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*Research Article*

# Toeplitz Operators on the Weighted Pluriharmonic Bergman Space with Radial Symbols

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We construct an operator $R$ whose restriction onto weighted pluriharmonic Bergman Space $b_\mu^2(\mathbb{B}^n)$ is an isometric isomorphism between $b_\mu^2(\mathbb{B}^n)$ and $l_2^\#$. Furthermore, using the operator $R$ we prove that each Toeplitz operator $T_a$ with radial symbols is unitary to the multication operator $\gamma_{a,\mu}I$. Meanwhile, the Wick function of a Toeplitz operator with radial symbol gives complete information about the operator, providing its spectral decomposition.

## 1. Introduction

Let $\mathbb{B}^n$ be the open unit ball in the complex vector space $\mathbb{C}^n$. For any $z = (z_1, \dots, z_n)$ and $\xi = (\xi_1, \dots, \xi_n)$ in $\mathbb{C}^n$, let $z\cdot\xi = \sum_{j=1}^n z_j\overline{\xi_j}$, where $\overline{\xi_j}$ is the complex conjugate of $\xi_j$ and $|z| = \sqrt{z\cdot\overline{z}}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, where $\alpha_k \in Z_+ = \mathbb{N}\cup\{0\}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is its length, $\alpha! = \alpha_1!\cdots\alpha_n!$.

The weighted pluriharmonic Bergman space $b_\mu^2(\mathbb{B}^n)$ is the subspace of the weighted space $L_\mu^2(\mathbb{B}^n)$ consisting of all pluriharmonic functions on $\mathbb{B}^n$. A pluriharmonic function in the unit ball is the sum of a holomorphic function and the conjugate of a holomorphic functions. It is known that $b_\mu^2(\mathbb{B}^n)$ is a closed subspace of $L_\mu^2(\mathbb{B}^n)$ and hence is a Hilbert space. Let $Q_{\mathbb{B}^n}^\mu$ be the Hilbert space orthogonal projection from $L_\mu^2(\mathbb{B}^n)$ onto $b_\mu^2(\mathbb{B}^n)$. For a function $u \in L_\mu^2(\mathbb{B}^n)$, the Toeplitz operator $T_u : b_\mu^2(\mathbb{B}^n) \to b_\mu^2(\mathbb{B}^n)$ with symbol $u$ is the linear operator defined by

$$T_u f = Q_{\mathbb{B}^n}^\mu(uf), \quad f \in b_\mu^2(\mathbb{B}^n). \tag{1.1}$$

$T_u$ is densely defined and not bounded in general.

The boundedness and compactness of Toeplitz operators on Bergman type spaces have been studied intensively in recent years. The fact that the product of two harmonic functions is no longer harmonic adds some mystery in the study of Toeplitz operators on harmonic Bergman space. Many methods which work for the operator on analytic Bergman spaces lost their effectiveness on harmonic Bergman space. Therefore new ideas and methods are needed. We refer to [1–3] for references about the results of Toeplitz operator on harmonic Bergman space. The paper [3] characterizes compact Toeplitz operators in the case of the unit disk $\mathbb{D}$. In [2], the authors consider Toeplitz operators acting on the pluriharmonic Bergman space and study the problem of when the commutator or semicommutator of certain Toeplitz operators is zero. Lee [1] proved that two Toplitz operators acting on the pluriharmonic Bergman space with radial symbols and pluriharmonic symbol, respectively, commute only in an obvious case.

The authors in [4] analyze the influence of the radial component of a symbol to spectral, compactness and Fredholm properties of Toeplitz operators on Bergman space on unit disk $\mathbb{D}$. In [5], they are devoted to study Toeplitz operators with radial symbols on the weighted Bergman spaces on the unit ball in $\mathbb{C}^n$.

In this paper, we will be concerned with the question of Toeplitz operators with radial symbols on the weighted pluriharmonic Bergman space. Based on the techniques in [4–6], we construct an operator $R$ whose restriction onto weighted pluriharmonic Bergman space $b^2_\mu(\mathbb{B}^n)$ is an isometric isomorphism between $b^2_\mu(\mathbb{B}^n)$ and $l_2^\#$, and

$$\begin{aligned} RR^* &= I : l_2^\# \longrightarrow l_2^\#, \\ R^*R &= Q^\mu_{\mathbb{B}^n} : L^2_\mu(\mathbb{B}^n) \longrightarrow b^2_\mu(\mathbb{B}^n); \end{aligned} \tag{1.2}$$

where $l_2^\#$ is the subspace of $l_2$. Using the operator $R$ we prove that each Toeplitz operator $T_a$ with radial symbols is unitary to the multication operator $\gamma_{a,\mu}I$ acting on $l_2^\#$. Next, we use the Berezin concept of Wick and anti-Wick symbols. It turns out that in our particular (radial symbols) case the Wick symbols of a Toeplitz operator give complete information about the operator, providing its spectral decomposition.

## 2. Pluriharmonic Bergman Space and Orthogonal Projection

We start this section with a decomposition of the space $L^2_\mu(\mathbb{B}^n)$. Consider a nonnegative measurable function $\mu(r), r \in (0,1)$, such that $\operatorname{mes}\{r \in (0,1) : \mu(r) > 0\} = 1$, and

$$\int_{\mathbb{B}^n} \mu(|z|)dv(z) = \left|S^{2n-1}\right| \int_0^1 \mu(r) r^{2n-1} dr < \infty, \tag{2.1}$$

where $|S^{2n-1}| = 2\pi^{n-(1/2)}\Gamma^{-1}(n-(1/2))$ is the surface area of unit sphere $S^{2n-1}$ and $\Gamma(z)$ is the Gamma function.

Introduce the weighted space

$$L^2_\mu(\mathbb{B}^n) = \left\{ f : \|f\|^2_{L^2_\mu(\mathbb{B}^n)} = \int_{\mathbb{B}^n} \left|f(z)\right|^2 \mu(|z|)dv(z) < \infty \right\}, \tag{2.2}$$

where $dv(z)$ is the usual Lebesgue volume measure and $L_2(S^{2n-1})$ is the space with the usual Lebesgue surface measure.

The space $L_2(S^{2n-1})$ is the direct sum of mutually orthogonal spaces $\mathcal{H}_k$, that is,

$$L_2\left(S^{2n-1}\right) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k, \tag{2.3}$$

where $\mathcal{H}_k$ denotes the space of spherical harmonics of order $k$. Meanwhile, each space $\mathcal{H}_k$ is the direct sum (under the identification $\mathbb{C}^n = \mathbb{R}^{2n}$) of the mutually orthogonal spaces $\mathcal{H}_{p,q}$ (see, e.g., [7]):

$$\mathcal{H}_k = \bigoplus_{\substack{p+q=k \\ p,q\in\mathbb{Z}_+}} H_{p,q}, \quad k \in \mathbb{Z}_+, \tag{2.4}$$

where $H_{p,q}$, for each $p, q = 0, 1, \ldots$, is the space of harmonic polynomials (their restrictions to the unit sphere) of complete order $p$ in the variable $z$ and complete order $q$ in the conjugate variable $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$. Thus, we can get

$$L_2\left(S^{2n-1}\right) = \bigoplus_{p,q\in\mathbb{Z}_+} H_{p,q}. \tag{2.5}$$

The Hardy space $H^2(\mathbb{B}^n)$ in the unit ball $\mathbb{B}^n$ is a closed subspace of $L_2(S^{2n-1})$. Denote by $P_{S^{2n-1}}$ the Szegö orthogonal projection of $L_2(S^{2n-1})$ onto the Hardy space $H^2(\mathbb{B}^n)$. It is well known that $H^2(\mathbb{B}^n) = \bigoplus_{p=0}^{\infty} H_{p,0}$. The standard orthonormal base in $H^2(\mathbb{B}^n)$ has the form

$$e_\alpha(\omega) = d_{n,\alpha}\omega^\alpha, \qquad d_{n,\alpha} = \sqrt{\frac{(n-1+|\alpha|)!}{|S^{2n-1}|(n-1)!\alpha!}} \quad \text{for } |\alpha| = 0, 1, \ldots. \tag{2.6}$$

Fix an orthonormal basis $\{e_{\alpha,\beta}(\omega)\}_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{Z}_+^n$, in the space $L_2(S^{2n-1})$ so that $e_{\alpha,0}(\omega) \equiv e_\alpha(\omega)$, $e_{0,\alpha}(\omega) \equiv \overline{e_{\alpha,0}(\omega)} \equiv \overline{e_\alpha(w)}$, $|\alpha| = 0, 1, \ldots$.

Passing to the spherical coordinates in the unit ball we have

$$L^2_\mu(\mathbb{B}^n) = L_2\left((0,1), \mu(r)r^{2n-1}dr\right) \otimes L_2\left(S^{2n-1}\right). \tag{2.7}$$

For any function $f(z) \in L^2_\mu(\mathbb{B}^n)$ have the decomposition

$$f(z) = \sum_{|\alpha|+|\beta|=0}^{\infty} c_{\alpha,\beta}(r)e_{\alpha,\beta}(\omega), \quad r = |z|, \ \omega = \frac{z}{r}, \tag{2.8}$$

with the coefficients $c_{\alpha,\beta}(r)$ satisfying the condition

$$\|f\|^2_{L^2_\mu(\mathbb{B}^n)} = \sum_{|\alpha|+|\beta|=0}^{\infty} \int_0^1 \left|c_{\alpha,\beta}(r)\right|^2 \mu(r) r^{2n-1} dr < \infty. \tag{2.9}$$

According to the decomposition (2.7), (2.8) together with Parseval's equality, we can define the unitary operator

$$\begin{aligned} U_1 : L_2\Big((0,1), \mu(r) r^{2n-1} dr\Big) \otimes L_2\Big(S^{2n-1}\Big) &\longrightarrow L_2\Big((0,1), \mu(r) r^{2n-1} dr\Big) \otimes l_2 \\ &\equiv l_2\Big(L_2\Big((0,1), \mu(r) r^{2n-1} dr\Big)\Big), \end{aligned} \tag{2.10}$$

by the rule $U_1 : f(z) \longrightarrow \{c_{\alpha,\beta}(r)\}$, and

$$\|f\|^2_{L^2_\mu(\mathbb{B}^n)} = \left\|c_{\alpha,\beta}(r)\right\|^2_{l_2(L_2((0,1),\mu(r)r^{2n-1}dr))} = \sum_{|\alpha|+|\beta|=0}^{\infty} \left\|c_{\alpha,\beta}(r)\right\|^2_{L_2((0,1),\mu(r)r^{2n-1}dr)}. \tag{2.11}$$

Let $f(z)$ be a pluriharmonic in the unit ball $\mathbb{B}^n$ and write $f = g + \overline{h}$, where the functions $g$, $h$ are holomorphic in $\mathbb{B}^n$. Suppose

$$g(z) = \sum_{|\alpha|=0}^{\infty} c_\alpha z^\alpha, \qquad h(z) = \sum_{|\beta|=0}^{\infty} c_\beta z^\beta \tag{2.12}$$

are their power series representations of $g$ and $h$, respectively. We have

$$f(z) = \sum_{|\alpha|=0}^{\infty} c_\alpha z^\alpha + \sum_{|\beta|=0}^{\infty} \overline{c_\beta z^\beta} = \sum_{|\alpha|=0}^{\infty} c_\alpha(r) e_\alpha(\omega) + \sum_{|\beta|=0}^{\infty} \overline{c_\beta(r) e_\beta(\omega)}, \tag{2.13}$$

where $c_\alpha(r) = c_\alpha d^{-1}_{n,\alpha} r^{|\alpha|}$, $c_\beta(r) = c_\beta d^{-1}_{n,\beta} r^{|\beta|}$, $r = |z|$, $\omega = (z/r)$.

Let $b^2_\mu(\mathbb{B}^n)$ be the pluriharmonic Bergman space in $\mathbb{B}^n$ from $L^2_\mu(\mathbb{B}^n)$. Denote by $Q^\mu_{\mathbb{B}^n}$ the pluriharmonic Bergman orthogonal projection of $L^2_\mu(\mathbb{B}^n)$ onto the Bergman space $b^2_\mu(\mathbb{B}^n)$. From the above it follows that to characterize a function $f(z) \in b^2_\mu(\mathbb{B}^n)$ and considering its decomposition according to (2.13), one can restrict to the function having the representation

$$f(z) = g(z) + \overline{h(z)} = \sum_{|\alpha|=0}^{\infty} c_{\alpha,0}(r) e_{\alpha,0}(\omega) + \sum_{|\beta|=0}^{\infty} c_{0,\beta}(r) e_{0,\beta}(\omega). \tag{2.14}$$

Now let us take an arbitrary $f(z)$ from $b^2_\mu(\mathbb{B}^n)$ in the form (2.14). It will satisfy the Cauchy-Riemann equations, that is,

$$\frac{\partial}{\partial \overline{z}_k} g(z) \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right) g(z) = 0, \quad k = 1, \dots, n, \ z \in \mathbb{B}^n,$$
$$\frac{\partial}{\partial z_k} \overline{h}(z) \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \overline{h}(z) = 0, \quad k = 1, \dots, n, \ z \in \mathbb{B}^n. \tag{2.15}$$

Applying $\partial/\partial\overline{z}_k$, $\partial/\partial z_k$ to $g$ and $\overline{h}$, respectively, we have

$$\frac{\partial}{\partial \overline{z}_k} \sum_{|\alpha|=0}^{\infty} c_{\alpha,0}(r) e_{\alpha,0}(\omega) = \frac{z_k}{2r} \sum_{|\alpha|=0}^{\infty} \left( \frac{d}{dr} c_{\alpha,0}(r) - \frac{|\alpha|}{r} c_{\alpha,0}(r) \right) e_{\alpha,0}(\omega),$$
$$\frac{\partial}{\partial z_k} \sum_{|\beta|=0}^{\infty} c_{0,\beta}(r) e_{0,\beta}(\omega) = \frac{\overline{z}_k}{2r} \sum_{|\beta|=0}^{\infty} \left( \frac{d}{dr} c_{0,\beta}(r) - \frac{|\beta|}{r} c_{0,\beta}(r) \right) e_{0,\beta}(\omega), \tag{2.16}$$

where $k = 0, \dots, n$, and we come to the infinite system of ordinary linear differential equations

$$\frac{d}{dr} c_{\alpha,0}(r) - \frac{|\alpha|}{r} c_{\alpha,0}(r) = 0, \quad |\alpha| = 0, 1, \dots$$
$$\frac{d}{dr} c_{0,\beta}(r) - \frac{|\beta|}{r} c_{0,\beta}(r) = 0, \quad |\beta| = 0, 1, \dots. \tag{2.17}$$

Their general solution has the form $c_{\alpha,0} = b_\alpha r^{|\alpha|} = \lambda(n, |\alpha|) c_{\alpha,0} r^{|\alpha|}$, $c_{0,\beta} = b_\beta r^{|\beta|} = \lambda(n, |\beta|) c_{0,\beta} r^{|\beta|}$, with $\lambda(n, m) = (\int_0^1 t^{2m+2n-1} \mu(t) dt)^{-1/2}$. Hence, for any $f(z) \in b^2_\mu(\mathbb{B}^n)$ we have

$$f(z) = \sum_{|\alpha|=0}^{\infty} c_{\alpha,0} \lambda(n, |\alpha|) r^{|\alpha|} e_{\alpha,0} + \sum_{|\beta|=0}^{\infty} c_{0,\beta} \lambda(n, |\beta|) r^{|\beta|} e_{0,\beta}. \tag{2.18}$$

And, it is easy to verify $\|f\|^2_{L^2_\mu(\mathbb{B}^n)} = \sum_{|\alpha|=0}^{\infty} |c_{\alpha,0}|^2 + \sum_{|\beta|=0}^{\infty} |c_{0,\beta}|^2$. Thus the image $b^2_{1,\mu}(\mathbb{B}^n) = U_1(b^2_\mu(\mathbb{B}^n))$ is characterized as the closed subspace of

$$L_2\left((0,1), \mu(r) r^{2n-1} dr\right) \otimes l_2 = l_2\left(L_2\left((0,1), \mu(r) r^{2n-1} dr\right)\right) \tag{2.19}$$

which consists of all sequences $\{c_{\alpha,\beta}(r)\}$ of the form

$$c_{\alpha,\beta}(r) = \begin{cases} \lambda(n, |\alpha|) c_{\alpha,0} r^{|\alpha|}, & |\beta| = 0 \\ \lambda(n, |\beta|) c_{0,\beta} r^{|\beta|}, & |\alpha| = 0 \\ 0, & |\alpha| \neq 0, \ |\beta| \neq 0. \end{cases} \tag{2.20}$$

For each $m \in \mathbb{Z}_+$ introduce the function

$$\varphi_m(\rho) = \lambda(n,m)^{1/n}\left(\int_0^\rho r^{2m+2n-1}\mu(r)dr\right)^{1/2n}, \quad \rho \in [0,1]. \tag{2.21}$$

Obviously, there exists the inverse function for the function $\varphi_m(\rho)$ on $[0,1]$, which we will denote by $\phi_m(r)$. Introduce the operator

$$(u_m f)(r) = \frac{\sqrt{2n}}{\lambda(n,m)}\phi_m^{-m}(r)f(\phi_m(r)). \tag{2.22}$$

By Proposition 2.1 in [5], the operator $u_m$ maps unitary $L_2((0,1),\mu(r)r^{2n-1})$ onto $L_2((0,1),r^{2n-1}dr)$ in such a way that

$$u_m(\lambda(n,m)r^m) = \sqrt{2n}, \quad m \in \mathbb{Z}_+. \tag{2.23}$$

Intoduce the unitary operator

$$U_2 : l_2\Big(L_2\Big((0,1),\mu(r)r^{2n-1}dr\Big)\Big) \longrightarrow l_2\Big(L_2\Big((0,1),r^{2n-1}dr\Big)\Big), \tag{2.24}$$

where

$$U_2 : \{c_{\alpha,\beta}(r)\} \longrightarrow \{(u_{|\alpha|+|\beta|}c_{\alpha,\beta})(r)\}. \tag{2.25}$$

By (2.23), we can get the space $b^2_{2,\mu} = U_2(b^2_{1,\mu})$ coincides with the space of all sequences $\{k_{\alpha,\beta}\}$ for which

$$k_{\alpha,\beta} = \begin{cases} \sqrt{2n}c_{\alpha,0}, & |\beta| = 0 \\ \sqrt{2n}c_{0,\beta}, & |\alpha| = 0 \\ 0, & |\alpha| \neq 0, \ |\beta| \neq 0. \end{cases} \tag{2.26}$$

Let $l_0(r) = \sqrt{2n}$. We have $l_0(r) \in L_2((0,1),r^{2n-1}dr)$ and $\|l_0\|_{L_2((0,1),r^{2n-1})} = 1$. Denote by $L_0$ the one-dimensional subspace of $L_2((0,1),r^{2n-1}dr)$ generated by $l_0(r)$. The orthogonal projection $P_0$ of $L_2((0,1),r^{2n-1}dr)$ onto $L_0$ has the form

$$(P_0 f)(r) = \langle f, l_0\rangle l_0 = \sqrt{2n}\int_0^1 f(\rho)\sqrt{2n}\rho^{2n-1}d\rho. \tag{2.27}$$

Let $d_{\alpha,\beta} = k_{\alpha,\beta}(\sqrt{2n})^{-1}$. Denote by $l_2^{\#}$ the subspace of $l_2$ consisting of all sequences $\{d_{\alpha,\beta}\}$. And let $p^{\#}$ be the orthogonal projections of $l_2$ onto $l_2^{\#}$, then $p^{\#} = \chi_+(\alpha,\beta)I$, where $\chi_+(\alpha,\beta) = 0$, if $|\alpha||\beta| > 0$ and $\chi_+(\alpha,\beta) = 1$, if $|\alpha||\beta| = 0$.

Observe that $b^2_{2,\mu} = L_0 \otimes l^{\#}_2$ and the orthogonal projection $B_2$ of

$$l_2\Big(L_2\Big((0,1), r^{2n-1}dr\Big)\Big) \equiv L_2\Big((0,1)r^{2n-1}dr\Big) \otimes l_2 \tag{2.28}$$

onto $b^2_{2,\mu}$ has the form $B_2 = P_0 \otimes p^{\#}$. This leads to the following theorem.

**Theorem 2.1.** *The unitary operator $U = U_1U_2$ gives an isometric isomorphism of the space $L^2_\mu(\mathbb{B}^n)$ onto $l_2(L_2((0,1), r^{2n-1}dr)) \equiv L_2((0,1), r^{2n-1}dr) \otimes l_2$ such that*

*(1) the pluriharmonic Bergman space $b^2_\mu(\mathbb{B}^n)$ is mapped onto $L_0 \otimes l^{\#}_2$,*

$$U : b^2_\mu(\mathbb{B}^n) \longrightarrow L_0 \otimes l^{\#}_2, \tag{2.29}$$

*where $L_0$ is the one-dimensional subspace of $L_2((0,1), r^{2n-1}dr)$, generated by the function $l_0(r) = \sqrt{2n}$;*

*(2) the pluriharmonic Bergman projection $Q^\mu_{\mathbb{B}^n}$ is unitary equivalent to*

$$UQ^\mu_{\mathbb{B}^n}U^{-1} = P_0 \otimes p^{\#}, \tag{2.30}$$

*where $P_0$ is the one-dimensional projection* (2.27) *of $L_2((0,1), r^{2n-1}dr)$ onto $L_0$.*

Introduce the operator

$$R_0 : l^{\#}_2 \longrightarrow L_2\Big((0,1), r^{2n-1}dr\Big) \otimes l_2 \tag{2.31}$$

by the rule

$$R_0 : \{d_{\alpha,\beta}\} \longrightarrow l_0(r)\{d_{\alpha,\beta}\}. \tag{2.32}$$

The mapping $R_0$ is an isometric embedding, and the image of $R_0$ coincides with the space $b^2_{2,\mu}$. The adjoint operator

$$R^*_0 : L_2\Big((0,1), r^{2n-1}dr\Big) \otimes l_2 \longrightarrow l^{\#}_2 \tag{2.33}$$

is given by

$$\begin{gathered} R^*_0 : \{c_{\alpha,\beta}(r)\} \longrightarrow \left\{\chi_+(\alpha,\beta)\int_0^1 c_{\alpha,\beta}(\rho)\sqrt{2n}\rho^{2n-1}d\rho\right\}, \\ R^*_0R_0 = I : l^{\#}_2 \longrightarrow l^{\#}_2, \\ R_0R^*_0 = B_2 : L_2\Big((0,1), r^{2n-1}dr\Big) \otimes l_2 \longrightarrow b^2_{2,\mu}. \end{gathered} \tag{2.34}$$

Meanwhile the operator $R = R_0^* U$ maps the space $L_\mu^2(\mathbb{B}^n)$ onto $l_2^\#$, and its restriction

$$R \mid b_\mu^2(\mathbb{B}^n) : b_\mu^2(\mathbb{B}^n) \longrightarrow l_2^\# \tag{2.35}$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : l_2^\# \longrightarrow b_\mu^2(\mathbb{B}^n) \subset L_\mu^2(\mathbb{B}^n) \tag{2.36}$$

is isometric isomorphism of $l_2^\#$ onto the subspace $b_\mu^2(\mathbb{B}^n)$ of $L_\mu^2(\mathbb{B}^n)$.

*Remark 2.2.* We have

$$RR^* = I : l_2^\# \longrightarrow l_2^\#, \qquad R^* R = Q_{\mathbb{B}^n}^\mu : L_\mu^2(\mathbb{B}^n) \longrightarrow b_\mu^2(\mathbb{B}^n). \tag{2.37}$$

**Theorem 2.3.** *The isometric isomorphism $R^* = U^* R_0 : l_2^\# \to b_\mu^2(\mathbb{B}^n)$ is given by*

$$R^* : \{d_{\alpha,\beta}\} \longmapsto \sum_{|\alpha|=0}^{\infty} \lambda(n,|\alpha|) c_{\alpha,0} r^{|\alpha|} e_{\alpha,0}(\omega) + \sum_{|\beta|=1}^{\infty} \lambda(n,|\beta|) c_{0,\beta} r^{|\beta|} e_{0,\beta}(\omega). \tag{2.38}$$

*Proof.* Let $\{d_{\alpha,\beta}\} \in l_2^\#$, we can get

$$\begin{aligned} R^* = U_1^* U_2^* R_0 : \{d_{\alpha,\beta}\} &\longmapsto U_1^* U_2^* \left(\left\{\sqrt{2n}\, d_{\alpha,\beta}\right\}\right) \\ &= U_1^* \left(\left\{\lambda(n,|\alpha|) c_{\alpha,0} r^{|\alpha|}\right\} + \left\{\lambda(n,|\beta|) c_{0,\beta} r^{|\beta|}\right\}\right) \\ &= \sum_{|\alpha|=0}^{\infty} \lambda(n,|\alpha|) c_{\alpha,0} r^{|\alpha|} e_{\alpha,0}(\omega) + \sum_{|\beta|=1}^{\infty} \lambda(n,|\beta|) c_{0,\beta} r^{|\beta|} e_{0,\beta}(\omega). \end{aligned} \tag{2.39}$$

□

**Corollary 2.4.** *The inverse isomorphism $R : b_\mu^2(\mathbb{B}^n) \to l_2^\#$ is given by*

$$R : \varphi(z) \longmapsto \{d_{\alpha,\beta}\} = \left\{\left(\sqrt{2n}\right)^{-1} k_{\alpha,\beta}\right\}, \tag{2.40}$$

*where $c_{\alpha,0} = \langle \varphi, \widetilde{e}_{\alpha,0}^\mu \rangle = \lambda(n,|\alpha|) d_{n,\alpha} \int_{\mathbb{B}^n} \varphi(z) \overline{z}^\alpha dv(z)$, $c_{0,\beta} = \langle \varphi, \widetilde{e}_{0,\beta}^\mu \rangle = \lambda(n,|\beta|) d_{n,\beta} \int_{\mathbb{B}^n} \varphi(z) z^\beta dv(z)$, $|\alpha|, |\beta| \in \mathbb{Z}_+$, and $\{\widetilde{e}_{\alpha,0}^\mu\}_{|\alpha|=0}^\infty \cup \{\widetilde{e}_{0,\beta}^\mu\}_{|\beta|=1}^\infty$ is the standard basis for the pluriharmonic Bergman space $b_\mu^2(\mathbb{B}^n)$; that is,*

$$\widetilde{e}_{\alpha,0}^\mu = d_{n,\alpha} \lambda(n,|\alpha|) z^\alpha, \qquad \widetilde{e}_{0,\beta}^\mu = d_{n,\beta} \lambda(n,|\beta|) \overline{z}^\beta. \tag{2.41}$$

## 3. Toeplitz Operator with Radial Symbols on $b_{\mu}^2(\mathbb{B}^n)$

In this section we will study the Toeplitz operators $T_a = Q_{\mathbb{B}^n}^{\mu} a : \varphi \in b_{\mu}^2(\mathbb{B}^n) \mapsto Q_{\mathbb{B}^n}^{\mu} a\varphi \in b_{\mu}^2(\mathbb{B}^n)$ with radial symbols $a = a(r)$.

**Theorem 3.1.** *Let $a(r)$ be a measurable function on the segment $[0,1]$. Then the Toeplitz operator $T_a$ acting on $b_{\mu}^2(\mathbb{B}^n)$ is unitary equivalent to the multication operator $\gamma_{a,\mu} I$ acting on $l_2^{\#}$. The sequence $\gamma_{a,\mu} = \{\chi_+(\alpha,\beta)\gamma_{a,\mu}(|\alpha|+|\beta|)\}$ is given by*

$$\gamma_{a,\mu}(m) = \lambda^2(n,m) \int_0^1 a(r) r^{2m+2n-1} \mu(r) dr, \quad m \in Z_+. \tag{3.1}$$

*Proof.* By means of Remark 2.2, the operator $T_a$ is unitary equivalent to the operator

$$\begin{aligned} RT_a R^* &= RQ_{\mathbb{B}^n}^{\mu} a Q_{\mathbb{B}^n}^{\mu} R^* = R(R^* R) a (R^* R) R^* = (RR^*) R a R^* RR^* = RaR^* \\ &= R_0^* U_2 U_1 a(r) U_1^{-1} U_2^{-1} R_0 = R_0^* U_2 a(r) U_2^{-1} R_0 \\ &= R_0^* \{\chi_+(\alpha,\beta) a(\phi_{|\alpha|+|\beta|}(r))\} R_0. \end{aligned} \tag{3.2}$$

Further, let $\{d_{\alpha,\beta}\}$ be a sequence from $l_2^{\#}$. By (2.21), we have

$$\begin{aligned} &R_0^* \{\chi_+(\alpha,\beta) a(\phi_{|\alpha|+|\beta|}(r))\} R_0 \{d_{\alpha,\beta}\} \\ &\quad = R_0^* \left\{ \sqrt{2n} d_{\alpha,\beta} \chi_+(\alpha,\beta) a(\phi_{|\alpha|+|\beta|}(r)) \right\} \\ &\quad = \left\{ \int_0^1 \chi_+(\alpha,\beta) a(\phi_{|\alpha|+|\beta|}(r)) 2n d_{\alpha,\beta} r^{2n-1} dr \right\} \\ &\quad = \left\{ \chi_+(\alpha,\beta) d_{\alpha,\beta} \int_0^1 a(y) d\varphi_{|\alpha|+|\beta|}^{2n}(y) \right\} \\ &\quad = \left\{ \chi_+(\alpha,\beta) d_{\alpha,\beta} \lambda^2(n,|\alpha|+|\beta|) \int_0^1 a(y) y^{2(|\alpha|+|\beta|)+2n-1} \mu(y) dy \right\} \\ &\quad = \{\chi_+(\alpha,\beta) d_{\alpha,\beta} \gamma_{a,\mu}(|\alpha|+|\beta|)\}. \end{aligned} \tag{3.3}$$

□

**Corollary 3.2.** (i) *The Toeplitz operator $T_a$ with measurable radial symbol $a(r)$ is bounded on $b_{\mu}^2(\mathbb{B}^n)$ if and only if $\sup_{m\in\mathbb{Z}_+} |\gamma_{a,\mu}(m)| < \infty$. Moreover,*

$$\|T_a\| = \sup_{m\in\mathbb{Z}_+} |\gamma_{a,\mu}(m)|. \tag{3.4}$$

(ii) *The Toeplitz operator* $T_a$ *is compact if and only if* $\lim_{m\to\infty}\gamma_{a,\mu}(m) = 0$. *The spectrum of the bounded operator* $T_a$ *is given by*

$$\mathrm{sp}T_a = \overline{\{\gamma_{a,\mu}(m) : m \in \mathbb{Z}_+\}}, \tag{3.5}$$

*and its essential spectrum ess-sp*$T_a$ *coincides with the set of all limits points of the sequence* $\{\gamma_{a,\mu}(m)\}_{m\in\mathbb{Z}_+}$.

Let $H$ be a Hilbert space and $\{\varphi_g\}_{g\in G}$ a subset of elements of $H$ parameterized by elements $g$ of some set $G$ with measure $d\mu$.

Then $\{\varphi_g\}_{g\in G}$ is called a system of coherent states, if for all $\varphi \in H$,

$$\|\varphi\|^2 = (\varphi, \varphi) = \int_G |(\varphi, \varphi_g)|^2 d\mu, \tag{3.6}$$

or equivalently, if for all $\varphi_1, \varphi_2 \in H$,

$$(\varphi_1, \varphi_2) = \int_G (\varphi_1, \varphi_g)\overline{(\varphi_2, \varphi_g)} d\mu. \tag{3.7}$$

We define the isomorphic inclusion $V : H \to L_2(G)$ by the rule

$$V : \varphi \in H \longmapsto f = f(g) = (\varphi, \varphi_g) \in L_2(G). \tag{3.8}$$

By (3.7) we have $(\varphi_1, \varphi_2) = \langle f_1, f_2\rangle$, where $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are the scalar products on $H$ and $L_2(G)$, respectively, and $f_h(g) = \overline{f_g(h)}$. Let $H_2(G) = V(H) \subset L_2(G)$. A function $f \in L_2(G)$ is an element of $H_2(G)$ if and only if for all $h \in G$, $\langle f, f_h\rangle = f(h)$. The operator $(Pf)(g) = \int_G (\varphi_t, \varphi_g) f(t) d\mu(t)$ is the orthogonal projection of $L_2(G)$ onto $H_2(G)$.

The function $a(g)$, $g \in G$, is called the anti-Wick (or contravariant) symbol of an operator $T : H \to H$ if

$$VTV^{-1} \mid H_2(G) = Pa(g)P = Pa(g)I \mid H_2(G) : H_2(G) \longrightarrow H_2(G), \tag{3.9}$$

or, in other terminology, the operator $VAV^{-1} \mid H_2(G)$ is the Toeplitz operator

$$T_{a(g)} = Pa(g)I \mid H_2(G) : H_2(G) \longrightarrow H_2(G) \tag{3.10}$$

with the symbols $a(g)$.

Given an operator $T : H \to H$, introduce the (Wick) function

$$\tilde{a}(g, h) = \frac{(T\varphi_h, \varphi_g)}{(\varphi_h, \varphi_g)}, \quad g, h \in G. \tag{3.11}$$

If the operator $T$ has an anti-Wick symbols, that is, $VTV^{-1} = T_{a(g)}$ for some function $a = a(g)$, then

$$\widetilde{a}(g,h) = \frac{\langle T_a f_h, f_g\rangle}{\langle f_h, f_g\rangle}, \quad g,h \in G. \tag{3.12}$$

And the operator $T_a$ admits the following representation in terms of its Wick function:

$$\begin{aligned}(T_a f)(g) &= \int_G a(t) f(t) f_t(g) d\mu(t) = \int_G a(t) f_t(g) d\mu(t) \int_G f(h) f_h(t) d\mu(h)\\ &= \int_G f(h) d\mu(h) \int_G a(t) f_t(g) f_h(t) d\mu(t)\\ &= \int_G f(h) d\mu(h) \frac{f_h(g)}{\langle f_h, f_g\rangle} \int_G a(t) f_h(t) \overline{f_g(t)} d\mu(t)\\ &= \int_G \widetilde{a}(g,h) f(h) f_h(g) d\mu(h).\end{aligned} \tag{3.13}$$

Interchanging the integrals above, we understand them in a weak sense.

The restriction of the function $\widetilde{a}(g,h)$ onto the diagonal

$$\widetilde{a}(g) = \widetilde{a}(g,g) = \frac{(T\varphi_g, \varphi_g)}{(\varphi_g, \varphi_g)}, \quad g \in G, \tag{3.14}$$

is called the Wick (or covariant or Berezin) symbols of the operator $T : H \to H$.

The Wick and anti-Wick symbols of an operator $T : H \to H$ are connected by the Berezin transform

$$\widetilde{a}(g) = \widetilde{a}(g,g) = \frac{(T\varphi_g, \varphi_g)}{(\varphi_g, \varphi_g)} = \frac{\langle T_a f_g, f_g\rangle}{\langle f_g, f_g\rangle} = \frac{\int_G a(t) \left| f_g(t)\right|^2 d\mu(t)}{\int_G \left| f_g(t)\right|^2 d\mu(t)}. \tag{3.15}$$

The pluriharmonic Bergman reproducing kernel in the space $b^2_\mu(\mathbb{B}^n)$ has the form

$$R_z(w) = K_z(w) + \overline{K_z(w)} - d^2_{n,\mathbb{0}} \lambda^2(n,0) = \sum_{|\alpha|=0}^{\infty} \widetilde{e}^{\mu}_{\alpha}(w) \overline{\widetilde{e}^{\mu}_{\alpha}(z)} + \sum_{|\alpha|=0}^{\infty} \widetilde{e}^{\mu}_{\alpha}(z) \overline{\widetilde{e}^{\mu}_{\alpha}(w)} - d^2_{n,\mathbb{0}} \lambda^2(n,0), \tag{3.16}$$

where $\alpha = \mathbb{0} = (0,\ldots,0)$. For $f \in b^2_\mu(\mathbb{B}^n)$, the reproducing property

$$f(z) = \left(Q^{\mu}_{\mathbb{B}^n} f\right)(z) = \int_{\mathbb{B}^n} f(w) \overline{R_z(w)} \mu(|w|) dv(w) \tag{3.17}$$

shows that the system of functions $R_z(w)$, $w \in \mathbb{B}^n$, forms a system of coherent states in the space $b_\mu^2(\mathbb{B}^n)$. In our context, we have $G = \mathbb{B}^n$, $d\mu = \mu(|z|)dxdy$, $H = H_2(G) = b_\mu^2(\mathbb{B}^n)$, $L_2(G) = L_\mu^2(\mathbb{B}^n)$, $\varphi_g = f_g = R_g$, where $g = z \in \mathbb{B}^n$.

**Lemma 3.3.** *Let $T_a$ be the Toeplitz operator with a radial symbol $a = a(r)$. Then the corresponding Wick function* (3.11) *has the form*

$$\widetilde{a}(z,w) = R_w^{-1}(z)\left(\sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(w)}\widetilde{e}_\alpha^\mu(z)\gamma_{a,\mu}(|\alpha|) + \sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(z)}\widetilde{e}_\alpha^\mu(w)\gamma_{a,\mu}(|\alpha|) - d_{n,\mathbb{0}}^2\lambda^2(n,0)\gamma_{a,\mu}(0)\right). \tag{3.18}$$

*Proof.* By (3.11) and (3.16), we have

$$\begin{aligned}
\widetilde{a}(z,w) &= \frac{\langle T_a R_w, R_z\rangle}{\langle R_w, R_z\rangle} = R_w^{-1}(z)\langle aR_w, R_z\rangle \\
&= R_w^{-1}(z)\left(\langle aK_w, K_z\rangle + \left\langle a\overline{K_w}, \overline{K_z}\right\rangle - d_{n,\mathbb{0}}^4\lambda^4(n,0)\langle a,1\rangle\right) \\
&= R_w^{-1}(z)\left(\sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(w)}\widetilde{e}_\alpha^\mu(z)\left\langle a\widetilde{e}_\alpha^\mu, \widetilde{e}_\alpha^\mu\right\rangle + \sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(z)}\widetilde{e}_\alpha^\mu(w)\left\langle a\widetilde{e}_\alpha^\mu, \widetilde{e}_\alpha^\mu\right\rangle - d_{n,\mathbb{0}}^4\lambda^4(n,0)\langle a,1\rangle\right) \\
&= R_w^{-1}(z)\left(\sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(w)}\widetilde{e}_\alpha^\mu(z)\gamma_{a,\mu}(|\alpha|) + \sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(z)}\widetilde{e}_\alpha^\mu(w)\gamma_{a,\mu}(|\alpha|) - d_{n,\mathbb{0}}^2\lambda^2(n,0)\gamma_{a,\mu}(0)\right).
\end{aligned} \tag{3.19}$$

□

Denote by $L_\alpha^\mu$ the one-dimensional subspace of $b_\mu^2(\mathbb{B}^n)$ generated by the base element $\widetilde{e}_\alpha^\mu(z)$, $|\alpha| \in \mathbb{Z}_+$. Then the one-dimensional projection $P_\alpha^\mu$ of $b_\mu^2(\mathbb{B}^n)$ onto $L_\alpha^\mu$ has obviously the form

$$P_\alpha^\mu f = \left\langle f, \widetilde{e}_\alpha^\mu\right\rangle\widetilde{e}_\alpha^\mu = \widetilde{e}_\alpha^\mu(z)\int_{\mathbb{B}^n} f(w)\overline{\widetilde{e}_\alpha^\mu(w)}\mu(|w|)dv(w). \tag{3.20}$$

In the similar method, $\overline{L_\alpha^\mu}$ denote the one-dimensional subspace of $b_\mu^2(\mathbb{B}^n)$ generated by the base element $\overline{\widetilde{e}_\alpha^\mu(z)}$. Let $\overline{P_\alpha^\mu}$ be the projection from $b_\mu^2(\mathbb{B}^n)$ onto $\overline{L_\alpha^\mu}$, and the projection can be rewritten as

$$\overline{P_\alpha^\mu}f(z) = \left\langle f, \overline{\widetilde{e}_\alpha^\mu}\right\rangle\overline{\widetilde{e}_\alpha^\mu(z)} = \overline{\widetilde{e}_\alpha^\mu(z)}\int_{\mathbb{B}^n} f(w)\widetilde{e}_\alpha^\mu(w)\mu(|w|)dv(w). \tag{3.21}$$

**Theorem 3.4.** *Let $T_a$ be a bounded Toeplitz operator having radial symbol $a(r)$. Then one can get the spectral decomposition of the operator $T_a$:*

$$T_a = \sum_{|\alpha|=0}^{\infty} \gamma_{a,\mu}(|\alpha|) P_\alpha^\mu + \sum_{|\alpha|=0}^{\infty} \gamma_{a,\mu}(|\alpha|) \overline{P_\alpha^\mu} - \gamma_{a,\mu}(0) P_{\mathbb{0}}^\mu. \tag{3.22}$$

*Proof.* According to (3.13), (3.20), (3.21), and Lemma 3.3, we get

$$\begin{aligned}
(T_a f)(z) &= \int_{\mathbb{B}^n} \widetilde{a}(z,w) f(w) R_w(z) \mu(|w|) dv(w). \\
&= \int_{\mathbb{B}^n} \left( \sum_{|\alpha|=0}^{\infty} \overline{\widetilde{e}_\alpha^\mu(w)} \widetilde{e}_\alpha^\mu(z) \gamma_{a,\mu}(|\alpha|) + \sum_{|\alpha|=0}^{\infty} \overline{\widetilde{e}_\alpha^\mu(z)} \widetilde{e}_\alpha^\mu(w) \gamma_{a,\mu}(|\alpha|) \right. \\
&\qquad \left. - d_{n,\mathbb{0}}^2 \lambda^2(n,0) \gamma_{a,\mu}(0) \right) f(w) \mu(|w|) dv(w). \\
&= \sum_{|\alpha|=0}^{\infty} \gamma_{a,\mu}(|\alpha|) P_\alpha^\mu f(z) + \sum_{|\alpha|=0}^{\infty} \gamma_{a,\mu}(|\alpha|) \overline{P_\alpha^\mu} f(z) - \gamma_{a,\mu}(0) P_{\mathbb{0}}^\mu f(z).
\end{aligned} \tag{3.23}$$

□

The value $\gamma_{a,\mu}(|\alpha|)$ depends only on $|\alpha|$. Collecting the terms with the same $|\alpha|$ and using the formula

$$(z \cdot \overline{w})^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} z^\alpha \overline{w}^\alpha \tag{3.24}$$

we obtain

$$\widetilde{a}(z,w) = R_w^{-1}(z) \left[ \sum_{m=0}^{\infty} l(m,n) \gamma_{a,\mu}(m) \left( (z \cdot \overline{w})^m + (w \cdot \overline{z})^m \right) - d_{n,\mathbb{0}}^2 \lambda^2(n,0) \gamma_{a,\mu}(0) \right], \tag{3.25}$$

where $(l(m,n) = (m+n-1)!/|S^{2n-1}|m!(n-1)!)\lambda^2(n,m)$. The orthogonal projection of $b_\mu^2(\mathbb{B}^n)$ onto the subspace generated by all element $\widetilde{e}_\alpha^\mu$ with $|\alpha| = m$, $m \in \mathbb{Z}_+$ can be written as

$$\left(P_{(m)}^\mu f\right)(z) = l(m,n) \int_{\mathbb{B}^n} f(w) (z \cdot \overline{w})^m \mu(|w|) dv(w); \tag{3.26}$$

similarly,

$$\left(\overline{P_{(m)}^\mu} f\right)(z) = l(m,n) \int_{\mathbb{B}^n} f(w) (w \cdot \overline{z})^m \mu(|w|) dv(w) \tag{3.27}$$

denotes the orthogonal projection from $b^2_\mu(\mathbb{B}^n)$ onto the subspace generated by all elements $\overline{\widetilde{e}^{\mu}_{\alpha}}$ with $|\alpha| = m$. Therefore, (3.22) has the form

$$T_a = \sum_{m=0}^{\infty} \gamma_{a,\mu}(m) P^{\mu}_{(m)} + \sum_{m=0}^{\infty} \gamma_{a,\mu}(m) \overline{P^{\mu}_{(m)}} - \gamma_{a,\mu}(0) P^{\mu}_{\mathbb{0}}. \tag{3.28}$$

In view of (3.25), we can get the following useful corollary.

**Corollary 3.5.** *Let $T_a$ be a bounded Toeplitz operator having radial symbol $a(r)$. Then the Wick symbol of the operator $T_a$ is radial as well and is given by the formula*

$$\widetilde{a}(z) = \widetilde{a}(r) = R_z^{-1}(z)\left(2\sum_{m=0}^{\infty} l(m,n)\gamma_{a,\mu}(m) r^{2m} - d^2_{n,\mathbb{0}}\lambda^2(n,0)\gamma_{a,\mu}(0)\right), \tag{3.29}$$

*where $R_z(z) = 2\sum_{m=0}^{\infty} l(m,n) r^{2m} - d^2_{n,\mathbb{0}}\lambda^2(n,0)$.*

In terms of Wick function the composition formula for Toeplitz operators is quite transparent.

**Corollary 3.6.** *Let $T_a$, $T_b$ be the Toeplitz operators with the Wick function*

$$\begin{aligned}
\widetilde{a}(z,w) &= R_w^{-1}(z)\left[\sum_{m=0}^{\infty} l(m,n)\gamma_{a,\mu}(m)\left((z\cdot\overline{w})^m + (w\cdot\overline{z})^m\right) - d^2_{n,\mathbb{0}}\lambda^2(n,0)\gamma_{a,\mu}(0)\right], \\
\widetilde{b}(z,w) &= R_w^{-1}(z)\left[\sum_{m=0}^{\infty} l(m,n)\gamma_{b,\mu}(m)\left((z\cdot\overline{w})^m + (w\cdot\overline{z})^m\right) - d^2_{n,\mathbb{0}}\lambda^2(n,0)\gamma_{b,\mu}(0)\right],
\end{aligned} \tag{3.30}$$

*respectively. Then the Wick function $\widetilde{c}(z,w)$ of the composition $T = T_a T_b$ is given by*

$$\widetilde{c}(z,w) = R_w^{-1}(z)\left[\sum_{m=0}^{\infty} l(m,n)\gamma_{b,\mu}(m)\gamma_{a,\mu}(m)\left((z\cdot\overline{w})^m + (w\cdot\overline{z})^m\right) - d^2_{n,\mathbb{0}}\lambda^2(n,0)\gamma_{b,\mu}(0)\gamma_{a,\mu}(0)\right]. \tag{3.31}$$

*Proof.* According to Lemma 3.3 and (3.25), we have

$$
\begin{aligned}
\widetilde{c}(z,w) &= \frac{\langle T_a T_b R_w, R_z\rangle}{\langle R_w, R_z\rangle} = R_w^{-1}(z)\langle T_b R_w, aR_z\rangle \\
&= R_w^{-1}(z)\int_{\mathbb{B}^n}(T_b R_w)(u)\overline{R_z(u)}a(|u|)\mu(|u|)dv(u) \\
&= R_w^{-1}(z)\int_{\mathbb{B}^n}\langle T_b R_w, R_u\rangle\overline{R_z(u)}a(|u|)\mu(|u|)dv(u) \\
&= R_w^{-1}(z)\Bigg(\sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(w)}e_\alpha^\mu(z)\gamma_{b,\mu}(|\alpha|)\gamma_{a,\mu}(|\alpha|) + \sum_{|\alpha|=0}^{\infty}\overline{\widetilde{e}_\alpha^\mu(z)}e_\alpha^\mu(w)\gamma_{b,\mu}(|\alpha|)\gamma_{a,\mu}(|\alpha|) \\
&\qquad - d_{n,0}^2\lambda^2(n,0)\gamma_{b,\mu}(0)\gamma_{a,\mu}(0)\Bigg) \\
&= R_w^{-1}(z)\left[\sum_{m=0}^{\infty} l(m,n)\gamma_{b,\mu}(m)\gamma_{a,\mu}(m)\left((z\cdot\overline{w})^m + (w\cdot\overline{z})^m\right) - d_{n,0}^2\lambda^2(n,0)\gamma_{b,\mu}(0)\gamma_{a,\mu}(0)\right].
\end{aligned}
\tag{3.32}
$$

□

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