_k}\| - \|Tz_0 - Tz_{n_k}\| \\ &\geq \|z_0 - z_{n_k}\| - \|z_0 - y_{n_k}\|\alpha \\ &= \|z_0 - z_{n_k}\|(1 - \alpha) \\ &> d(1 - \alpha), \end{aligned} \tag{3.13}$$

Because A and B are convex, $((z + z_{n_k})/2) \in A$, and $Tz + Tz_{n_k}/2 \in B$, the following inequality leads to a contradiction:

$$\begin{aligned} \left\|\frac{z + z_{n_k}}{2} - \frac{Tz + Tz_{n_k}}{2}\right\| &\leq (d(A, B) + \epsilon)\left[1 - \delta_X\left(\frac{d(1 - \alpha)}{d(A, B) + \epsilon}\right)\right] \\ &< d(A, B). \end{aligned} \tag{3.14}$$

□

References

[1] M. A. Al-Thagafi and N. Shahzad, "Best proximity pairs and equilibrium pairs for Kakutani multimaps," *Nonlinear Analysis*, vol. 70, no. 3, pp. 1209–1216, 2009.

[2] I. Singer, *Best Approximations Normed Linear Spaces by Elements of Linear Subspaces*, Springer, 1970.

[3] T. Suzuki, M. Kikkawa, and C. Vetro, "The existence of best proximity points in metric spaces with the property UC," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2918–2926, 2009.

[4] X. B. Xu, "A result on best proximity pair of two sets," *Journal of Approximation Theory*, vol. 54, no. 3, pp. 322–325, 1988.

[5] W. A. Kirk, S. Reich, and P. Veeramani, "Proximinal retracts and best proximity pair theorems," *Numerical Functional Analysis and Optimization*, vol. 24, no. 7-8, pp. 851–862, 2003.

[6] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[7] M. A. Al-Thagafi and N. Shahzad, "Convergence and existence results for best proximity points," *Nonlinear Analysis*, vol. 70, no. 10, pp. 3665–3671, 2009.

[8] G. Beer and D. Pai, "Proximal maps, prox maps and coincidence points," *Numerical Functional Analysis and Optimization*, vol. 11, no. 5-6, pp. 429–448, 1990.

[9] M. Derafshpour, S. Rezapour, and N. Shahzad, "On the existence of best proximity points of cyclic contractions," *Advances in Dynamical Systems and Applications*, vol. 6, no. 1, pp. 33–40, 2011.

[10] S. A. M. Mohsenalhosseini, H. Mazaheri, and M. A. Dehghan, "Approximate best proximity pairs in metric space," *Abstract and Applied Analysis*, vol. 2011, Article ID 596971, 9 pages, 2011.

[11] K. Fan, "Extensions of two fixed point theorems of F. E. Browder," *Mathematische Zeitschrift*, vol. 112, pp. 234–240, 1969.

[12] W. K. Kim and K. H. Lee, "Existence of best proximity pairs and equilibrium pairs," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 433–446, 2006.

[13] Sh. Rezapour, M. Derafshpour, and N. Shahzad, "Best proximity points of cyclic $\emptyset$-contractions on reflexive Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 946178, 7 pages, 2010.

[14] S. Sadiq Basha, N. Shahzad, and R. Jeyaraj, "Common best proximity points: global optimization of multi-objective functions," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 883–886, 2011.