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Research Article

Stochastic Dynamics of Nonautonomous Cohen-Grossberg Neural Networks

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This paper is devoted to the study of the stochastic stability of a class of Cohen-Grossberg neural networks, in which the interconnections and delays are time-varying. With the help of Lyapunov function, Burkholder-Davids-Gundy inequality, and Borel-Cantell's theory, a set of novel sufficient conditions on pth moment exponential stability and almost sure exponential stability for the trivial solution of the system is derived. Compared with the previous published results, our method does not resort to the Razumikhin-type theorem and the semimartingale convergence theorem. Results of the development as presented in this paper are more general than those reported in some previously published papers. An illustrative example is also given to show the effectiveness of the obtained results.

1. Introduction

For decades, the studies of neural networks have attracted considerable multidisciplinary research interest. Ranging from signal processing, pattern recognition, programming problems, and static image processing, neural networks have witnessed a large amount of successful applications in many fields [1–7]. These applications rely crucially on the analysis of the dynamical behavior of the models [8–16]. Most existing literature on theoretical studies of neural networks is predominantly concerned with deterministic differential equations.

Recently, studies have been intensively focused on stochastic models [17–24]; it has been realized that the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes, and it is of great significance to consider stochastic effects on the stability of neural networks described by stochastic functional differential equations, see [25–34].

In [17], Liao and Mao studied mean square exponential stability and instability of cellular neural networks (CNNs). In [18, 26], the authors continued their research to discuss almost sure exponential stability for a class of stochastic neural networks with discrete delays by using the nonnegative semimartingale convergence theorem. In [25], exponential stability of stochastic Cohen-Grossberg neural networks (CGNNs) with time-varying delays via Razumikihin-type technique were investigated. In [19], Wan and Sun investigated mean square exponential stability of stochastic delayed Hopfield neural networks (HNNs) by using the method of variation of constants. Also with the help of the method of variation of constants, Sun and Cao in [29] investigated pth moment exponential stability of stochastic recurrent neural networks with time-varying delays.

However, to the best of our knowledge, few authors have considered the problem of pth moment exponential stability and almost sure exponential stability of stochastic nonautonomous Cohen-Grossberg neural networks. In fact, in the process of the electronic circuits' applications, assuring constant connection matrix and delays are unrealistic. Therefore, in this sense, time-varying connection matrix and delays will be better candidates for modeling neural information processing.

Motivated by the above discussions, in this paper, we consider the stochastic Cohen-Grossberg Neural Networks (SCGNN) with time-varying connection matrix and delays described by the following non-autonomous stochastic functional differential equations:

$$\begin{aligned} \mathrm{d}x_i(t) = &-h_i(x_i(t))\left[c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}(t) g_j(x_j(t-\tau_j(t)))\right]\mathrm{d}t \\ &+ \sum_{j=1}^{n} \sigma_{ij}(x_j(t))\mathrm{d}\omega_j(t), \quad i = 1,\dots,n, \end{aligned} \tag{1.1}$$

or

$$\mathrm{d}x(t) = -H(x(t))[C(x(t)) - A(t)F(x(t)) - B(t)G(x_\tau(t))]dt + \sigma(x(t))\mathrm{d}\omega(t), \tag{1.2}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, $H(x(t)) = \mathrm{diag}(h_1(x_1(t)), h_2(x_2(t)), \dots, h_n(x_n(t)))$, $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times n}$, $G(x_\tau(t)) = (g_1(x_1(t-\tau_1(t))), \dots, g_n(x_n(t-\tau_n(t))))^T$, $F(x(t)) = (f_1(x_1(t)), \dots, f_n(x_n(t)))^T$, and $\sigma(x(t)) = (\sigma_{ij}(x_j(t)))_{n\times n}$. Here $x_i(t)$ denotes the state variable associated with the ith neuron at time t; $h_i(\cdot)$ represent an amplification function; $c_i(\cdot)$ is an appropriately behaved function; $f_j(\cdot)$ and $g_j(\cdot)$ are activation functions; $A(t) = (a_{ij}(t))_{n\times n}$ and $B(t) = (b_{ij}(t))_{n\times n}$ represents the strength of the neuron interconnection within the network; $\tau_j(t)$ corresponds to the time delay required in processing, $0 \le \tau_j(t) \le \tau$; $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{n\times n}$ is the diffusion coefficient matrix and $\omega(t) = (\omega_1(t), \dots, \omega_n(t))^T$ is an n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ (i.e., $\mathcal{F}_t = \sigma\{w(s) : 0 \le s \le t\}$).

Obviously, model (1.1) or (1.2) is quite general, and it includes several well-known neural networks models as its special cases such as Hopfield neural networks, cellular neural networks, and bidirectional association memory neural networks [10, 16, 27, 28]. There are at least three different types of stochastic stability to describe limiting behaviors of stochastic differential equations: stability in probability, moment stability and almost sure stability (see [35]). When designing an associative memory neural network, we should make convergence speed as high as possible to ensure the quick convergence of the network operation. Therefore, pth moment ($p \geq 2$) exponential stability and almost sure exponential stability are most useful concepts as they imply that the solutions will tend to the trivial solution exponentially fast. This motivates us to study pth moment exponential stability, and almost sure exponential stability for System (1.1) in this paper.

The remainder of this paper is organized as follows. In Section 2, the basic assumptions and preliminaries are introduced. After establishing the criteria for the pth moment ($p \geq 2$) exponential stability and almost sure exponential stability for System (1.1) by using the Lyapunov function method, Burkholder-Davids-Gundy inequality and Borel-Cantell's theory in Section 3, an illustrative example and its simulations are given in Section 4.

2. Preliminaries

Throughout this article, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\mathcal{C} = C((-\infty, 0], R^n)$ be the Banach space of continuous functions which map into R^n with the topology of uniform convergence. For any $x(t) = (x_1(t), \ldots, x_n(t))^T \in R^n$, we define $\|x(t)\| = \|x(t)\|_p = (\sum_{i=1}^{n} |x_i(t)|^p)^{1/p}$, $(1 \leq p < \infty)$.

The initial conditions for system (1.1) are $x(s) = \varphi(s), -\tau \leq s \leq 0, \varphi \in L^p_{\mathcal{F}_0}((-\tau, 0], R^n)$; here $L^p_{\mathcal{F}_0}((-\tau, 0], R^n)$ is R^n-valued stochastic process $\varphi(s), -\tau \leq s \leq 0$, $\varphi(s)$ is $\mathcal{F}_0$-measurable, $\int_{-\tau}^{0} E|\varphi(s)|^p ds < \infty$. For the sake of convenience, throughout this paper, we assure $f_j(0) = g_j(0) = \sigma_{ij}(0) = 0$, which implies that system (1.1) admits an equilibrium solution $x(t) \equiv 0$.

If $V \in C^{2,1}([-\tau, \infty) \times R^n; R_+)$, according to the Itô formula, define an operator $\mathcal{L}V$ associated with (1.2) as

$$\begin{aligned} \mathcal{L}V(t,x) = {} & V_t + V_x\{-H(x(t))[C(x(t)) - A(t)F(x(t)) - B(t)G(x_\tau(t))]\} \\ & + \frac{1}{2}\operatorname{trace}\left[\sigma^T V_{xx}\sigma\right], \end{aligned} \tag{2.1}$$

where $V_t = \partial V(t,x)/\partial t$, $V_x = (\partial V(t,x)/\partial x_1, \ldots, \partial V(t,x)/\partial x_n)$, and $V_{xx} = (\partial^2 V(t,x)/\partial x_i \partial x_j)_{n\times n}$.

To establish the main results of the model given in (1.1), some of the standing assumptions are formulated as follows:

(H_1) there exist positive constants $\underline{h}_i$, $\overline{h}_i$, such that

$$0 < \underline{h}_i \leq h_i(x) \leq \overline{h}_i < +\infty, \quad \forall x \in R,\ i = 1, 2, \ldots, n; \tag{2.2}$$

(H_2) for each $i = 1, 2, \ldots, n$, there exist positive functions $\alpha_i(t) > 0$, such that

$$x_i(t)c_i(x_i(t)) \geq \alpha_i(t)x_i^2(t); \tag{2.3}$$

(H_3) there exist positive constants $\beta_j, \gamma_j, i, j = 1, 2, \ldots, n$, such that

$$\left|f_j(u) - f_j(v)\right| \leq \beta_j |u - v|, \qquad \left|g_j(u) - g_j(v)\right| \leq \gamma_j |u - v|; \tag{2.4}$$

(H_4) each $\sigma_{ij}(x)$ satisfies the Lipschitz condition, and there exist positive constants $\mu_i, i = 1, 2, \ldots, n$, such that

$$\operatorname{trace}\left\{\sigma^T(x)\sigma(x)\right\} \leq \sum_{i=1}^{n} \mu_i x_i^2. \tag{2.5}$$

Remark 2.1. The activation functions are typically assumed to be continuous, differentiable, and monotonically increasing, such as the functions of sigmoid type. These restrictive conditions are no longer needed in this paper. Instead, only the Lipschitz condition is imposed in Assumption (H_3). Note that the type of activation functions in (H_3) have already been used in numerous papers, see [5, 10] and references therein.

Remark 2.2. We remark here that non-autonomous conditions (H_2)–(H_4) replace the usual autonomous conditions which is more useful for practical purpose; please refer to [4, 13] and references therein.

Remark 2.3. The delay functions $\tau_j(t)$ considered in this paper only needed to be bounded; they can be time-varying, nondifferentiable functions. This generalized some recently published results in [4, 13, 26–29]. Different from the models considered in [4, 13, 29], in this paper, we have removed the following condition: (H_0) For each $j = 1, 2, \ldots, n, \tau_j(t)$ is a differentiable function, namely, there exists ξ such that

$$\dot{\tau}_j(t) \leq \xi < 1. \tag{2.6}$$

Definition 2.4 (see [35]). The trivial solution of (1.1) is said to be pth moment exponential stability if there is a pair of positive constants λ and C such that

$$E\|x(t, t_0, \mathrm{x}_0)\|^p \leq C\|x_0\|^p e^{-\lambda(t-t_0)}, \quad \text{on } \ t \geq t_0, \quad \forall x_0 \in R^n, \tag{2.7}$$

where $p \geq 2$ is a constant; when $p = 2$, it is usually said to be exponential stability in mean square.

Definition 2.5 (see [35]). The trivial solution of (1.1) is said to be almost sure exponential stability if for almost all sample paths of the solution $x(t)$, we have

$$\limsup_{t \to \infty} \frac{1}{t} \log\|x(t)\| < 0. \tag{2.8}$$

Lemma 2.6 ([35] Burkholder-Davids-Gundy inequality). *There exists a universal constant K_p for any $0 < p < \infty$ such that for every continuous local martingale M vanishing at zero and any stopping time η,*

$$E\left(\sup_{0\le s\le \eta}|M_s|^p\right) \le K_p E\left(\langle M, M\rangle_\eta\right)^{p/2}, \tag{2.9}$$

where $\langle M, M\rangle_\eta$ is the cross-variation of M. In particular, one may have $K_p = (32/p)^{p/2}$ if $0 < P < 2$ and $K_2 = 4$ if $p = 2$; although they may not be optimal, for example, one could have $K_1 = 4\sqrt{2}$.

Lemma 2.7 ([35] Chebyshev's inequality).

$$P\{\omega : |X(\omega)| \ge c\} \le c^{-p}E|X|^p \tag{2.10}$$

if $c > 0$, $p > 0$, $X \in L^p$.

Lemma 2.8 ([36] Borel-Cantell's lemma). *Let $\{A_n, n \ge 1\}$ be a sequence of events in some probability space, then*

(i) *if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n, i.o.) = 0$;*

(ii) *moreover, if $\{A_n, n \ge 1\}$ are independent of each other, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies*

$$P(A_n, i.o.) = 1, \tag{2.11}$$

where $\{A_n, i.o.\}$ denotes occurring infinitely often within $\{A_n, n \ge 1\}$, that is, $\{A_n, i.o.\} = \cap_{k=1}^{\infty}\cup_{n=k}^{\infty}A_n$. "i.o." is the abbreviation of "infinitely often".

3. Main Results

Theorem 3.1. *Under the assumptions (H_1)–(H_4), if there are a positive diagonal matrix $M = \mathrm{diag}(m_1, \ldots, m_n)$ and two constants $0 < N_2$, $0 \le \mu < 1$, such that*

$$0 < N_2 \le N_2(t) \le \mu N_1(t), \quad \text{for } t \ge t_0, \tag{3.1}$$

where

$$\begin{aligned} N_1(t) = \min_{1\le i\le n}\Bigg\{ & p\underline{h}_i\alpha_i(t) - \sum_{j=1}^{n}\overline{h}_i(p-1)\left|a_{ij}(t)\right|\beta_j - \sum_{j=1}^{n}\frac{m_j}{m_i}\overline{h}_j\left|a_{ji}(t)\right|\beta_i \\ & -\sum_{j=1}^{n}\overline{h}_i(p-1)\left|b_{ij}(t)\right|\gamma_j - \sum_{j=1}^{n}\frac{(p-1)(p-2)}{2}\mu_j - \sum_{j=1}^{n}\frac{m_j}{m_i}(p-1)\mu_i\Bigg\}, \end{aligned} \tag{3.2}$$

$$N_2(t) = \max_{1\le i\le n}\sum_{j=1}^{n}\frac{m_j}{m_i}\overline{h}_j\left|b_{ji}(t)\right|\gamma_i,$$

then the trivial solution of system (1.1) *is pth moment exponential stability, where* $p \geq 2$ *denotes a positive constant. When* $p = 2$*, the trivial solution of system* (1.1) *is exponential stability in mean square.*

Proof. Consider the following Lyapunov function:

$$V(x,t) = \sum_{i=1}^{n} m_i \left| x_i^p(t) \right|. \tag{3.3}$$

As $p \geq 2$ denotes a positive constant, we can get the following inequality: if a and b denote nonnegative real numbers, then $pa^{p-1}b \leq (p-1)a^p + b^p$, $a^{p-2}b^2 \leq (p-2)a^p/p + 2b^p/p$. Using this inequality, then the operator associated with system (1.1) has the form as follows:

$$\begin{aligned}
\mathcal{L}V(x,t) = &-p\sum_{i=1}^{n} m_i x_i^{P-2}(t) x_i(t) h_i(x_i(t)) c_i(x_i(t)) \operatorname{sgn}\{x_i(t)\} \\
&+ p\sum_{i=1}^{n} m_i x_i^{P-1}(t) h_i(x_i(t)) \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) \operatorname{sgn}\{x_i(t)\} \\
&+ p\sum_{i=1}^{n} m_i x_i^{P-1}(t) h_i(x_i(t)) \sum_{j=1}^{n} b_{ij}(t) g_j(x_j(t-\tau_j)) \operatorname{sgn}\{x_i(t)\} \\
&+ \frac{p(p-1)}{2} \sum_{i=1}^{n} m_i x_i^{P-2}(t) \sum_{j=1}^{n} \sigma_{ij}^2 \operatorname{sgn}\{x_i(t)\} \\
\leq &-p\sum_{i=1}^{n} m_i \left| x_i^{P-2}(t) \right| \underline{h}_i \alpha_i(t) x_i^2(t) \\
&+ p\sum_{i=1}^{n} m_i \left| x_i^{P-1}(t) \right| \overline{h}_i \sum_{j=1}^{n} |a_{ij}(t)| \beta_j |x_j(t)| \\
&+ p\sum_{i=1}^{n} m_i \left| x_i^{P-1}(t) \right| \overline{h}_i \sum_{j=1}^{n} |b_{ij}(t)| \gamma_j |x_j(t-\tau_j(t))| \\
&+ \frac{p(p-1)}{2} \sum_{i=1}^{n} m_i \left| x_i^{P-2}(t) \right| \sum_{j=1}^{n} \mu_j x_j^2(t) \\
\leq &-N_1(t) V(x(t),t) + N_2(t) \sup_{t-\tau \leq s \leq t} \{V(x(s),s)\},
\end{aligned} \tag{3.4}$$

where

$$N_1(t) = \min_{1\le i\le n}\left\{ p\underline{h}_i\alpha_i(t) - \sum_{j=1}^{n}\overline{h}_i(p-1)|a_{ij}(t)|\beta_j - \sum_{j=1}^{n}\frac{m_j}{m_i}\overline{h}_j|a_{ji}(t)|\beta_i \right.$$
$$\left. -\sum_{j=1}^{n}\overline{h}_i(p-1)|b_{ij}(t)|\gamma_j - \sum_{j=1}^{n}\frac{(p-1)(p-2)}{2}\mu_j - \sum_{j=1}^{n}\frac{m_j}{m_i}(p-1)\mu_i \right\}, \tag{3.5}$$
$$N_2(t) = \max_{1\le i\le n}\sum_{j=1}^{n}\frac{m_j}{m_i}\overline{h}_j|b_{ji}(t)|\gamma_i.$$

The remaining part of the proof is similar to that of Theorem 3.3 in [33]; we omit it. □

In Theorem 3.1, if we let M be the identity matrix, we can easily obtain the following corollary.

Corollary 3.2. *Under the assumptions* (H_1)*–*(H_4)*, if there are two constants* $0 < N_2, 0 \le \mu < 1$*, such that*

$$0 < N_2 \le N_2(t) \le \mu N_1(t), \quad \forall t \ge t_0, \tag{3.6}$$

where

$$N_1(t) = \min_{1\le i\le n}\left\{ p\underline{h}_i\alpha_i(t) - \sum_{j=1}^{n}\overline{h}_i(p-1)|a_{ij}(t)|\beta_j - \sum_{j=1}^{n}\overline{h}_j|a_{ji}(t)|\beta_i \right.$$
$$\left. -\sum_{j=1}^{n}\overline{h}_i(p-1)|b_{ij}(t)|\gamma_j - \sum_{j=1}^{n}\frac{(p-1)(p-2)}{2}\mu_j - \sum_{j=1}^{n}(p-1)\mu_i \right\}, \tag{3.7}$$
$$N_2(t) = \max_{1\le i\le n}\sum_{j=1}^{n}\overline{h}_j|b_{ji}(t)|\gamma_i,$$

then the trivial solution of system (1.1) *is pth moment exponentially stability.*

Remark 3.3. Compared with [10, 12], our method does not resort to the Razumikhin-type theorem or Halanay inequality.

Theorem 3.4. *Suppose system* (1.1) *satisfies assumptions* (H_1)*–*(H_4) *and the inequality* (3.1) *hold; if* $a_{ij}(t)$*,* $b_{ij}(t)$*, and* $\alpha_i(t)$ *are bounded functions for all* i*,* j*, then the trivial solution of* (1.1) *is almost sure exponential stability.*

Proof. Let N be an integer such that $N - \tau \ge 1$ and $I_N = [N, N+1]$; consider the following Lyapunov function:

$$V(x(t)) = \|x(t)\|^p. \tag{3.8}$$

Using the Itô formula, we have

$$\begin{aligned}
\mathrm{d}\|x(t)\|^p = \sum_{i=1}^{n}\Bigg\{ & -p|x_i(t)|^{p-2}x_i(t)h_i(x_i(t)) \\
& \times\left[c_i(x_i(t)) - \sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t)) - \sum_{j=1}^{n}b_{ij}(t)g_j(x_j(t-\tau_j(t)))\right]\Bigg\}\mathrm{d}t \\
& + \frac{p(p-1)}{2}\sum_{i=1}^{n}|x_i(t)|^{p-2}\operatorname{trace}\Big(\sigma^T(x(t))\sigma(x(t))\Big)\mathrm{d}t \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n}p|x_i(t)|^{p-2}x_i(t)\sigma_{ij}(x_j(t))\mathrm{d}\omega_j(t).
\end{aligned}\tag{3.9}$$

Calculating the integral of (3.9) from N to t, we have

$$\begin{aligned}
\|x(t)\|^p - \|x(N)\|^p = \int_N^t\Bigg\{ & \sum_{i=1}^{n} - p|x_i(s)|^{p-2}x_i(s)h_i(x_i(s)) \\
& \times\left[c_i(x_i(s)) - \sum_{j=1}^{n}a_{ij}(t)f_j(x_j(s)) - \sum_{j=1}^{n}b_{ij}(t)g_j(x_j(s-\tau_j(t)))\right] \\
& + \frac{p(p-1)}{2}\sum_{i=1}^{n}|x_i(s)|^{p-2}\operatorname{trace}\Big(\sigma^T(x(s))\sigma(x(s))\Big)\Bigg\}\mathrm{d}s \\
& + \int_N^t\sum_{i=1}^{n}\sum_{j=1}^{n}p|x_i(s)|^{p-2}x_i(s)\sigma_{ij}(x_j(s))\mathrm{d}\omega_j(s),
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
E\sup_{t\in I_N}\|x(t)\|^p = E\sup_{t\in I_N}\Bigg\{ & \|x(N)\|^p + \int_N^t\Bigg\{\sum_{i=1}^{n} - p|x_i(s)|^{p-2}x_i(s)x_i(s)h_i(x_i(s)) \\
& \times\Bigg[c_i(x_i(s)) - \sum_{j=1}^{n}a_{ij}(s)f_j(x_j(s)) \\
& - \sum_{j=1}^{n}b_{ij}(s)g_j(x_j(s-\tau_j(s)))\Bigg] \\
& + \frac{p(p-1)}{2}\sum_{i=1}^{n}|x_i(s)|^{p-2}\operatorname{trace}\Big(\sigma^T(x(s))\sigma(x(s))\Big)\Bigg\}\mathrm{d}s \\
& + \int_N^t\sum_{i=1}^{n}\sum_{j=1}^{n}p|x_i(s)|^{p-2}x_i(s)\sigma_{ij}(x_j(s))\mathrm{d}\omega_j(s)\Bigg\} \\
\le E\|x(N)\|^p + M_1 + M_2, &
\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}
M_1 = E\sup_{t\in I_N}\int_N^t \Bigg\{ &\sum_{i=1}^{n} - p|x_i(s)|^{p-2}x_i(s)h_i(x(s)) \\
&\times\left[c_i(x_i(s)) - \sum_{j=1}^{n}a_{ij}(s)f_j(x_j(s)) - \sum_{j=1}^{n}b_{ij}(s)g_j(x_j(s-\tau_j(s)))\right] \\
&+\frac{p(p-1)}{2}\sum_{i=1}^{n}|x_i(s)|^{p-2}\mathrm{trace}\Big(\sigma^T(x(s))\sigma(x(s))\Big)\Bigg\}\mathrm{d}s, \\
M_2 = E\sup_{t\in I_N}\int_N^t &\sum_{i=1}^{n}\sum_{j=1}^{n}p|x_i(\mathrm{s})|^{p-2}x_i(s)\sigma_{ij}(x_j(s))\mathrm{d}\omega_j(s).
\end{aligned} \tag{3.12}$$

From Theorem 3.1, there exists a pair of positive constants λ and δ, such that

$$E\|x(t)\|^p \le \delta\|x_0\|^p e^{-\lambda(t-t_0)}, \quad \text{on } t \ge t_0. \tag{3.13}$$

Furthermore, from (H_1)–(H_4) and inequality (3.13), we have

$$\begin{aligned}
M_1 \le E\int_N^{N+1}\Bigg\{&\sum_{i=1}^{n}\Bigg[-p|x_i(s)|^p\underline{h}_i\alpha_i(t) + \sum_{j=1}^{n}p|x_i(s)|^{p-2}|a_{ij}(s)|\overline{h}_i\beta_j|x_i(s)x_j(s)| \\
&+\sum_{j=1}^{n}p|x_i(s)|^{p-2}|b_{ij}(s)|\overline{h}_i\gamma_j|x_i(s)x_j(s-\tau_j(s))|\Bigg] \\
&+\frac{p(p-1)}{2}\sum_{i=1}^{n}\mu_{\mathrm{i}}\left|x_i^p(s)\right|\Bigg\}\mathrm{d}s \\
\le E\int_N^{N+1}\Bigg\{&\sum_{i=1}^{n}\Bigg[-p\underline{h}_i\alpha_i(t)|x_i(s)|^p + \sum_{j=1}^{n}|a_{ij}(s)|\overline{h}_i\beta_j\left[(p-1)|x_i(s)|^p + |x_j(s)|^p\right] \\
&+\sum_{j=1}^{n}|b_{ij}(s)|\overline{h}_i\gamma_j\left[(p-1)|x_i(s)|^p + |x_j(s-\tau_j(s))|^p\right]\Bigg] \\
&+\frac{p(p-1)}{2}\sum_{i=1}^{n}\mu_i|x_i(s)|^p\Bigg\}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_N^{N+1}\left\{\sum_{i=1}^n\left[p\underline{h}_i\alpha_i(t)+\sum_{j=1}^n p\left|a_{ij}(s)\right|\overline{h}_i\beta_j\right.\right.\\
&\qquad\left.\left.+\sum_{j=1}^n(p-1)\left|b_{ij}(s)\right|\overline{h}_i\gamma_j+\frac{p(p-1)}{2}\sum_{i=1}^n\mu_i\right]E\|x(s)\|^p\right\}\mathrm{d}s\\
&\quad+\int_N^{N+1}\left\{\sum_{i=1}^n\sum_{j=1}^n\left|b_{ij}(s)\right|\overline{h}_i\gamma_j E\left\|x\left(s-\tau_j(s)\right)\right\|^p\right\}\mathrm{d}s\\
&\leq \int_N^{N+1}\left\{\sum_{i=1}^n\left[p\underline{h}_i\alpha_i(t)+p\sum_{j=1}^n\left|a_{ij}(s)\right|\overline{h}_i\beta_j\right.\right.\\
&\qquad\left.\left.+\sum_{j=1}^n\left|b_{ij}(s)\right|\overline{h}_i\gamma_j\left(p-1+e^{\lambda\tau}\right)+\frac{p(p-1)}{2}\mu_i\right]e^{\lambda t_0}\right\}\delta\|x_0\|^p e^{-\lambda s}\mathrm{d}s\\
&\leq I_1\delta\|x_0\|^p e^{-\lambda N},
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
I_1=\sup_{t\in R}&\left\{\sum_{i=1}^n\left[p\underline{h}_i\alpha_i(t)+p\sum_{j=1}^n\left|a_{ij}(t)\right|\overline{h}_i\beta_j\right.\right.\\
&\left.\left.+\sum_{j=1}^n\left|b_{ij}(t)\right|\overline{h}_i\gamma_j\left(p-1+e^{\lambda\tau}\right)+\frac{p(p-1)}{2}\mu_i\right]e^{\lambda t_0}\right\}.
\end{aligned}
\tag{3.15}
$$

For any two different norms $\|\cdot\|_2$, $\|\cdot\|_{p-2}$, $(1<p<\infty)$, for all $x\in R^n$, as the space R^n is a finite dimensional space, there exist two positive constants ζ_1, ζ_2, such that

$$
\begin{aligned}
\|x\|_2^2&\leq\zeta_1\|x\|_p^2,\\
\|x\|_{p-2}^{\mathrm{p}-2}&\leq\zeta_2\|x\|_p^{\mathrm{p}-2}.
\end{aligned}
\tag{3.16}
$$

As $M=\int_N^t\sum_{i=1}^n\sum_{j=1}^n p|x_i(s)|^{p-2}x_i(s)\sigma_{ij}(x_j(s))\mathrm{d}\omega_j(s)$ is continuous local martingale, then from Lemma 2.6, (H_4), and (3.16), it follows that

$$
\begin{aligned}
M_2&\leq K_1E\left\{\int_N^{N+1}\sum_{j=1}^n\left[\sum_{i=1}^n p|x_i(s)|^{p-1}\left|\sigma_{ij}\left(x_j(s)\right)\right|\right]^2\mathrm{d}s\right\}^{1/2}\\
&\leq 4\sqrt{2}E\left\{\int_N^{N+1}\sum_{j=1}^n\left\{\left[\sum_{i=1}^n|x_i(s)|^{2p-2}\right]\left[\sum_{i=1}^n p^2\sigma_{ij}^2\left(x_j(s)\right)\right]\right\}\mathrm{d}s\right\}^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq 4\sqrt{2}E\left\{\int_{N}^{N+1} p^2\sum_{j=1}^{n}\mu_j\left|x_j(t)\right|^2\left[\sum_{i=1}^{n}|x_i(s)|^{2p-2}\right]\mathrm{d}s\right\}^{1/2} \\
&\leq 4\sqrt{2}E\left\{\int_{N}^{N+1} p^2\max_{1\leq i\leq n}\{\mu_i\}\zeta_1\|x\|_p^2\zeta_2\|x\|_p^{\mathrm{p}-2}\mathrm{d}s\right\}^{1/2} \\
&\leq 2E\left\{\left[\frac{1}{2}\sup_{s\in I_N}\|x(s)\|^p\right]\int_{N}^{N+1} 16p^2\zeta_1\zeta_2\max_{1\leq i\leq n}\{\mu_i\}\|x(s)\|^p\mathrm{d}s\right\}^{1/2} \\
&\leq \frac{1}{2}E\sup_{t\in I_N}\|x(t)\|^p + 16p^2\zeta_1\zeta_2\max_{1\leq i\leq n}\{\mu_i\}\int_{N}^{N+1} E\{\|x(s)\|^p\}ds.
\end{aligned}
\tag{3.17}
$$

According to (3.11), (3.13), (3.14), and (3.17), we have the following inequality:

$$
\begin{aligned}
E\sup_{t\in I_N}\|x(t)\|^p &\leq E\|x(N)\|^p + M_1 + M_2 \\
&\leq \delta\|x_0\|^p e^{-\lambda(N-t_0)} + I_1\delta\|x_0\|^p e^{-\lambda N} + \frac{1}{2}E\sup_{t\in I_N}\|x(t)\|^p \\
&\quad + 16p^2\zeta_1\zeta_2\max_{1\leq i\leq n}\{\mu_i\}\int_{N}^{N+1} \delta\|x_0\|^p e^{-\lambda(s-t_0)}ds \\
&\leq \frac{1}{2}E\sup_{t\in I_N}\|x(t)\|^p + \left\{e^{\lambda t_0} + I_1 + \frac{16p^2\zeta_1\zeta_2}{\lambda}\max_{1\leq i\leq n}\{\mu_i\}e^{\lambda t_0}\right\}\delta e^{-\lambda N}.
\end{aligned}
\tag{3.18}
$$

Therefore,

$$
E\sup_{t\in I_N}\|x(t)\|^p \leq \theta e^{-\lambda N},
\tag{3.19}
$$

where

$$
\theta = 2\left\{e^{\lambda t_0} + I_1 + \frac{16p^2\zeta_1\zeta_2}{\lambda}\max_{1\leq i\leq n}\{\mu_i\}e^{\lambda t_0}\right\}\delta.
\tag{3.20}
$$

For each integer N, we set $\varepsilon_N = e^{-\lambda N/2}$. Then, from Lemma 2.7, we have

$$
P\left\{\omega : \sup_{t\in I_N}\|x(t)\|^p > \varepsilon_N\right\} \leq \frac{E\sup_{t\in I_N}\|x(t)\|^p}{\varepsilon_N} \leq \theta e^{-\lambda N/2}.
\tag{3.21}
$$

Therefore, in view of Lemma 2.8, for almost all $\omega \in \Omega$, we have that

$$\sup_{t \in I_N} \|x(t)\|^p \leq e^{-\lambda N/2}, \tag{3.22}$$

holds for all but finitely many N. Hence, there exists an $N_0 = N_0(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, for which (3.21) holds whenever $N \geq N_0$. Consequently, for almost all $\omega \in \Omega$,

$$\frac{1}{t} \log\|x(t)\| = \frac{1}{pt} \log \|x(t)\|^p \leq -\frac{\lambda}{2p}, \tag{3.23}$$

if $N \leq t \leq N+1$, $N \geq N_0$. Hence

$$\limsup_{t \to \infty} \frac{1}{t} \log\|x(t)\| = \frac{1}{pt} \leq -\frac{\lambda}{p} \quad \text{a.s.} \tag{3.24}$$

Therefore, the trivial solution of (1.1) is almost sure exponential stability. □

Remark 3.5. Compared with [26, 32], our method does not resort to the semimartingale convergence theorem. Since system (1.1) does not require the delays to be constants, furthermore, the model is non-autonomous, it is clear that the results obtained in [19, 25–32, 34] cannot be applicable to system (1.1). This implies that the results of this paper are essentially new and complement some corresponding ones already known.

Remark 3.6. By Theorems 3.1 and 3.4, the stability of system (1.1) is dependent on the magnitude of noise, and therefore, stochastic noise fluctuation is one of the very important aspects in designing a stable network and should to be considered adequately.

It should be noted that the assumptions of the boundedness of $a_{ij}(t)$, $b_{ij}(t)$, and $\alpha_i(t)$ in Theorem 3.4 are not necessary; we use these assumptions just to simplify the process of the proof. In fact, in view of (3.15), (3.20), (3.21), and (3.22), similar to the proof of Theorem 3.4, we have the following theorem.

Theorem 3.7. *Suppose system* (1.1) *satisfies assumptions* (H_1)*–*(H_4) *and the inequality* (3.1) *hold, if there exist positive constants* ρ_{ij}*,* ρ_{ij}*,* ϑ_{ij} *such that for any* t*, we have*

$$|a_{ij}(t)| \bigvee |b_{ij}(t)| \bigvee \alpha_i(t) \leq \rho_{ij} t^{\rho_{ij}} + \vartheta_{ij}, \tag{3.25}$$

then the trivial solution of (1.1) *is almost sure exponential stability.*

Remark 3.8. Furthermore, the derived conditions for stability of the following stochastic delayed recurrent neural networks can be viewed as byproducts of our results. The significant of this paper does offer a wider selection on the networks parameters in order to achieve some necessary convergence in practice.

Remark 3.9. For system (1.1), when $h_i(x_i(t)) = 1$, $c_i(x_i(t)) = c_i x_i(t)(c_i > 0)$, and $a_{ij}(t) = a_{ij}, b_{ij}(t) = b_{ij}$, then it turns out to be following stochastic delayed recurrent neural networks with time-varying delays

$$\mathrm{d}x_i(t) = -c_i x_i(t)dt + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right)dt + \sum_{j=1}^{n} b_{ij} g_j\left(x_j\left(t-\tau_j(t)\right)\right)\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}\left(x_j(t)\right)\mathrm{d}\omega_j(t). \tag{3.26}$$

Using Theorems 3.1 and 3.4, one can easily get a set of similar corollary for checking the pth moment exponential stability and almost sure exponential stability for the trivial solution of this system.

4. An Illustrative Example

In this section, an example is presented to demonstrate the correctness and effectiveness of the main obtained results.

Example 4.1. Consider the following stochastic Cohen-Grossberg neural networks with time-varying delays:

$$\begin{aligned}
\mathrm{d}x(t) &= \begin{pmatrix} 3-\cos(x_1(t)) & 0 \\ 0 & 3-\sin(x_2(t)) \end{pmatrix} \\
&\quad \times \left[-\begin{pmatrix} 5+0.002t & 0 \\ 0 & 5+0.003t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \begin{pmatrix} 0.2\tanh(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{pmatrix} \right) \\
&\quad + \begin{pmatrix} -0.8 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0.2\tanh(x_1(t-\tau_1(t))) \\ 0.2\tanh(x_2(t-\tau_2(t))) \end{pmatrix} \Bigg] \mathrm{d}t + \sigma(x(t))\mathrm{d}w(t), \quad t \geq 0,
\end{aligned} \tag{4.1}$$

where $\tau(t) = (\tau_1(t), \tau_2(t))^T$ and $\tau_i(t)$ is any bounded positive function for $i = 1, 2$. Each $\sigma_{ij}(x)$ satisfies the Lipschitz condition, and there exist positive constants $\mu_1 = \mu_2 = 2$, such that

$$\operatorname{trace}\left\{\sigma^T(x)\sigma(x)\right\} \leq 2\left(x_1^2 + x_2^2\right). \tag{4.2}$$

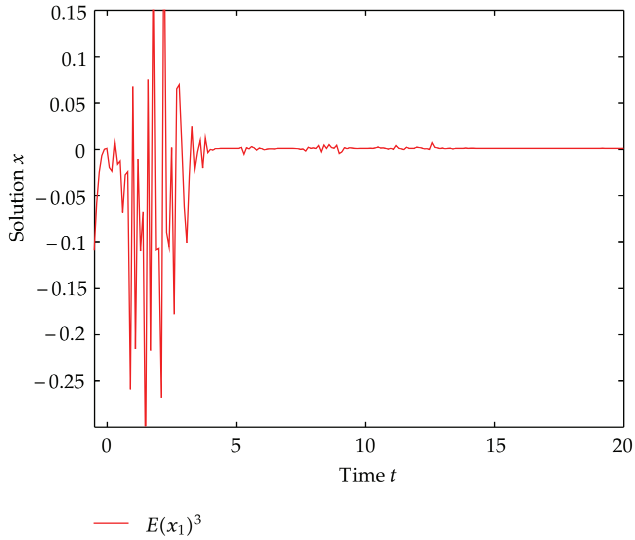

Figure 1: Numerical solution $E(x_1^3(t))$ of system (4.1).

In the example, let $p = 3$; by simple computation, we obtain

$$\begin{aligned} N_1(t) &= \min_{1\le i\le n}\left\{ p\underline{h}_i\alpha_i(t) - \sum_{j=1}^{n}\overline{h}_i(p-1)\left|a_{ij}(t)\right|\beta_j - \sum_{j=1}^{n}\frac{m_j}{m_i}\overline{h}_j\left|a_{ji}(t)\right|\beta_i \right. \\ &\qquad\left. -\sum_{j=1}^{n}\overline{h}_i(p-1)\left|b_{ij}(t)\right|\gamma_j - \sum_{j=1}^{n}\frac{(p-1)(p-2)}{2}\mu_j - \sum_{j=1}^{n}\frac{m_j}{m_i}(p-1)\mu_i \right\} \\ &= 11.6 + 0.008t, \\ N_2(t) &= \max_{1\le i\le n}\sum_{j=1}^{n}\frac{m_j}{m_i}\overline{h}_j\left|b_{ji}(t)\right|\gamma_i = 8.8 + 0.0008t. \end{aligned} \tag{4.3}$$

Choosing $\mu = 8/9$, one can easily get that

$$0 < N_2 \le N_2(t) \le \mu N_1(t), \quad \text{for } t \ge 0. \tag{4.4}$$

Thus, it follows Theorem 3.7 that system (4.1) is the third moment exponentially stable and also almost sure exponentially stable. These conclusions can be verified by the following numerical simulations (Figures 1, 2, 3, and 4).

Remark 4.2. Let $\tau_1(t) = \tau_2(t) = 0.5\sin t + 1$; we can find that [29, Theorem 1] is not satisfied; therefore, they fail to conclude whether system (4.1) is pth moment exponentially stable even when the delay functions are differential and their derivatives are simultaneously required to be not greater than 1. It is obvious that the results in [19, 25–32, 34] and the references therein cannot be applicable to system (4.1).

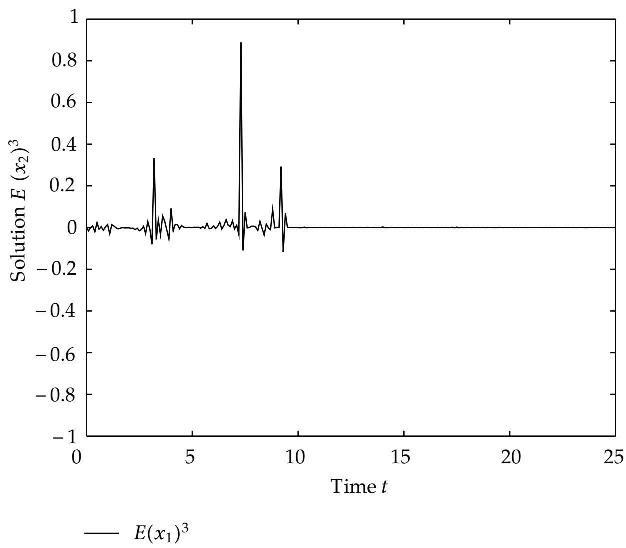

Figure 2: Numerical solution $E(x_2^3(t))$ of system (4.1).

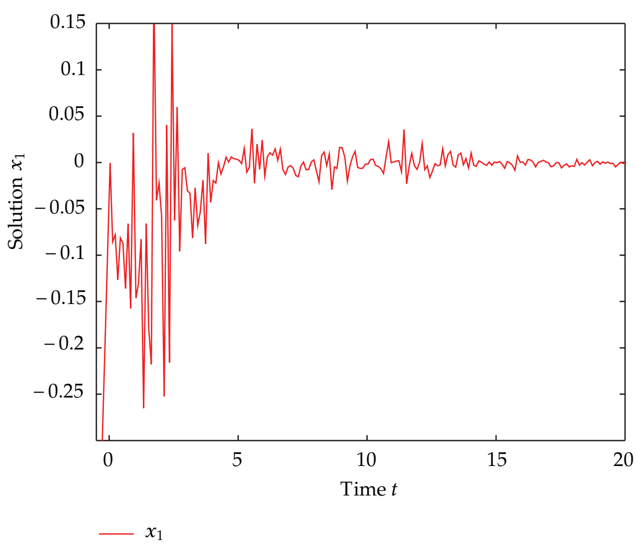

Figure 3: Numerical solution $x_1(t)$ of system (4.1).

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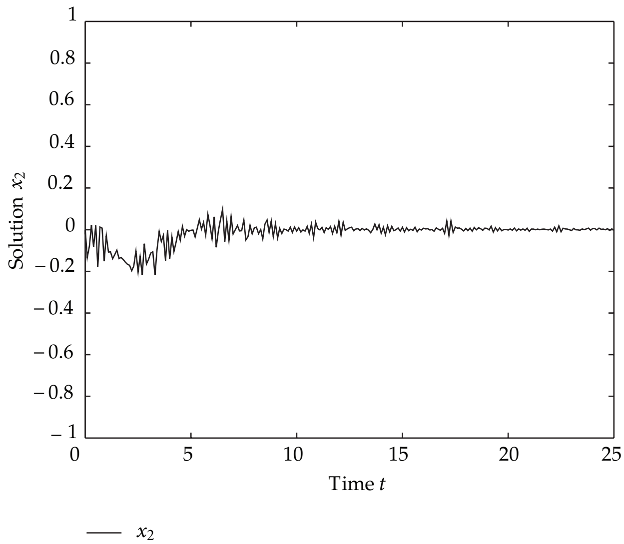

Figure 4: Numerical solution $x_2(t)$ of system (4.1).

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