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Research Article

On the Travelling Waves for the Generalized Nonlinear Schrödinger Equation

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This paper is devoted to the analysis of the travelling waves for a class of generalized nonlinear Schrödinger equations in a cylindric domain. Searching for travelling waves reduces the problem to the multiparameter eigenvalue problems for a class of perturbed p-Laplacians. We study dispersion relations between the eigenparameters, quantitative analysis of eigenfunctions and discuss some variational principles for eigenvalues of perturbed p-Laplacians. In this paper we analyze the Dirichlet, Neumann, No-flux, Robin and Steklov boundary value problems. Particularly, a "duality principle" between the Robin and the Steklov problems is presented.

1. Introduction

The main concerns of the paper are the travelling waves for the generalized nonlinear Schrödinger (NLS) equation with the free initial condition in the following form (see [1] for generalized NLS):

$$\begin{gathered} iv_t - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = \nu|v|^{q-2}v, \quad p>1, \\ v|_{\partial Q} = 0, \end{gathered} \tag{1.1}$$

where $v := v(t, x_1, x_2, \dots, x_{n+1})$ and ν is a parameter. $Q = \mathbb{R} \times \Omega$ is a cylinder, ∂Q is the lateral boundary of Q, $t > 0$, $x_1 \in \mathbb{R}$, and $(x_2, x_3, \dots, x_{n+1}) \in \Omega$. Assume that Ω is a bounded domain in $\mathbb{R}^n$ with the smooth boundary. Particularly, in the case of $p = 2$ we get

$$iv_t - \Delta v = \nu|v|^{q-2}v, \tag{1.2}$$

which is a nonlinear Scrödinger equation (see [1]). On the other hand problem (1.1) can be considered as an evolution pq-Laplacian equation. Different aspects of such kind of problems, with some initial conditions, have been studied in [2]. Thus problem (1.1) models the linear Schrödinger equations ($p = q = 2$), NLS ($p = 2$, $q \neq 2$), evolution pq-Laplacians, and generalized NLS. This definitely means that we have a good motivation for problem (1.1).

In this paper by the travelling waves we mean the solutions of (1.1) in the form $v = e^{i(wt-kx)}u(x_2,\dots,x_{n+1})$, where $x := x_1$ and u is a real-valued function. A simple computation yields $v_t = iwe^{i(wt-kx)}u$, $v_x = -ike^{i(wt-kx)}u$, and $v_{x_i} = e^{i(wt-kx)}u_{x_i}$, $i = 2,3,\dots,n+1$. Hence, $\nabla v := (v_x, v_{x_2},\dots,v_{x_{n+1}}) = (-iku,\nabla u)e^{i(wt-kx)}$ and $|\nabla v| = (k^2u^2+u_{x_2}^2+\cdots+u_{x_{n+1}}^2)^{1/2}$. By using the notation $\nabla_k u(x_2,x_3,\dots,x_{n+1}) := (ku, u_{x_2},\dots,u_{x_{n+1}})$ we obtain $|\nabla v| = |\nabla_k u|$. Finally, by setting all of these into (1.1) we can obtain the following nonstandard multiparameter eigenvalue problems for perturbed p-Laplacians:

$$\begin{gathered}-wu + k^2|\nabla_k u|^{p-2}u - \operatorname{div}\left(|\nabla_k u|^{p-2}\nabla u\right) = \nu|u|^{q-2}u, \quad p > 1,\\ u|_{\partial\Omega} = 0.\end{gathered} \tag{1.3}$$

In what follows, by shifting the variables $x_2,\dots,x_{n+1}$, we have used the following notations: $u := u(x_1,\dots,x_n)$, $|\nabla_k u| = (k^2u^2 + u_{x_1}^2 + \cdots + u_{x_n}^2)^{1/2}$, and $\nabla u = (u_{x_1}, u_{x_2},\dots,u_{x_n})$.

At this point we have to note that by searching for the standing waves $v = e^{iwt}u(x_1,x_2,\dots,x_{n+1})$ for the NLS equation

$$iv_t - \Delta v = \nu|v|^{q-2}v \tag{1.4}$$

we obtain the following eigenvalue problem:

$$-wu - \Delta u = \nu|u|^{q-2}u. \tag{1.5}$$

On the other hand by setting the travelling wave solutions of the form $v = e^{i(wt-kx)}u(x_2,\dots,x_{n+1})$ into the NLS equation we obtain

$$-\left(w + k^2\right)u - \Delta u = \nu|u|^{q-2}u. \tag{1.6}$$

Thus we obtain the same type eigenvalue problems for both standing and travelling waves for the NLS equation (see [3] and references therein for standing waves for NLS). However, standing and travelling waves for generalized NLS are associated with quite different type of eigenvalue problems. Particularly, the eigenvalue problem associated to the travelling wave solutions is problem (1.3), which is clearly a nonstandard multiparameter eigenvalue problem in the nonlinear analysis, and we are not aware of any known result for this problem.

A solution of (1.3) is a weak solution, defined in the following way.

Definition 1.1. $0 \neq u \in W_0^{1,p}(\Omega)$ is a solution of (1.3) if and only if

$$-w\int_\Omega uv\,dx + k^2\int_\Omega |\nabla_k u|^{p-2}uv\,dx + \int_\Omega |\nabla_k u|^{p-2}\nabla u\cdot\nabla v\,dx = \nu\int_\Omega |u|^{q-2}uv\,dx \tag{1.7}$$

holds for all $v \in W_0^{1,p}(\Omega)$, where $W_0^{1,p}(\Omega)$ is the Sobolev space (for Sobolev spaces, see [4]). In this case, we say that u is an eigenfunction, corresponding to the eigenpair (w,k) and ν, where w is a frequency, k is a wave number, and the parameter ν comes from the initial equation (1.1). We prefer to denote the test functions in (1.7) by v, which is clearly different from the notation that is used in (1.1). Let us define

$$\begin{gathered} F(u)=-\frac{w}{2}\int_{\Omega}u^2dx+\frac{1}{p}\int_{\Omega}|\nabla_k u|^p dx-\frac{\nu}{q}\int_{\Omega}|u|^q dx, \\ G_k(u)=\frac{1}{p}\int_{\Omega}|\nabla_k u|^p dx. \end{gathered} \tag{1.8}$$

We set $X := W_0^{1,p}(\Omega)$. Then, $u \in X$ is a solution of (1.7) if and only if u is a free critical point for $F(u)$, that is, $\langle F'(u), v\rangle = 0$, for all $v \in X$, where $F' : X \to X^*$ is the Fréchet derivative of F, X^* is the dual space, and $\langle F'(u), v\rangle$ denotes the value of the functional $F'(u)$ at $v \in X$. Indeed, the existence of Fréschet derivative implies the existence of directional (Gateaux) derivative. Using the definition of Gateaux derivative, we can obtain

$$\left\langle G_k'(u),v\right\rangle=\frac{d}{dt}G_k(u+tv)\bigg|_{t=0}=k^2\int_{\Omega}|\nabla_k u|^{p-2}uv\,dx+\int_{\Omega}|\nabla_k u|^{p-2}\nabla u\cdot\nabla v dx, \tag{1.9}$$

which is enough to see that (1.7) is the variational equation for the functional $F(u)$.

As $u \in W_0^{1,p}(\Omega)$, by Sobolev embedding theorems (see [4]) the functional $F(u)$ can be well defined if

(i) $p \geq n$ or

(ii) $1 < p < n$ and $\max\{2,q\} \leq np/(n-p)$.

In the next section two cases $\nu = 0$ and $\nu \neq 0$ will be studied separately. If $\nu = 0$, then we may rewrite (1.7) in the form

$$k^2\int_{\Omega}|\nabla_k u|^{p-2}uvdx+\int_{\Omega}|\nabla_k u|^{p-2}\nabla u\cdot\nabla v\,dx=w\int_{\Omega}uvdx, \tag{1.10}$$

which is the equation for free critical points of the functional

$$F(u)=\frac{1}{p}\int_{\Omega}|\nabla_k u|^p dx-\frac{w}{2}\int_{\Omega}u^2dx. \tag{1.11}$$

Evidently, there are not nontrivial solutions of (1.10) in the case of $w \leq 0$. Thus, $w > 0$, and by the scaling property, we obtain that if $p \neq 2$ and (1.10) has a nontrivial solution for some $w > 0$, then it has nontrivial solutions for all $w > 0$.

In what follows, $\|u\| := [\int_{\Omega}|\nabla u|^p dx]^{1/p}$ denotes the standard norm in $W_0^{1,p}(\Omega)$ and $\|u\|_k := [\int_{\Omega}|\nabla_k u|^p dx]^{1/p}$, which is equivalent to the norm $\|u\|$.

This paper consists of an introduction (Section 1) and two sections. In Section 2 we study the structure of the eigenparameters ν, k, w and the eigenfunctions for problem (1.3), including the dispersion relations between w, k, and ν and variational principles in some special cases. We consider separately two cases: $\nu = 0$ and $\nu \neq 0$. In the case of $\nu = 0$ we have estimated bounds for the set of eigenfunctions, proved the existence of infinitely many eigenfunctions, corresponding to an eigenpair (w, k), $w > 0$, and demonstrated that, in the case of $p > 2$, the set crit F is compact. For the general case $\nu \neq 0$ we study the existence of positive solutions and variational principles in some special cases. The proofs are based on the Sobolev imbedding theorems, the Palais-Smale condition, variational techniques, and the Ljusternik-Schnirelman critical point theory. Various boundary problems and some relations between them are studied in Section 3.

2. The Structure of Eigenparameters w, k, ν and Related Eigenfunctions

As mentioned above the problem of the existence of travelling waves and a quantitative analysis for travelling waves is a multiparameter eigenvalue problem given by (1.7) or (1.10). This section is devoted to these problems, and the techniques we use in this section are partially close to that used in [5].

We study separately two cases: $\nu = 0$ and $\nu \neq 0$.

Case 1 ($\nu = 0$). This subsection is devoted to the quantitative analysis of solutions of (1.10). We assume that

(i) $p \geq n$ or

(ii) $1 < p < n$ and $2 < np/(n-p)$.

Our first observation for eigenvalue problem (1.10) is given in the following proposition.

Proposition 2.1. *(a) Let $p = 2$. In this case, all the eigenpairs (w, k) of problem* (1.10) *lie in the parabola $\lambda_1 + k^2 \leq w$, where λ_1 is the first eigenvalue of $-\Delta$ in $L_2(\Omega)$ and $\lambda_1 > 0$. Moreover, for a fixed w, there is a finite number of k and for a fixed k, there is a countable number of $w_n(k)$, such that $w_n(k) \to +\infty$ as $n \to \infty$,*

(b) If $p > 2$, then for a fixed $(w, k) \in \mathbb{R} \times \mathbb{R}$, the solutions of (1.10) *are bounded and*

$$\|u\|_k \leq \left[C_{p,k}(\Omega)w\right]^{1/(p-2)} \tag{2.1}$$

holds for some $C_{p,k}(\Omega) > 0$,

(c) Let $p < 2$. In this case, one has

$$\|u\|_k \geq \left(\frac{1}{wC_{p,k}(\Omega)}\right)^{1/(p-2)} > 0. \tag{2.2}$$

Proof. The proof easily follows from (1.10), by using the Courant-Weyl variational principle

$$0 < \lambda_1 = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}, \tag{2.3}$$

the bounded embedding $W_0^{1,p}(\Omega) \hookrightarrow L_2(\Omega)$, and the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_k$. □

In the following theorem, we have proved some properties of the functional $F(u) = (1/p)\int_\Omega |\nabla_k u|^p dx - (w/2)\int_\Omega u^2\, dx$ in X, which guarantee the existence of nontrivial solutions of (1.10) for a fixed $(w,k) \in \mathbb{R}\times\mathbb{R}$, $w > 0$.

Theorem 2.2. *Let $p > 2$ and $(w,k) \in \mathbb{R}\times\mathbb{R}$, $w > 0$. Then*

(i)

$$\left(\frac{1}{p} - \frac{1}{2}\right)\left(wC_{p,k}(\Omega)\right)^{p/(p-2)} \le \inf_X F(u) < 0 \tag{2.4}$$

for some $C_{p,k}(\Omega) > 0$.

(ii) *F attains its infimum at a nontrivial vector $u_0 \in X$.*

Proof. One has $F(u) = (1/p)\int_\Omega |\nabla_k u|^p dx - w/2\int_\Omega u^2 dx$. As $p > 2$, the Poincaré inequality yields $\int_\Omega u^2 dx \le C_{p,k}(\Omega)\|u\|_k^2$. Hence,

$$F(u) = \frac{1}{p}\|u\|_k^p - \frac{w}{2}\int_\Omega u^2 dx \ge \frac{1}{p}\|u\|_k^p - \frac{w}{2}C_{p,k}(\Omega)\|u\|_k^2. \tag{2.5}$$

Let $f(x) = (1/p)x^p - (w/2)C_{p,k}(\Omega)x^2$. We have $f(0) = 0$ and $f(x) \to +\infty$ as $n \to \infty$. This indicates that f has a global minimum point, and clearly, this point is $x = [wC_{p,k}(\Omega)]^{1/(p-2)}$. Now, by putting the vectors u with $\|u\|_k = [wC_{p,k}(\Omega)]^{1/(p-2)}$ into $F(u)$, we obtain

$$\left(\frac{1}{p} - \frac{1}{2}\right)\left(wC_{p,k}(\Omega)\right)^{p/(p-2)} \le \inf_X F(u). \tag{2.6}$$

Now, we will show that $\inf_X F(u) < 0$. Let u be a vector such that $\|u\|_k = 1$. Subsequently, by setting tu in $F(u)$, we obtain $F(tu) = (t^p/p) - (w/2)t^2 c$, where $c = \int_\Omega u^2 dx$. Thus $F(tu) < 0$ if $0 < t < [(p/2)wc]^{1/(p-2)}$.

(ii) We have to show that $\inf_X F(u)$ is attained. Let $\inf_X F(u) = \alpha$. Then, there exists a sequence $u_n \in X$ such that $F(u_n) \to \alpha$ as $n \to \infty$. The sequence u_n should be bounded, because

$$F(u) \ge \frac{1}{p}\|u\|_k^p - \frac{w}{2}C_{p,k}(\Omega)\|u\|_k^2 = \|u\|_k^2\left(\frac{1}{p}\|u\|_k^{p-2} - \frac{w}{2}C_{p,k}(\Omega)\right) \tag{2.7}$$

and $F(u) \to +\infty$ as $\|u\|_k \to +\infty$, which means that F is coercive. However, $X = W_0^{1,p}(\Omega)$ is a reflexive Banach space. Consequently, $u_n \rightharpoonup u_0$ for some $u_0 \in X$, where "$\rightharpoonup$" denotes the weak convergence in X. Evidently, $F(u)$ is sequentially lower semicontinuous, that is,

$$u_n \rightharpoonup u_0 \text{ implies } F(u_0) \le \lim_{n\to\infty}\inf F(u_n). \tag{2.8}$$

Indeed, $\int_\Omega |\nabla_k u|^p dx = \|u\|_k^p$, and it is known that the norm is sequentially lower semicontinuous. The second term of $F(u)$ is $\int_\Omega u^2 dx$, and this term is sequentially continuous, because the embedding $W_0^{1,p}(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Now, it follows from (2.8) that $F(u_0) \leq \alpha$, which means $\inf_X F(u) = F(u_0)$. Finally, $u_0 \neq 0$ because $\inf_X F(u) < 0$ by (i) and $F(0) = 0$. □

Corollary 2.3. *In the case of $p > 2$, all pairs $(w, k) \in \mathbb{R} \times \mathbb{R}$, $w > 0$, are eigenpairs of problem* (1.10).

Proof. This immediately follows from (ii) of Theorem 2.2. □

Now, we will prove that there are infinitely many solutions of (1.10) for all $(w, k) \in \mathbb{R} \times \mathbb{R}$, $w > 0$. For this, our main component will be Proposition 2.5 ([6], p. 324 Proposition 44.18) about free critical points of a functional, that is, about the solutions of the operator equation

$$F'(u) = 0, \quad u \in X. \tag{2.9}$$

First, we will give some definitions, including the Palais-Smale (PS-condition) which are crucial in the theory of nonlinear eigenvalue problems (see [6, 7]).

Definition 2.4. Let $F \in C'(X, \mathbb{R})$. F satisfies the PS-condition at a point $c \in \mathbb{R}$ if each sequence $u_n \in X$, such that $F(u_n) \to c$ and $F'(u_n) \to 0$ in X^* has a convergent subsequence.

Particularly, F satisfies $(\mathrm{PS})_-$ if and only if it satisfies the PS-condition for all $c < 0$.

Let us denote by $\mathcal{K}_m$ the class of all compact, symmetric, and zero-free subsets K of X, such that $\operatorname{gen} K \geq m$. Here, $\operatorname{gen} K$ is defined as the smallest natural number $n \geq 1$ for which there exists an odd and continuous function $f : K \to \mathbb{R}^n \setminus \{0\}$. Let

$$c_m = \inf_{K \subset \mathcal{K}_m} \sup_{u \in K} F(u), \quad m = 1, 2, \ldots. \tag{2.10}$$

Suppose that

$(H1)$ X is a real B-space,

$(H2)$ F is an even functional with $F \in C'(X, \mathbb{R})$,

$(H3)$ F satisfies $(\mathrm{PS})_-$ with respect to X and $F(0) = 0$.

As mentioned earlier, our main component will be the following proposition.

Proposition 2.5. *If $(H1)$, $(H2)$, and $(H3)$ hold and $-\infty < c_m < 0$, then F has a pair of critical points $(u, -u)$ on X such that $F(\pm u) = c_m$, to which solutions of* (2.9) *correspond. Moreover, if $-\infty < c_m = c_{m+1} = \cdots = c_{m+p} < 0$, $p \geq 1$, then $\operatorname{gen}(\operatorname{crit}_{X,c_m} F) \geq p + 1$, where $\operatorname{crit}_{X,c_m} F = \{u \in X \mid F'(u) = 0,\ F(u) = c_m\}$.*

Now, we are ready to prove the following theorem.

Theorem 2.6. *Let $p > 2$. (a) For each (w, k), $w > 0$, problem* (1.10) *has an infinite number of nontrivial solutions.*

(b) The set $\operatorname{crit} F$ is compact, where $\operatorname{crit} F := \{u \in X \mid F'(u) = 0\}$.

Proof. Our proof is based on the previous proposition. Clearly, conditions $(H1)$ and $(H2)$ are satisfied. The fact that F satisfies $(\mathrm{PS})_{-}$ is standard (see [5]). In our case F satisfies PS-condition for all $c \in \mathbb{R}$. It needs to be demonstrated that $c_m < 0$, $m = 1, 2, \ldots$. By the definition of "$inf\,sup$", it is adequate to show the existence of a set $K \in \mathcal{K}_m$ such that $\sup_K F(u) < 0$. We have

$$F(u) = \frac{1}{p}\int_{\Omega} |\nabla_k u|^p dx - \frac{w}{2}\int_{\Omega} u^2 dx. \tag{2.11}$$

Let X_m be an m-dimensional subspace of X and S_1 the unit sphere in X. We can choose $u \in X_m \cap S_1$ and define $F(tu) = (t^p/p) - (w/2)t^2 \int_{\Omega} u^2 dx$. As $X_m \cap S_1$ is compact, $\inf_{u \in X_m \cap S_1} \int_{\Omega} u^2 dx := \alpha(m) > 0$. Hence,

$$F(tu) \le \frac{t^p}{p} - \frac{w}{2} t^2 \alpha(m), \quad \forall u \in X_m \cap S_1. \tag{2.12}$$

Moreover, $\lim_{t \to 0} F(tu) = 0$ and $F(tu) < 0$ provided $0 < t < [(p/2)w\alpha(m)]^{1/(p-2)}$. Using this fact we obtain that for every m there exist $\varepsilon_m > 0$ and $t_m > 0$ such that $F(t_m u) < -\varepsilon_m$ for all $u \in X_m \cap S_1$. Clearly $t_m u \in S_{t_m}$ and $\mathrm{gen}(X_m \cap S_{t_m}) = m$. Now, set $K := X_m \cap S_{t_m}$. Then $\sup_K F(u) \le -\varepsilon_m < 0$. Consequently, $\inf_{K \subset \mathcal{K}_m} \sup_{u \in K} F(u) < 0$. By Theorem 2.2 F is bounded below. Hence,

$$-\infty < c_m = \inf_{K \subset \mathcal{K}_m} \sup_{u \in K} F(u) < 0, \tag{2.13}$$

and the statements of the theorem in (a) follow from Proposition 2.5.

(b) Let us prove that the set $\mathrm{crit}\, F$ is compact. Let $u_n \in \mathrm{crit}\, F$ be a sequence. Then

$$\langle F'(u_n), u_n \rangle = \int_{\Omega} |\nabla_k u_n|^p dx - \frac{w}{2}\int_{\Omega} u_n^2 dx = 0. \tag{2.14}$$

However,

$$F(u_n) = \frac{1}{p}\int_{\Omega} |\nabla_k u_n|^p dx - \frac{w}{2}\int_{\Omega} u_n^2 dx \le \frac{1}{2}\langle F'(u_n), u_n \rangle = 0, \tag{2.15}$$

and by Theorem 2.2, F is bounded below. Thus, $F(u_n)$ is bounded, and consequently, it has a convergent subsequence (denoted again by $F(u_n)$). We have $F(u_n) \to c$ and $F'(u_n) = 0$. Hence, by PS-condition, u_n has a convergent subsequence. The limit points of u_n belong to $\mathrm{crit}\, F$ because, by PS-condition, the set $\mathrm{crit}\, F$ is closed. □

The case $p < 2$. This case is standard, and by similar methods that are given in [5] and earlier for the case $p > 2$, one can establish the existence of nontrivial solutions of (1.10) for all (w, k), $w > 0$.

Case 2 ($\nu \neq 0$). We first look at the following problem:

$$k^2 \int_\Omega |\nabla_k u|^{p-2} uv dx + \int_\Omega |\nabla_k u|^{p-2} \nabla u \cdot \nabla v dx - \nu \int_\Omega |u|^{p-2} uv dx = w \int_\Omega uv dx, \qquad u > 0 \text{ in } \Omega, \tag{2.16}$$

where $u \in W_0^{1,p}(\Omega)$ and (2.16) holds for all $v \in W_0^{1,p}(\Omega)$. The main result is as follows.

Proposition 2.7. *Let* $2 < np/(n-p)$. *Then (a) for all* $\nu < \nu_1(k)$ *there is a positive solution to problem* (2.16), *where* $0 < \nu_1(k) = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \int_\Omega |\nabla_k u|^p / \int_\Omega |u|^p dx$,

(b) for each $\nu \in \mathbb{R}$ *there is a number* k^* *such that for all* $k > k^*$ *problem* (2.16) *has a positive solution.*

Proof. (a) As a result of the compact imbedding $W_0^{1,p}(\Omega) \hookrightarrow L_2(\Omega)$ and the fact that $\int_\Omega |\nabla_k u|^p dx$ is a norm in $W_0^{1,p}(\Omega)$, the functional

$$F(u) = \frac{1}{p} \int_\Omega |\nabla_k u|^p dx - \frac{\nu}{p} \int_\Omega |u|^p dx, \tag{2.17}$$

is coercive and lower semicontinuous on the weakly closed set $M := \{u \mid \int_\Omega u^2 = 1\}$. From these properties, by using the condition $\nu < \nu_1(k)$ we obtain the existence of a nonnegative solution. The positivity follows from the maximum principle.(b) This fact follows from (a) and the relation $k^p < \nu_1(k) \to \infty$ as $k \to \infty$. $\square$

Note 1. The case $2 = np/(n-p)$ is the critical case: lack of compactness, which is a subject that deserves a separate study.

Now, our concern is the following typical eigenvalue problem:

$$k^2 \int_\Omega |\nabla_k u|^{p-2} uv\, dx + \int_\Omega |\nabla_k u|^{p-2} \nabla u \cdot \nabla v\, dx = \nu \int_\Omega |u|^{p-2} uv\, dx. \tag{2.18}$$

Let us look at problem (2.18) with respect to ν for a fixed k. This problem is a typical eigenvalue problem. If $k = 0$, then we get the p-Laplacian eigenvalue problem, and these questions have been studied by many authors (see [8, 9] and the references therein). Particularly, it has been shown in [8] that there is a sequence of "variational eigenvalues" which can be described by the Ljusternik-Schnirelman type variational principles. Our aim is to get the similar results for perturbed p-Laplacian eigenvalue problem (2.18). In our case $k \neq 0$, and we can apply two methods.

Method 1. For the Diriclet problem the norms: $\|u\| := [\int_\Omega |\nabla u|^p dx]^{1/p}$, which is the standard norm in $W_0^{1,p}(\Omega)$, and $\|u\|_k := [\int_\Omega |\nabla_k u|^p dx]^{1/p}$ are equivalent. Then it is enough to replace $\langle X, \|u\| \rangle$ by the Banach space $\langle X, \|u\|_k \rangle$ and follow the methods of [8, 9] to get the needed results.

Method 2. One can construct a Ljusternik-Schnirelman deformation (see [6, 7]) and check Palais-Smale condition for the functional

$$F(u) = \frac{1}{p} \int_{\Omega} |u|^p dx \tag{2.19}$$

on the manifold

$$G_k = \left\{ u \mid \int_{\Omega} |\nabla_k u|^p dx = 1 \right\}. \tag{2.20}$$

A such construction was given in our previous paper (see [10]). We follow our construction and just give the final result.

Theorem 2.8. *For a fixed $k \in \mathbb{R}$, there exists a sequence of eigenvalues of problem* (2.18)*, depending on k, which is given by*

$$\frac{1}{\nu_n(k)} = \sup_{K \subset \mathcal{K}_n(k)} \inf_{u \in K} F(u). \quad \textit{Moreover,} \ \nu_n(k) \longrightarrow \infty, \ \textit{as} \ n \longrightarrow \infty, \tag{2.21}$$

where one denotes by $\mathcal{K}_n(k)$ the class of all compact, symmetric subsets K of G_k, such that $\operatorname{gen} K \geq n$.

3. On the Neumann, No-Flux, Robin, and Steklov Boundary Value Problems

At the end of the paper we briefly discuss the other boundary problems, such as Neumman, No-flux, Robin, and Steklov. We note that all of the above given results are related to problem (1.3) with the Diriclet boundary condition; however the similar results are valid for the following boundary conditions too:

Neumann problem:

$$\begin{gathered} -wu + k^2 |\nabla_k u|^{p-2} u - \operatorname{div}\left(|\nabla_k u|^{p-2} \nabla u \right) = \nu |u|^{q-2} u, \quad p > 1, \\ \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega} = 0. \end{gathered} \tag{3.1}$$

No-flux problem:

$$\begin{gathered} -wu + k^2 |\nabla_k u|^{p-2} u - \operatorname{div}\left(|\nabla_k u|^{p-2} \nabla u \right) = \nu |u|^{q-2} u, \quad p > 1, \\ u|_{\partial\Omega} = \text{constant}, \qquad \int_{\partial\Omega} |\nabla_k u|^{p-2} \frac{\partial u}{\partial n} ds = 0. \end{gathered} \tag{3.2}$$

Robin problem:

$$-wu + k^2 |\nabla_k u|^{p-2} u - \operatorname{div}\left(|\nabla_k u|^{p-2} \nabla u \right) = \nu |u|^{q-2} u, \quad p > 1,$$
$$\left(|\nabla_k u|^{p-2} \frac{\partial u}{\partial n} + \beta(x) |u|^{p-2} u \right) \Big|_{\partial\Omega} = 0. \tag{3.3}$$

Steklov problem:

$$-wu + k^2 |\nabla_k u|^{p-2} u - \operatorname{div}\left(|\nabla_k u|^{p-2} \nabla u \right) = |u|^{q-2} u, \quad p > 1,$$
$$|\nabla_k u|^{p-2} \frac{\partial u}{\partial n} = \nu |u|^{p-2} u \quad \text{on } \partial\Omega. \tag{3.4}$$

Evidently, the energy space X (the Banach space, we use in the critical point theory) for Dirichlet, Neuman, No-flux, Robin, and Steklov problems is $W_0^{1,2}(\Omega)$, $W^{1,2}(\Omega)$, $W_0^{1,2}(\Omega) \oplus \mathbb{R}$, $W^{1,2}(\Omega)$, and $W^{1,2}(\Omega)$, respectively. In the case of $w = 0$, $k = 0$, and $p = q$ we obtain the standard eigenvalue problems for p-Laplacians, which have been studied in detail in [8] for all of the above given boundary value problems. Many results for standard p-Laplacians, including the regularity results, may be extended to the perturbed p-Laplacians by the similar techniques that are used in [8]. However, we omit these questions in this paper.

Our simple observation between Robin and Steklov problems is as follows: (w, k, ν) *is an eigentriple for Steklov problem if and only if* $(w, k, 1)$ *is an eigentriple for Robin problem at* $\beta = -\nu$.

Finally, we use a similar connection between the typical Robin and Steklov eigenvalue problems to prove the existence of negative eigenvalues for the Robin problem. For sake of simplicity we choose $k = 0$ and consider the following standard eigenvalue problems for p-Laplacians:

$$\text{Robin problem:} \ -\operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = \nu |u|^{p-2} u, \quad p > 1,$$
$$\left(|\nabla u|^{p-2} \frac{\partial u}{\partial n} + \beta |u|^{p-2} u \right) \Big|_{\partial\Omega} = 0,$$
$$\text{Steklov problem:} \ \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = |u|^{p-2} u, \quad p > 1,$$
$$|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \nu |u|^{p-2} u \quad \text{on } \partial\Omega. \tag{3.5}$$

Problems (3.5) can be rewritten in the following variational forms:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \; dx + \beta \int_{\partial\Omega} |u|^{p-2} uv ds = \nu \int_\Omega |u|^{p-2} uv,$$
$$u \in W^{1,2}(\Omega), \quad \forall v \in W^{1,2}(\Omega), \tag{3.6}$$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} uv = \nu \int_{\partial\Omega} |u|^{p-2} uv \, ds,$$
$$u \in W^{1,2}(\Omega), \quad \forall v \in W^{1,2}(\Omega), \tag{3.7}$$

respectively.

It is known that (see [8])

(I) if $\beta \geq 0$ then the Robin problem has a sequence of positive eigenvalues $\nu_n(\beta)$ such that $\nu_n(\beta) \rightarrow +\infty$ as $n \rightarrow \infty$;

(II) the Steklov problem also has a sequence of positive eigenvalues ν_n such that $\nu_n \rightarrow +\infty$ as $n \rightarrow \infty$.

An Inverse Problem

Now let us be given $\nu < 0$. Our question is as follows: for what values of β the given number $\nu < 0$ will be an eigenvalue for Robin problem (3.6). To answer this question we use a "duality principle" between Robin and Steklov problems and give the final result in the following theorem.

Theorem 3.1. *For a given $\nu < 0$ there exists a sequence $\beta_n \rightarrow -\infty$, such that the number $\nu < 0$ will be an eigenvalue for the Robin problem at $\beta = \beta_n$, $n = 1, 2, \ldots$. Moreover, $\beta_n = -\nu_n$ and ν_n are the eigenvalues of the Steklov problem.*

Proof. The proof is based on the relations between the Robin and Steklov problems. To answer this question we consider the Steklov problem in the form

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \beta |u|^{p-2} u, \quad p > 1,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} = \nu |u|^{p-2} u \quad \text{on } \partial\Omega. \tag{3.8}$$

Then the variational problem (3.7) is replaced by

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \beta \int_\Omega |u|^{p-2} uv = \nu \int_{\partial\Omega} |u|^{p-2} uv ds. \tag{3.9}$$

By comparing (3.6) and (3.9) we obtain that (β, ν) is an eigenpair for the Steklov problem if and only if $(-\nu, -\beta)$ is an eigenpair for the Robin problem. We know that (see [8]) for a positive number β Steklov problem (3.9) has a sequence of positive eigenvalues ν_n such that $\nu_n \rightarrow +\infty$ as $n \rightarrow \infty$. Thus (β, ν_n), $n = 1, 2, \ldots$ are eigenpairs for (3.9). Then it follows that

$(-\nu_n, -\beta)$, $n = 1, 2, \ldots$ are eigenpairs for the Robin problem. To end the proof we notice that $\nu = -\beta$ and $\beta_n = -\nu_n$. □

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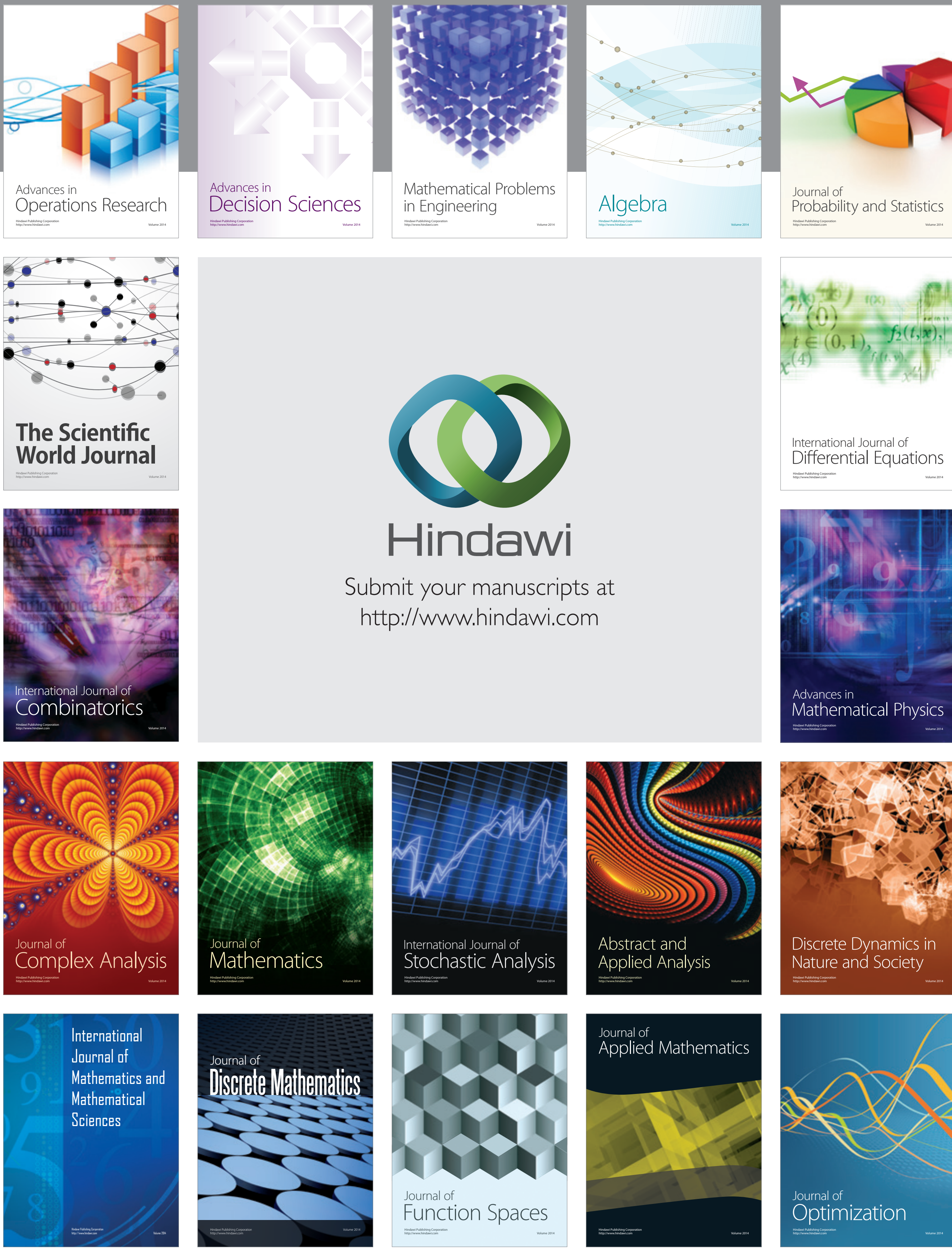
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