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Research Article

Strong Convergence Theorems for Equilibrium Problems and k-Strict Pseudocontractions in Hilbert Spaces

Dao-Jun Wen

School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

Correspondence should be addressed to Dao-Jun Wen, wendaojun@ctbu.edu.cn

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We introduce a new iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed point of a finite family of k-strictly pseudo-contractive nonself-mappings. Strong convergence theorems are established in a real Hilbert space under some suitable conditions. Our theorems presented in this paper improve and extend the corresponding results announced by many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of H. Let F be a bifunction from $K \times K$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following problem: Find $x \in K$ such that

$$F(x, y) \geq 0, \quad \forall y \in K, \tag{1.1}$$

which is called equilibrium problem. We use $\mathrm{EP}(F)$ to denote the set of solution of the problem (1.1). Given a mapping $T : K \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in K$. Then, $z \in \mathrm{EP}(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in K$; that is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem (see, e.g., [1–3]).

Recall that a nonself-mapping $T : K \to H$ is called a k-strict pseudocontraction if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall\, x, y \in K. \tag{1.2}$$

We use $F(T)$ to denote the fixed point set of the mapping T, that is, $F(T) := \{x \in K : Tx = x\}$. As $k = 0$, T is said to be nonexpansive, that is, $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$. T is said to be pseudocontractive if $k = 1$ and is also said to be strongly pseudocontractive if there exists a positive constant $\lambda \in (0,1)$ such that $T + \lambda I$ is pseudocontractive. Clearly, the class of k-strict pseudocontractions falls into the one between classes of nonexpansive mappings and pseudocontractions. We remark also that the class of strongly pseudocontractive mappings is independent of the class of k-strict pseudocontractions (see, e.g., [4, 5]).

Iterative methods for equilibrium problem and nonexpansive mappings have been extensively investigated; see, for example, [1–18] and the references therein. However, iterative methods for strict pseudocontractions are far less developed than those for non-expansive mappings though Browder and Petryshyn [5] initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1.2) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudocontraction T. On the other hand, strict pseudocontractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see, e.g., [6]). Therefore, it is interesting to develop the theory of iterative methods for equilibrium problem and strict pseudocontractions.

In 2007, Acedo and Xu [12] proposed the following parallel algorithm for a finite family of k_i-strict pseudocontractions $\{T_i\}_{i=1}^N$ in Hilbert space H:

$$\forall x_0 \in K, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{i=1}^{N} \lambda_i T_i x_n, \tag{1.3}$$

where $\{\alpha_n\} \subset (0,1)$, and $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i = 1$. They proved that the sequence $\{x_n\}$ defined by (1.3) converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$ under some appropriate conditions. Moreover, by applying additional projections, they further proved that algorithm can be modified to have strong convergence.

Recently, S. Takahashi and W. Takahashi [13] studied the equilibrium problem and fixed point of nonexpansive self-mappings T in Hilbert spaces by a viscosity approximation methods for finding an element of $\mathrm{EP}(F) \bigcap F(T)$. Very recently, by using the general approximation method, Qin et al. [14] obtained a strong convergence theorem for finding an element of $F(T)$. On the other hand, Ceng et al. [16] proposed an iterative scheme for finding an element of $\mathrm{EP}(F) \bigcap F(T)$ and then obtained some weak and strong convergence theorems.

In this paper, inspired and motivated by research going in this area, we introduce a modified parallel iteration, which is defined in the following way:

$$F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in K,$$

$$y_n = \alpha_n u_n + (1-\alpha_n)\sum_{i=1}^{N}\eta_i^{(n)} T_i u_n,$$
$$x_{n+1} = \beta_n u + \gamma_n x_n + (1-\beta_n-\gamma_n) y_n, \quad n \geq 0, \tag{1.4}$$

where $u \in K$ is a given point, $\{T_i\}_{i=i}^{N} : K \to H$ is a finite family of k_i-strictly pseudocontractive nonself-mappings, $\{\eta_i^{(n)}\}_{i=1}^{N}$ is a finite sequences of positive numbers, $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are some sequences in (0,1).

Our purpose is not only to modify the parallel algorithm (1.3) to the case of equilibrium problems and common fixed point for a finite family of k_i-strictly pseudocontractive nonself-mappings, but also to establish strong convergence theorems in a real Hilbert space under some different conditions. Our theorems presented in this paper improve and extend the main results of [9, 12–14, 16].

2. Preliminaries

Let K be a nonempty closed and convex subset of a Hilbert space H. We use P_K to denote the metric or nearest point projection of H onto K; that is, for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$. we write $x_n \rightharpoonup x$ and $x_n \to x$ indicate that the sequence $\{x_n\}$ convergence weakly and strongly to x, respectively.

It is well known that Hilbert space H satisfies Opial's condition [8], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and every $y \in H$ with $y \neq x$, we have

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|. \tag{2.1}$$

To study the equilibrium problem (1.1), we may assume that the bifunction F of $K \times K$ into $\mathbb{R}$ satisfies the following conditions.

(A1) $F(x,x) = 0$ for all $x \in K$.

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in K$.

(A3) For each $x, y, z \in K$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$.

(A4) For each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

In order to prove our main results, we need the following Lemmas and Propositions.

Lemma 2.1 (see [1, 3]). *Let F be a bifunction from $K \times K$ into $\mathbb{R}$ satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in K$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K. \tag{2.2}$$

Further, if $T_r x = \{z \in K : F(z,y) + (1/r)\langle y - z, z - x\rangle \geq 0,$ for all $y \in K\}$, then the following holds.

(1) T_r *is single-valued.*

(2) T_r *is firmly nonexpansive, that is,* $\|T_rx - T_ry\|^2 \leq \langle T_rx - T_ry, x - y\rangle$, *for all* $x, y \in H$.

(3) $F(T_r) = \mathrm{EP}(F)$.

(4) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.2 (see [7]). *Let* $(E, \langle \cdot, \cdot \rangle)$ *be an inner product space. Then, for all* $x, y, z \in E$ *and* $\alpha, \beta, \gamma \in [0,1]$ *with* $\alpha + \beta + \gamma = 1$, *we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \tag{2.3}$$

Lemma 2.3 (see [19]). *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequence in Banach space* E, *and let* $\{\lambda_n\}$ *be a sequence in* $[0,1]$ *such that* $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 1$. *Suppose* $x_{n+1} = \lambda_n x_n + (1 - \lambda_n)z_n$ *and*

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0, \quad \forall n \geq 0. \tag{2.4}$$

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 2.4 (see [2, 10]). *Let* $T : K \to H$ *be a* k*-strict pseudocontraction. For* $\lambda \in [k, 1)$, *define* $S : K \to H$ *by* $Sx = \lambda x + (1 - \lambda)Tx$ *for each* $x \in K$. *Then, as* $\lambda \in [k, 1)$, S *is a nonexpansive mapping such that* $F(S) = F(T)$.

Lemma 2.5 (see [10]). *If* $T : K \to H$ *is a* k*-strict pseudocontraction, then the fixed point set* $F(T)$ *is closed convex so that the projection* $P_{F(T)}$ *is well defined.*

Lemma 2.6 (see [9]). *Let* K *be a nonempty bounded closed convex subset of* H. *Given* $x \in H$ *and* $z \in K$, *then* $z = P_K x$ *if and only if there holds the relation:*

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in K. \tag{2.5}$$

Lemma 2.7 (see [20]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0, \tag{2.6}$$

where $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a real sequence such that*

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

Proposition 2.8 (see, e.g., Acedo and Xu [12]). *Let* K *be a nonempty closed convex subset of Hilbert space* H. *Given an integer* $N \geq 1$, *assume that* $\{T_i\}_{i=1}^{N} : K \to H$ *is a finite family of* k_i*-strict pseudocontractions. Suppose that* $\{\lambda_i\}_{i=1}^{N}$ *is a positive sequence such that* $\sum_{i=1}^{N}\lambda_i = 1$. *Then* $\sum_{i=1}^{N}\lambda_i T_i$ *is a* k*-strict pseudocontraction with* $k = \max\{k_i : 1 \leq i \leq N\}$.

Proposition 2.9 (see, e.g., Acedo and Xu [12]). *Let* $\{T_i\}_{i=1}^N$ *and* $\{\lambda_i\}_{i=1}^N$ *be given as in Proposition 2.8 above. Then* $F(\sum_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N F(T_i)$.

3. Main Results

Theorem 3.1. *Let* K *be a nonempty closed convex subset of Hilbert space* H, *and let* F *be a bifunction from* $K \times K$ *into* $\mathbb{R}$ *satisfying (A1)–(A4). Let* $\{T_i\}_{i=1}^N : K \to H$ *be a finite family of* k_i*-strict pseudocontractions such that* $k = \max\{k_i : 1 \leq i \leq N\}$ *and* $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \bigcap \mathrm{EP}(F) \neq \phi$. *Assume* $\{\eta_i^{(n)}\}_{i=1}^N$ *is a finite sequences of positive numbers such that* $\sum_{i=1}^N \eta_i^{(n)} = 1$ *for all* $n \geq 0$. *Given* $u \in K$ *and* $x_0 \in K$, $\{\alpha_n\}, \{\beta_n\}$, *and* $\{\gamma_n\}$ *are some sequences in (0,1); the following control conditions are satisfied.*

(i) $k \leq \alpha_n \leq \lambda < 1$ *for all* $n \geq 0$ *and* $\lim_{n\to\infty}\alpha_n = \alpha$,

(ii) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$,

(iv) $\lim_{n\to\infty}|\eta_i^{(n+1)} - \eta_i^{(n)}| = 0$.

Then the sequence $\{x_n\}$ *generated by* (1.4) *converges strongly to* $q \in \mathcal{F}$, *where* $q = P_{\mathcal{F}}u$.

Proof. From Lemma 2.1, we see that $\mathrm{EP}(F) = F(T_r)$, and note that u_n can be rewritten as $u_n = T_r x_n$. Putting $A_n = \sum_{i=1}^N \eta_i^{(n)} T_i$, we have $A_n : K \to H$ is a k-strict pseudocontraction and $F(A_n) = \bigcap_{i=1}^N F(T_i)$ by Propositions 2.8 and 2.9, where $k = \max\{k_i : 1 \leq i \leq N\}$.

From (1.4), condition (i), and Lemma 2.2, taking a point $p \in \mathcal{F}$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(u_n - p) + (1-\alpha_n)(A_n u_n - p)\|^2 \\
&= \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|A_n u_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - A_n u_n\|^2 \\
&\leq \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\left[\|u_n - p\|^2 + k\|u_n - A_n u_n\|^2\right] \\
&\quad - \alpha_n(1-\alpha_n)\|u_n - A_n u_n\|^2 \\
&= \|u_n - p\|^2 - (1-\alpha_n)(\alpha_n - k)\|u_n - A_n u_n\|^2 \\
&\leq \|T_r x_n - T_r p\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{3.1}$$

Furthermore, we have

$$\|y_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.2}$$

It follows from (1.4) and (3.2) that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\beta_n u+\gamma_n x_n+(1-\beta_n-\gamma_n)y_n-p\| \\
&\le \beta_n\|u-p\|+\gamma_n\|x_n-p\|+(1-\beta_n-\gamma_n)\|y_n-p\| \\
&\le \beta_n\|u-p\|+(1-\beta_n)\|x_n-p\| \\
&\le \max\{\|u-p\|,\|x_0-p\|\}.
\end{aligned} \tag{3.3}$$

Consequently, sequence $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{y_n\}$.

Define a mapping $T_n x := \alpha_n x + (1-\alpha_n)A_n x$ for each $x \in K$. Then $T_n : K \to H$ is nonexpansive. Indeed, by using (1.1), condition (i), and Lemma 2.2, we have for all $x, y \in K$ that

$$\begin{aligned}
\|T_n x - T_n y\|^2 &= \|\alpha_n(x-y)+(1-\alpha_n)(A_n x - A_n y)\|^2 \\
&= \alpha_n\|x-y\|^2+(1-\alpha_n)\|A_n x - A_n y\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x-A_n x-(y-A_n y)\|^2 \\
&\le \alpha_n\|x-y\|^2+(1-\alpha_n)\left[\|x-y\|^2+k\|x-A_n x-(y-A_n y)\|^2\right] \\
&\quad -\alpha_n(1-\alpha_n)\|x-A_n x-(y-A_n y)\|^2 \\
&= \|x-y\|^2-(1-\alpha_n)(\alpha_n-k)\|x-A_n x-(y-A_n y)\|^2 \\
&\le \|x-y\|^2,
\end{aligned} \tag{3.4}$$

which shows that $T_n : K \to H$ is nonexpansive.

Next we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Setting $x_{n+1} = \gamma_n x_n + (1-\gamma_n)z_n$, we have

$$\begin{aligned}
z_{n+1}-z_n &= \frac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}}-\frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n} \\
&= \frac{\beta_{n+1}u+(1-\beta_{n+1}-\gamma_{n+1})y_{n+1}}{1-\gamma_{n+1}}-\frac{\beta_n u+(1-\beta_n-\gamma_n)y_n}{1-\gamma_n} \\
&= \frac{\beta_{n+1}}{1-\gamma_{n+1}}(u-y_{n+1})+(y_{n+1}-y_n)-\frac{\beta_n}{1-\gamma_n}(u-y_n).
\end{aligned} \tag{3.5}$$

It follows that

$$\|z_{n+1}-z_n\| \le \frac{\beta_{n+1}}{1-\gamma_{n+1}}\|u-y_{n+1}\|+\|y_{n+1}-y_n\|+\frac{\beta_n}{1-\gamma_n}\|u-y_n\|. \tag{3.6}$$

From (1.4), we have $y_n = T_n u_n$ and

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|T_{n+1}u_{n+1} - T_{n+1}u_n\| + \|T_{n+1}u_n - T_n u_n\| \\ &\le \|u_{n+1} - u_n\| + \|\alpha_{n+1}u_n + (1-\alpha_{n+1})A_{n+1}u_n - [\alpha_n u_n + (1-\alpha_n)A_n u_n]\| \\ &\le \|u_{n+1} - u_n\| + |\alpha_{n+1} - \alpha_n|\ \|u_n - A_n u_n\| + (1-\alpha_{n+1})\|A_{n+1}u_n - A_n u_n\| \\ &\le \|u_{n+1} - u_n\| + |\alpha_{n+1} - \alpha_n|\|u_n - A_n u_n\| + (1-\alpha_{n+1})\sum_{i=1}^{N}\left|\eta_i^{(n+1)} - \eta_i^{(n)}\right|\|T_i u_n\|.\end{aligned} \tag{3.7}$$

By Lemma 2.1, $u_n = T_r x_n$ and $u_{n+1} = T_r x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in K, \tag{3.8}$$

$$F(u_{n+1}, y) + \frac{1}{r}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in K. \tag{3.9}$$

Putting $y = u_{n+1}$ in (3.8) and $y = u_n$ in (3.9), we obtain

$$\begin{gathered}F(u_n, u_{n+1}) + \frac{1}{r}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0\ , \\ F(u_{n+1}, u_n) + \frac{1}{r}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0.\end{gathered} \tag{3.10}$$

So, from (A2) and $r > 0$, we have

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - (u_{n+1} - x_{n+1})\rangle \ge 0, \tag{3.11}$$

and hence

$$\|u_{n+1} - u_n\|^2 \le \langle u_{n+1} - u_n, x_{n+1} - x_n\rangle, \tag{3.12}$$

which implies that

$$\|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\|. \tag{3.13}$$

Combining (3.6), (3.7), and (3.13), we have

$$\begin{aligned}\|z_{n+1} - z_n\| \le{}& \frac{\beta_{n+1}}{1-\gamma_{n+1}}\|u - y_{n+1}\| + \|x_{n+1} - x_n\| + \frac{\beta_n}{1-\gamma_n}\|u - y_n\| \\ &+ |\alpha_{n+1} - \alpha_n|\|u_n - A_n u_n\| + (1-\alpha_{n+1})\sum_{i=1}^{N}\left|\eta_i^{(n+1)} - \eta_i^{(n)}\right|\|T_i u_n\|.\end{aligned} \tag{3.14}$$

This together with (i), (ii) and (iv) imply that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0. \tag{3.15}$$

Hence, by Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.16}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\gamma_n)\|z_n-x_n\|=0. \tag{3.17}$$

On the other hand, by (1.4) and (iii), we have

$$\|x_{n+1}-y_n\|\leq\beta_n\|u-y_n\|+\gamma_n\|x_n-x_{n+1}\|+\gamma_n\|x_{n+1}-y_n\|, \tag{3.18}$$

which implies that

$$\|x_{n+1}-y_n\|\leq\frac{\beta_n}{1-\gamma_n}\|u-y_n\|+\frac{\gamma_n}{1-\gamma_n}\|x_n-x_{n+1}\|. \tag{3.19}$$

Combining (ii), (3.17), and (3.19), we have

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{3.20}$$

Note that

$$\|x_n-y_n\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|, \tag{3.21}$$

which together with (3.17) and (3.20) implies

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.22}$$

Moreover, for $p\in\mathcal{F}=\bigcap_{i=1}^{N}F(T_i)\bigcap\mathrm{EP}(F)$, we have

$$\begin{aligned}\|u_n-p\|^2&=\|T_rx_n-T_rp\|^2\leq\langle x_n-p,u_n-p\rangle\\&=\frac{1}{2}\left(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\right),\end{aligned} \tag{3.23}$$

and hence

$$\|u_n-p\|^2\leq\|x_n-p\|^2-\|x_n-u_n\|^2. \tag{3.24}$$

From Lemma 2.2, (3.2) and (3.24), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\beta_n u+\gamma_n x_n+(1-\beta_n-\gamma_n)y_n-p\|^2\\ &\le \beta_n\|u-p\|^2+\gamma_n\|x_n-p\|^2+(1-\beta_n-\gamma_n)\|y_n-p\|^2\\ &\le \beta_n\|u-p\|^2+\gamma_n\|x_n-p\|^2+(1-\beta_n-\gamma_n)\left(\|x_n-p\|^2-\|x_n-u_n\|^2\right)\\ &\le \beta_n\|u-p\|^2+\|x_n-p\|^2-(1-\beta_n-\gamma_n)\|x_n-u_n\|^2,\end{aligned}\tag{3.25}$$

and hence

$$\begin{aligned}(1-\beta_n-\gamma_n)\|x_n-u_n\|^2 &\le \beta_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le \beta_n\|u-p\|^2+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|).\end{aligned}\tag{3.26}$$

By (ii) and (3.17), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{3.27}$$

It follows from (3.22) and (3.27) that

$$\lim_{n\to\infty}\|y_n-u_n\|=0.\tag{3.28}$$

Define $S_n : K \to H$ by $S_n x = \alpha x+(1-\alpha)A_n x$. Then, S_n is a nonexpansive with $F(S_n)=F(A_n)$ by Lemma 2.4. Note that $\lim_{n\to\infty}\alpha_n=\alpha\in[k,1)$ by condition (i) and

$$\begin{aligned}\|u_n-S_n u_n\| &\le \|u_n-y_n\|+\|y_n-S_n u_n\|\\ &\le \|u_n-y_n\|+\|\alpha_n u_n+(1-\alpha_n)A_n u_n-[\alpha u_n+(1-\alpha)A_n u_n]\|\\ &\le \|u_n-y_n\|+|\alpha_n-\alpha|\|u_n-A_n u_n\|,\end{aligned}\tag{3.29}$$

which combines with condition (i) and (3.28) yielding that

$$\lim_{n\to\infty}\|u_n-S_n u_n\|=0.\tag{3.30}$$

We now show that $\limsup_{n\to\infty}\langle u-q, x_n-q\rangle \le 0$, where $q=P_{\mathcal{F}}u$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-q, x_n-q\rangle=\lim_{i\to\infty}\langle u-q, x_{n_i}-q\rangle.\tag{3.31}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ converging weakly to u^*. Without loss of generality, we assume that $u_{n_i} \rightharpoonup u^*$ as $i\to\infty$. Form (3.27), we obtain $x_{n_i}\rightharpoonup u^*$

as $i \to \infty$. Since K is closed and convex, K is weakly closed. So, we have $u^* \in K$ and $u^* \in F(S_n)$. Otherwise, from $u^* \neq S_n u^*$ and Opial's condition, we obtain

$$\begin{aligned}\liminf_{i\to\infty}\|u_{n_i} - u^*\| &< \liminf_{i\to\infty}\|u_{n_i} - S_{n_i}u^*\| \\ &\le \liminf_{i\to\infty}(\|u_{n_i} - S_{n_i}u_{n_i}\| + \|S_{n_i}u_{n_i} - S_{n_i}u^*\|) \\ &\le \liminf_{i\to\infty}\|u_{n_i} - u^*\|.\end{aligned} \tag{3.32}$$

This is a contradiction. Hence, we get $u^* \in F(S_n) = F(A_n)$. Moreover, by $u_n = T_r x_n$, we have

$$F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in K. \tag{3.33}$$

It follows from (A2) that

$$\frac{1}{r}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n). \tag{3.34}$$

Replacing n by n_i, we have

$$\frac{1}{r}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \ge F(y, u_{n_i}). \tag{3.35}$$

Since $u_{n_i} - x_{n_i} \to 0$ and $u_{n_i} \rightharpoonup u^*$, it follows from (A4) that $F(y, u^*) \le 0$ for all $y \in K$. Put $z_t = ty + (1-t)u^*$ for all $t \in (0,1]$ and $y \in K$. Then, we have $z_t \in K$, and hence, $F(z_t, u^*) \le 0$. By (A1) and (A4), we have

$$0 = F(z_t, z_t) \le tF(z_t, y) + (1-t)F(z_t, u^*) \le tF(z_t, y), \tag{3.36}$$

which implies $F(z_t, y) \ge 0$. From (A3), we have $F(u^*, y) \ge 0$ for all $y \in K$, and hence, $u^* \in \mathrm{EP}(F)$. Therefore, $u^* \in F(S_n) \cap \mathrm{EP}(F)$. From Lemma 2.6, we know that

$$\langle u - P_{\mathcal{F}}u, u^* - P_{\mathcal{F}}u\rangle \le 0. \tag{3.37}$$

It follows from (3.31) and (3.37) that

$$\limsup_{n\to\infty}\langle u - q, x_n - q\rangle = \lim_{i\to\infty}\langle u - q, x_{n_i} - q\rangle = \langle u - q, u^* - q\rangle \le 0. \tag{3.38}$$

Finally, we prove that $x_n \to q = P_{\mathcal{F}}u$ as $n \to \infty$. From (1.4) again, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \langle \beta_n u + \gamma_n x_n + (1-\beta_n-\gamma_n)y_n - q, x_{n+1}-q\rangle \\
&= \beta_n\langle u-q, x_{n+1}-q\rangle + \gamma_n\langle x_n-q, x_{n+1}-q\rangle + (1-\beta_n-\gamma_n)\langle y_n-q, x_{n+1}-q\rangle \\
&\le \beta_n\langle u-q, x_{n+1}-q\rangle + \gamma_n\|x_n-q\|\|x_{n+1}-q\| + (1-\beta_n-\gamma_n)\|y_n-q\|\|x_{n+1}-q\| \\
&\le (1-\beta_n)\|x_n-q\|\|x_{n+1}-q\| + \beta_n\langle u-q, x_{n+1}-q\rangle \\
&\le \frac{1-\beta_n}{2}\left(\|x_n-q\|^2 + \|x_{n+1}-q\|^2\right) + \beta_n\langle u-q, x_{n+1}-q\rangle \\
&\le \frac{1-\beta_n}{2}\|x_n-q\|^2 + \frac{1}{2}\|x_{n+1}-q\|^2 + \beta_n\langle u-q, x_{n+1}-q\rangle,
\end{aligned} \tag{3.39}$$

which implies that

$$\|x_{n+1}-q\|^2 \le (1-\beta_n)\|x_n-q\|^2 + 2\beta_n\langle u-q, x_{n+1}-q\rangle. \tag{3.40}$$

It follows from (3.38), (3.40), and Lemma 2.7 that $\lim_{n\to\infty}\|x_n-q\| = 0$. This completes the proof. □

As $N = 1$, that is, $A_n = T$ and $\eta_i^{(n)} \equiv 1$ in Theorem 3.1, we have the following results immediately.

Theorem 3.2. *Let K be a nonempty closed convex subset of Hilbert space H, and let F be a bifunction from $K \times K$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $T : K \to H$ be a k-strict pseudocontractions such that $\mathcal{F} = F(T)\bigcap \mathrm{EP}(F) \neq \phi$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
F(u_n, y) + \frac{1}{r}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in K, \\
y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\
x_{n+1} = \beta_n u + \gamma_n x_n + (1-\beta_n-\gamma_n)y_n, \quad n \ge 0,
\end{gathered} \tag{3.41}$$

where $u \in K$ and $x_0 \in K$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are some sequences in (0,1). If the following control conditions are satisfied:

(i) $k \le \alpha_n \le \lambda < 1$ *for all* $n \ge 0$ *and* $\lim_{n\to\infty}\alpha_n = \alpha$,

(ii) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$,

then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}}u$.

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