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Research Article

Global Existence and Blowup Analysis to Single-Species Bacillus System with Free Boundary

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This paper is concerned with a reaction-diffusion equation which describes the dynamics of single bacillus population with free boundary. The local existence and uniqueness of the solution are first obtained by using the contraction mapping theorem. Then we exhibit an energy condition, involving the initial data, under which the solution blows up in finite time. Finally we examine the long time behavior of global solutions; the global fast solution and slow solution are given. Our results show that blowup occurs if the death rate is small and the initial value is large enough. If the initial value is small the solution is global and fast, which decays at an exponential rate while there is a global slow solution provided that the death rate is small and the initial value is suitably large.

1. Introduction

As we know, mathematical aspects of biological population have been considered widely. Most of the authors have studied growth and diffusions of biological population in a homogeneous or heterogeneous fixed environment [1, 2], and the nonlinear differential equations are described such as Logistic equation and Fisher equation.

In this paper, we consider the following single bacillus population model:

$$u_t - d\Delta u = Kau^2 - bu, \tag{1.1}$$

which was first proposed by Verhulst see [3]. Parameters a, b, d and K are positive constants. Ecologically, a represents the net birth rate, b is the death rate, d denotes the diffusion coefficient, and K measures the living resource for bacillus. In [4], Jin et al. considered the model and established a time-dependent dynamic basis to quantitatively clarify the biological wave behavior of the popular growth and propagation.

The present paper aims to investigate the parabolic equation with a moving boundary in one-dimensional space.

As in [5], assumed that the amount of the species flowing across the free boundary is increasing with respect to the moving length, the condition on the interface (free boundary) is $s'(t) = -\mu u_x(s(t),t)$ by using the Taylor expansion. Here μ is a positive constant and measures the ability of the bacillus disperse in a new area. The free boundary is regarded as the moving front, the detailed biological implication see Section 6 of [6] for the logistic model; the authors also compared their results in biological terms with some documented ecological observations there. In this way, we have the following problem for $u(x,t)$ and a free boundary $x = s(t)$ such that

$$\begin{gathered} u_t - du_{xx} = Kau^2 - bu, \quad 0 < x < s(t),\ 0 < t < T, \\ u(s(t),t) = 0, \quad 0 < t < T, \\ u_x(0,t) = 0, \quad 0 < t < T, \\ s(0) = s_0 > 0, \\ u(x,0) = u_0(x) \geqslant 0, \quad 0 \leqslant x \leqslant s(0), \\ s'(t) = -\mu u_x(s(t),t), \quad 0 < t < T, \end{gathered} \tag{1.2}$$

where the condition $u_x(0,t) = 0$ indicates that the habitat is semiunbounded domain and there is no migration cross the left boundary.

When $Ka = b = 0$, the problem is reduced to one phase Stefan problem, which accounts for phase transitions between solid and fluid states such as the melting of ice in contact with water [7]. Stefan problems have been studied by many authors. For example, the weak solution was considered by Oleĭnik in [8], and the existence of a classical solution was given by Kinderlehrer and Nirenberg in [9]. For the two-phase Stefan problem, the local classical solution was obtained in [10, 11], and the global classical solution was given by Borodin in [12].

The free boundary problems have been investigated in many areas, for example, the decrease of oxygen in a muscle in the vicinity of a clotted bloodvessel [13], the etching problem [14], the combustion process [15], the American option pricing problem [16, 17], chemical vapor deposition in hot wall reactor [18], image processing [19], wound healing [20], tumor growth [21–24] and the dynamics of population [5, 25, 26].

In this paper, we consider the free boundary problem (1.2) and focus on studying the blowup behavior of the solution and asymptotic behavior of the global solutions. We will give sufficient conditions to ensure the existence of fast solution and slow solution. Here if $T = +\infty$, we say the solution exists globally whereas if the solution ceases to exist for some finite time, that is, $T < +\infty$ and $\lim_{t\to T}\|u(x,t)\|_{L^\infty([0,s(t)]\times[0,t])} \to +\infty$, we say that the solution blows up. If $T = \infty$ and $\lim_{t\to\infty}s(t) < \infty$, the solution is called fast solution since that the solution decays uniformly to 0 at an exponential rate, while if $T = \infty$ and $\lim_{t\to\infty}s(t) = \infty$, it is called slow solution, see [27, 28] in detail.

The remainder of this paper is organized as follows. In Section 2, local existence and uniqueness will be given. Section 3 deals with the result of blowup behavior by constructing an energy condition. Section 4 is devoted to long time behaviors of global solutions, including the existence of global fast solution and slow solution.

2. Local Existence and Uniqueness

In this section, we prove the following local existence and uniqueness of the solution to (1.2) by contraction mapping principle.

Theorem 2.1. *For any given u_0 satisfying $u_0 \in C^{1+\alpha}([0,s_0])$ with $\alpha \in (0,1)$, $u_0'(0) = u_0(s_0) = 0$ and $u_0 > 0$ in $[0,s_0)$, there is a $T > 0$ such that problem (1.2) admits a unique solution*

$$(u,s) \in C^{1+\alpha,(1+\alpha)/2}([0,s(t)] \times [0,T]) \times C^{1+\alpha/2}([0,T]). \tag{2.1}$$

Furthermore,

$$\|u(x,t)\|_{C^{1+\alpha,(1+\alpha)/2}([0,s(t)]\times[0,T])} + \|s(t)\|_{C^{1+\alpha/2}([0,T])} \leqslant C, \tag{2.2}$$

where C and T depend only on α, s_0 and $\|u_0\|_{C^{1+\alpha}([0,s_0])}$.

Proof. We first make a change of variable to straighten the free boundary. Let

$$\xi = \frac{x}{s(t)}, \qquad u(x,t) = v(\xi,t). \tag{2.3}$$

Then the problem (1.2) is reduced to

$$\begin{gathered} v_t - \frac{s'(t)}{s(t)}\xi v_\xi - \frac{d}{s^2(t)} v_{\xi\xi} = Kav^2 - bv, \quad 0 < \xi < 1,\ 0 < t < T, \\ v(1,t) = 0, \quad 0 < t < T, \\ v_\xi(0,t) = 0, \quad 0 < t < T, \\ s(0) = s_0 > 0, \\ v(\xi,0) = v_0(\xi) := u_0(s_0\xi) \geqslant 0, \quad 0 \leqslant \xi \leqslant 1, \\ s'(t) = -\frac{\mu}{s(t)} v_\xi(1,t), \quad 0 < t < T. \end{gathered} \tag{2.4}$$

This transformation changes the free boundary $x = s(t)$ to the fixed line $\xi = 1$ at the expense of making the equation more complicated. In the first equation of (2.4), the coefficients contain the unknown $s(t)$.

Now we denote $s^* = -\mu v_0'(1)$ and set

$$\begin{gathered} S_T = \left\{ s \in C^1[0,T] : s(0) = s_0, s'(0)s_0 = s^*, 0 \leqslant s'(t)s(t) \leqslant s^* + 1 \right\}, \\ U_T = \left\{ v \in C([0,1] \times [0,T]) : v(\xi,0) = v_0(\xi), \|v - v_0\|_{C([0,1]\times[0,T])} \leqslant 1 \right\}. \end{gathered} \tag{2.5}$$

It is easy to see that $\Sigma_T := U_T \times S_T$ is a complete metric space with the metric

$$\mathfrak{D}((v_1, s_1), (v_2, s_2)) = \|v_1 - v_2\|_{C([0,1]\times[0,T])} + \left\|s_1' s_1 - s_2' s_2\right\|_{C([0,T])}. \tag{2.6}$$

Next applying standard L^p theory and the Sobolev imbedding theorem (see [29]), we then find that for any $(v, s) \in \Sigma_T$, the following initial boundary value problem:

$$\begin{gathered}
\widetilde{v}_t - \frac{s'(t)}{s(t)}\xi\widetilde{v}_\xi - \frac{d}{s^2(t)}\widetilde{v}_{\xi\xi} = Kav^2 - bv, \quad 0 < \xi < 1,\ 0 < t < T, \\
\widetilde{v}(1, t) = 0, \quad 0 < t < T, \\
\widetilde{v}_\xi(0, t) = 0, \quad 0 < t < T, \\
\widetilde{v}(\xi, 0) = v_0(\xi) \geqslant 0, \quad 0 \leqslant \xi \leqslant 1
\end{gathered} \tag{2.7}$$

admits a unique solution $\widetilde{v} \in C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T])$ and

$$\|\widetilde{v}\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T])} \leqslant C\|\widetilde{v}\|_{W^{2,1,p}([0,1]\times[0,T])} \leqslant C_1, \tag{2.8}$$

where $p = 3/(1-\alpha)$, C_1 is a constant dependent on α, s_0 and $\|u_0\|_{C^{1+\alpha}[0,s_0]}$.

Defining $\widetilde{s}$ by using the last equation of (2.4)

$$\widetilde{s}^2(t) = s_0^2 - 2\int_0^t \mu\widetilde{v}_\xi(1, \tau)\mathrm{d}\tau, \tag{2.9}$$

we have

$$\widetilde{s}'(t)\widetilde{s}(t) = -\mu\widetilde{v}_\xi(1, t), \qquad \widetilde{s}(0) = s_0, \qquad \widetilde{s}'(0)s_0 = -\mu\widetilde{v}_\xi(1, 0) = s^*, \tag{2.10}$$

and hence $\widetilde{s}' \in C^{\alpha/2}([0,T])$ with

$$\left\|\widetilde{s}'(t)\widetilde{s}(t)\right\|_{C^{\alpha/2}([0,T])} \leqslant C_2 := \mu C_1. \tag{2.11}$$

Define map $\mathcal{F}: \Sigma_T \to C([0,1]\times[0,T]) \times C^1[0,T]$ by

$$\mathcal{F}((v(\xi, t), s(t))) = (\widetilde{v}(\xi, t), \widetilde{s}(t)). \tag{2.12}$$

It is clear that $(v, s) \in \Sigma_T$ is a fixed point of $\mathcal{F}$ if and only if it solves (2.4).

By (2.10) and (2.11), we have

$$\begin{gathered}
\left\|\widetilde{s}'(t)s(t) - \widetilde{s}'(0)s(0)\right\|_{C[0,T]} \leqslant \left\|\widetilde{s}'(t)s(t)\right\|_{C^{\alpha/2}[0,T]} T^{\alpha/2} \leqslant C_2 T^{\alpha/2}, \\
\|\widetilde{v}(\xi, t) - v_0(\xi)\|_{C([0,1]\times[0,T])} \leqslant \|\widetilde{v} - v_0\|_{C^{0,(1+\alpha)/2}([0,1]\times[0,T])} T^{(1+\alpha)/2} \leqslant C_1 T^{(1+\alpha)/2}.
\end{gathered} \tag{2.13}$$

Therefore if we take $T \leqslant \min\{(C_2)^{-2/\alpha}, C_1^{-2/(1+\alpha)}\}$, then $\mathcal{F}$ maps Σ_T into itself.

To prove that $\mathcal{F}$ is a contraction mapping on Σ_T for $T > 0$ sufficiently small, we take $(v_i, s_i) \in \Sigma_T$ $(i = 1, 2)$ and denote $(\overline{v}_i, \overline{s}_i) = \mathcal{F}(v_i, s_i)$. Then it follows form (2.8) and (2.11) that

$$\|\overline{v}_i\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T])} \leqslant C\|\overline{v}_i\|_{W^{2,1,p}([0,1]\times[0,T])} \leqslant C_1, \qquad \left\|\overline{s}_i'(t)\overline{s}_i(t)\right\|_{C^{\alpha/2}[0,T]} \leqslant C_2. \tag{2.14}$$

By setting $V = \overline{v}_1 - \overline{v}_2$ it follows that $V(\xi, t)$ satisfies

$$\begin{gathered} V_t - \frac{s_1'}{s_1}\xi V_\xi - \frac{d}{s_1^2}V_{\xi\xi} - Ka(\overline{v}_1 + \overline{v}_2)V + bV \\ = \left(\frac{s_1'}{s_1} - \frac{s_2'}{s_2}\right)\xi\overline{v}_{2,\xi} + \left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)d\overline{v}_{2,\xi\xi}, \quad 0 < \xi < 1,\ 0 < t < T, \\ V(1,t) = 0, \qquad V_\xi(0,t) = 0, \quad 0 < t < T, \\ V(\xi, 0) = 0, \quad 0 \leqslant \xi \leqslant 1. \end{gathered} \tag{2.15}$$

Using the $W^{2,1,p}$ estimates for parabolic equations and Sobolev's imbedding yields

$$\|\overline{v}_1 - \overline{v}_2\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T])} \leqslant C_3\Big(\|v_1 - v_2\|_{C([0,1]\times[0,T])} + \|s_1 - s_2\|_{C^1[0,T]}\Big), \tag{2.16}$$

where C_3 is independent of T. Taking the difference of the equations for $\overline{s}_1'\overline{s}_1$, $\overline{s}_2'\overline{s}_2$ results in

$$\left\|\overline{s}_1'(t)\overline{s}_1(t) - \overline{s}_2'(t)\overline{s}_2(t)\right\|_{C^{\alpha/2}[0,T]} \leqslant \mu\left\|\overline{v}_{1,\xi}(\xi,t) - \overline{v}_{2,\xi}(\xi,t)\right\|_{C^{0,\alpha/2}([0,1]\times[0,T])}. \tag{2.17}$$

Combining inequalities (2.16) and (2.17), we obtain

$$\begin{aligned} &\|\overline{v}_1(\xi,t) - \overline{v}_2(\xi,t)\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T])} + \left\|\overline{s}_1'(t)\overline{s}_1(t) - \overline{s}_2'(t)\overline{s}_2(t)\right\|_{C^{\alpha/2}[0,T]} \\ &\quad \leqslant C_4\Big(\|v_1(\xi,t) - v_2(\xi,t)\|_{C([0,1]\times[0,T])} + \|s_1(t) - s_2(t)\|_{C^1[0,T]}\Big), \end{aligned} \tag{2.18}$$

where C_4 is independent of T. Using the property of norm

$$\left\|s_1'(t)s_1(t) - s_2'(t)s_2(t)\right\| \geqslant \left\|\left(s_1'(t) - s_2'(t)\right)s_1(t)\right\| - \left\|s_2'(t)\right\|\|s_1(t) - s_2(t)\|, \tag{2.19}$$

and the fact $s_1(0) = s_2(0), s_i(t) \geqslant s_0$ and $\overline{s}_2'(t) \leqslant (s^* + 1)/s_0$ give that

$$\begin{aligned} \left\|s_1' - s_2'\right\|_{C[0,T]} &\leqslant \frac{1}{s_0}\left\|s_1's_1 - s_2's_2\right\|_{C[0,T]} + \frac{s^* + 1}{s_0^2}T\left\|s_1' - s_2'\right\|_{C[0,T]} \\ &\leqslant \frac{2}{s_0}\left\|s_1'(t)s_1(t) - s_2'(t)s_2(t)\right\|_{C[0,T]} \end{aligned} \tag{2.20}$$

if $T \leqslant s_0^2/(2s^* + 2)$. Further,

$$\|s_1 - s_2\|_{C^1[0,T]} \leqslant (1+T)\left\|s_1' - s_2'\right\|_{C[0,T]} \leqslant \frac{2+2T}{s_0}\left\|s_1's_1 - s_2's_2\right\|_{C[0,T]}. \tag{2.21}$$

Hence, for

$$T := \min\left\{1, C_2^{-2/\alpha}, (C_1)^{-2/(1+\alpha)}, \frac{s_0^2}{2s^*+2}, \left[2C_4\left(1+\frac{4}{s_0}\right)\right]^{-2/\alpha}\right\}, \tag{2.22}$$

we have

$$\begin{aligned}
&\|\overline{v}_1(\xi,t) - \overline{v}_2(\xi,t)\|_{C([0,1]\times[0,T])} + \overline{s}_1'\overline{s}_1(t) - \overline{s}_2'(t)\overline{s}_2(t)_{C[0,T]} \\
&\quad\leqslant T^{(1+\alpha)/2}\|\overline{v}_1 - \overline{v}_2\|_{C^{(1+\alpha)/2,\ 1+\alpha}([0,1]\times[0,T])} + T^{\alpha/2}\left\|\overline{s}_1'\overline{s}_1(t) - \overline{s}_2'(t)\overline{s}_2(t)\right\|_{C^{\alpha/2}[0,T]} \\
&\quad\leqslant C_4T^{\alpha/2}\Big(\|v_1 - v_2\|_{C([0,1]\times[0,T])} + \|s_1 - s_2\|_{C^1[0,T]}\Big) \quad (2.18) \\
&\quad\leqslant C_4T^{\alpha/2}\left(\|v_1 - v_2\|_{C([0,1]\times[0,T])} + \frac{2+2T}{s_0}\left\|s_1's_1 - s_2's_2\right\|_{C[0,T]}\right) \quad (2.21) \\
&\quad\leqslant \frac{1}{2}\Big(\|v_1 - v_2\|_{C([0,1]\times[0,T])} + \left\|s_1's_1 - s_2's_2\right\|_{C[0,T]}\Big).
\end{aligned} \tag{2.23}$$

Thus for this T, $\mathcal{F}$ is a contraction. Now using the contraction mapping theorem gives the conclusion that there is a $(v(\xi,t), s(t))$ in Σ_T such that $\mathcal{F}(v(\xi,t), s(t)) = (v(\xi,t), s(t))$. In other words, $(v(\xi,t), s(t))$ is the solution of the problem (2.4) and therefore $(u(x,t), s(t))$ is the solution of the problem (1.2). Moreover, by using the Schauder estimates, we have additional regularity of the solution, $s(t) \in C^{1+\alpha/2}[0,T]$ and $u \in C^{2+\alpha,1+\alpha/2}((0,s(t)) \times (0,T])$. Thus $(u(x,t), s(t))$ is the classical solution of the problem (1.2). □

Now we give the monotone behavior of the free boundary $s(t)$.

Theorem 2.2. *The free boundary for the problem* (1.2) *is strictly monotone increasing, that is, for any solution in* $(0,T]$, *one has*

$$s'(t) > 0 \quad \text{for } 0 < t \leqslant T. \tag{2.24}$$

Proof. Using the Hopf lemma to the equation (1.2) yields that

$$u_x(s(t),t) < 0 \quad \text{for } 0 < t \leqslant T. \tag{2.25}$$

Thus, combining this inequality with the Stefan condition gives the result. □

Remark 2.3. If the initial function u_0 is smooth and satisfies the consistency condition:

$$-du_0''(s_0) + \mu u_0'(s_0)u_0'(s_0) = u_0(s_0)(Kau_0(s_0) - b), \tag{2.26}$$

then the solution $(u, s) \in C^{2+\alpha,\ 1+\alpha/2}([0, s(t)] \times [0, T]) \times C^{1+(1+\alpha)/2}([0, T])$.

3. Finite Time Blowup

In this section we discuss the blowup behavior. First we present the following lemma.

Lemma 3.1. *The solution of the problem* (1.2) *exists and is unique, and it can be extended to* $[0, T_{\max})$ *where* $T_{\max} \leqslant \infty$. *Moreover, if* $T_{\max} < \infty$, *one has*

$$\limsup_{t \to T_{\max}} \|u(x,t)\|_{L^\infty([0,s(t)]\times[0,t])} = \infty. \tag{3.1}$$

Proof. It follows from the uniqueness and Zorn's lemma that there is a number $T_{\max}$ such that $[0, T_{\max})$ is the maximal time interval in which the solution exists. In order to prove the present lemma, it suffices to show that, when $T_{\max} < \infty$,

$$\limsup_{t \to T_{\max}} \|u\|_{L^\infty([0,s(t)]\times[0,t])} = \infty. \tag{3.2}$$

In what follows we use the contradiction argument. Assume that $T_{\max} < \infty$ and $\|u\|_{L^\infty([0,s(t)]\times[0,t])} < \infty$. Since $s'(t)$ is bounded in $[0, T_{\max})$ by Theorem 2.1, using a bootstrap argument and Schauder's estimate yields a priori bound of $\|u(t,x)\|_{C^{1+\alpha}[0,s(t)]}$ for all $t \in [0, T_{\max})$. Let the bound be M^*. It follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending only on M^* such that the solution of the problem (1.2) with the initial time $T_{\max} - \tau/2$ can be extended uniquely to the time $T_{\max} - \tau/2 + \tau$, which contradicts the assumption. This completes the proof. □

In order to investigate the behavior of the free boundary, we introduce the energy of the solution u at t by

$$E(t) = \int_0^{s(t)} \left(\frac{d}{2}(u_x)^2 - \frac{Ka}{3}u^3 + \frac{b}{2}u^2\right)(x,t)\mathrm{d}x, \tag{3.3}$$

and its L^1-norm by $|u(t)|_1 = \int_0^{s(t)} u(x,t)\mathrm{d}x$. Then we have the following lemma.

Lemma 3.2. *Let u be the solution of the problem* (1.2). *Then one has the relations*

$$\frac{dE}{dt} = -\int_0^{s(t)} u_t^2(x,t)dx - \frac{d}{2\mu^2}s'^3(t), \tag{3.4}$$

$$|u(t)|_1 - |u_0|_1 = \frac{d}{\mu}(s_0 - s(t)) + \int_0^t\int_0^{s(\tau)} \left(Kau^2 - bu\right)(x,\tau)dx\,d\tau. \tag{3.5}$$

Proof. It is easy to see that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_0^{s(t)} \left(du_x u_{xt} - Kau^2 u_t + buu_t\right)(x,t)\mathrm{d}x + s'(t) \\ &\quad\times \left[\frac{d}{2}(u_x)^2(s(t),t) - \frac{Ka}{3}u^3(s(t),t) + \frac{b}{2}u^2(s(t),t)\right].\end{aligned} \tag{3.6}$$

Integrating by parts and using $u_x(0,t) = 0$ yield

$$\int_0^{s(t)} u_x u_{xt}\mathrm{d}x = -\int_0^{s(t)} u_{xx}u_t dx + u_x u_t(s(t),t). \tag{3.7}$$

Differentiating the second equation of (1.2) with respect to t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(s(t),t)) = s'(t)u_x(s(t),t) + u_t(s(t),t) = 0, \tag{3.8}$$

which implies that

$$u_x u_t(s(t),t) = -s'(t)u_x^2(s(t),t) = -\frac{s'^3(t)}{\mu^2}. \tag{3.9}$$

By substitution, we get

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= -\int_0^{s(t)} \left(du_{xx}u_t + Kau^2 u_t - buu_t\right)(x,t)\mathrm{d}x - \frac{d}{\mu^2}s'^3(t) + \frac{d}{2\mu^2}s'^3(t) \\ &= -\int_0^{s(t)} u_t^2(x,t)\mathrm{d}x - \frac{d}{2\mu^2}s'^3(t),\end{aligned} \tag{3.10}$$

that is (3.4).

Now we show (3.5). It is obvious that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{s(t)} u(x,t)\mathrm{d}x &= \int_0^{s(t)} u_t(x,t)\mathrm{d}x + s'(t)u(s(t),t) \\ &= \int_0^{s(t)} du_{xx}(x,t)\mathrm{d}x + \int_0^{s(t)} \left(Kau^2 - bu\right)(x,t)\mathrm{d}x \\ &= -\frac{d}{\mu}s'(t) + \int_0^{s(t)} \left(Kau^2 - bu\right)(x,t)\mathrm{d}x.\end{aligned} \tag{3.11}$$

Integrating the equation above, we get

$$|u(t)|_1 - |u_0|_1 = \frac{d}{\mu}(s_0 - s(t)) + \int_0^t \int_0^{s(t)} \left(Kau^2 - bu\right)(x,\tau)\mathrm{d}x\,\mathrm{d}\tau. \tag{3.12}$$

This completes the proof. □

Lemma 3.3. *Assume $T_{\max} = \infty$, and let $A = \int_0^\infty s'^3(t)dt$. If $b < \pi d^3/8(2ds_0 + \mu|u_0|_1)^2$, then one has $A \geqslant \mu^3 d^3 \pi^2 |u_0|_1^3/64(2ds_0 + \mu|u_0|_1)^4$.*

Proof. We see the following auxiliary free boundary problem:

$$\begin{gathered}
v_t - dv_{xx} = -bv, \quad 0 < x < h(t),\ 0 < t < \infty, \\
v(x,0) = u_0(x), \quad 0 \leqslant x \leqslant s_0,\ h(0) = s_0, \\
v(h(t),t) = v_x(0,t) = 0, \quad 0 < t < \infty, \\
h'(t) = -\mu v_x(h(t),t), \quad 0 < t < \infty.
\end{gathered} \tag{3.13}$$

By the same argument as in Theorem 2.1, the solution of the above problem exists for all $t > 0$ since the solution is bounded. Moreover, one can deduce from the maximum principle that $u \geqslant v \geqslant 0$ and $s(t) \geqslant h(t) \geqslant s_0$ on $(0, T_{\max})$. Similarly as in Lemma 3.2, denoting $|v(t)|_1 = \int_0^{h(t)} v(t,x)\mathrm{d}x$, we easily obtains

$$h(t) - s_0 = \frac{\mu}{d}(|u_0|_1 - |v(t)|_1) - \frac{\mu b}{d}\int_0^t |v(\tau)|_1 \mathrm{d}\tau. \tag{3.14}$$

Using Hölder's inequality and the fact that $s(t) \geqslant h(t) \geqslant s_0$ yields that for all $t \geqslant 0$,

$$\int_0^t s'(\tau)\mathrm{d}\tau \leqslant \left(\int_0^t (s'(\tau))^3 \mathrm{d}\tau\right)^{1/3} \left(\int_0^t \mathrm{d}\tau\right)^{2/3} \leqslant A^{1/3} t^{2/3}, \tag{3.15}$$

so we have

$$\begin{aligned}
A &\geqslant t^{-2}\left(\int_0^t s'(\tau)\mathrm{d}\tau\right)^3 = t^{-2}(s(t) - s_0)^3 \\
&\geqslant t^{-2}\left(\frac{\mu}{d}(|u_0|_1 - |v(t)|_1) - \frac{\mu b}{d}\int_0^t |v(\tau)|_1 \mathrm{d}\tau\right)^3.
\end{aligned} \tag{3.16}$$

On the other hand, by the maximum principle, we have $v \leqslant w$, where w is the solution of the following Cauchy problem:

$$
\begin{aligned}
& w_t - dw_{xx} = 0, \quad -\infty < x < \infty, \ t > 0, \\
& w(x,0) = \overline{u}_0(x) = \begin{cases} u_0(|x|), & -s_0 \leqslant x \leqslant s_0, \\ 0, & x \in \dfrac{\mathbb{R}}{[-s_0, s_0]}. \end{cases}
\end{aligned}
\tag{3.17}
$$

By the $L^1 - L^\infty$ estimate for the heat equation, we have

$$
\|v(t)\|_\infty \leqslant \|w(t)\|_\infty \leqslant (4d\pi t)^{-1/2} |w(0)|_1 = (d\pi t)^{-1/2} |u_0|_1, \tag{3.18}
$$

hence, by (3.14),

$$
\begin{aligned}
|v(t)|_1 &\leqslant h(t)\|v(t)\|_\infty \leqslant h(t)(d\pi t)^{-1/2}|u_0|_1 \\
&= \left(s_0 + \frac{\mu}{d}|u_0|_1 - \frac{\mu}{d}|v(t)|_1 - \frac{\mu b}{d}\int_0^t |v(t)|_1 d\tau \right)(d\pi t)^{-1/2}|u_0|_1 \\
&\leqslant \left(s_0 + \frac{\mu}{d}|u_0|_1 - \frac{\mu}{d}|v(t)|_1 \right)(d\pi t)^{-1/2}|u_0|_1.
\end{aligned}
\tag{3.19}
$$

Therefore, we have $|v(t_0)|_1 \leqslant |u_0|_1/2$ for $t_0 = (d\pi)^{-1}(2s_0 + (\mu/d)|u_0|_1)^2$. If $b < \pi d^3/8(2ds_0 + \mu|u_0|_1)^2$, we get $b\int_0^{t_0} |v(\tau)|_1 \mathrm{d}\tau \leqslant (1/4)|u_0|_1$. Taking $t = t_0$ in the inequality (3.16) yields the desired estimate. □

Theorem 3.4. *Let u be the solution of the problem* (1.2), *if $b < \pi d^3/8(2ds_0 + \mu|u_0|_1)^2$, then one has $T_{\max} < \infty$ whenever*

$$
E(0) < \frac{\mu d^4 \pi^2 |u_0|_1^3}{128(2ds_0 + \mu|u_0|_1)^4}. \tag{3.20}
$$

Proof. As in [28], define the function

$$
F(t) = \int_0^t \int_0^{s(\tau)} u^2(x,\tau)\mathrm{d}x\,\mathrm{d}\tau. \tag{3.21}
$$

Direct calculations show that $F'(t) = \int_0^{s(t)} u^2(x,t)dx$ and

$$\begin{aligned} F''(t) &= \int_0^{s(t)} 2uu_t(x,t)\mathrm{d}x + s'(t)u^2(s(t),t) = \int_0^{s(t)} 2uu_t(x,t)\mathrm{d}x \\ &= 2\int_0^{s(t)} \left(duu_{xx} + u^2(Kau-b)\right)(x,t)\mathrm{d}x \\ &= 2\int_0^{s(t)} \left(u^2(Kau-b) - du_x^2\right)(x,t)\mathrm{d}x + 2duu_x(x,t)\Big|_0^{s(t)} \\ &= -6E(t) + \int_0^{s(t)} \left(du_x^2 + bu^2\right)(x,t)\mathrm{d}x. \end{aligned} \tag{3.22}$$

It follows from the identity (3.4) that

$$\begin{aligned} F''(t) = 6\int_0^t\int_0^{s(\tau)} u_t^2(x,\tau)\mathrm{d}x\,\mathrm{d}\tau + \frac{3d}{\mu^2}\int_0^t s'^3(\tau)d\tau - 6E(0) \\ + \int_0^{s(t)} \left(du_x^2 + bu^2\right)(x,t)\mathrm{d}x. \end{aligned} \tag{3.23}$$

Now assume $T_{\max} = \infty$ by contradiction. The assumption (3.20), together with Lemma 3.3, implies that

$$E(0) < \frac{d}{2\mu^2}\int_0^t s'^3(\tau)\mathrm{d}\tau \tag{3.24}$$

for all $t \geqslant t_0$ sufficiently large, we then have

$$F''(t) > 6\int_0^t\int_0^{s(\tau)} u_t^2(x,\tau)\mathrm{d}x\,\mathrm{d}\tau, \quad t \geqslant t_0. \tag{3.25}$$

Applying the Cauchy-Schwarz inequality yields

$$\begin{aligned} F(t)F''(t) &\geqslant 6\int_0^t\int_0^{s(\tau)} u^2\mathrm{d}x\,\mathrm{d}\tau \int_0^t\int_0^{s(\tau)} u_t^2\mathrm{d}x\,\mathrm{d}\tau \\ &\geqslant 6\left(\int_0^t\int_0^{s(\tau)} uu_t\mathrm{d}x\,\mathrm{d}\tau\right)^2 = \frac{3}{2}\left(F'(t) - F'(0)\right)^2 \end{aligned} \tag{3.26}$$

since $F''(\tau) = \int_0^{s(\tau)} 2uu_t(x,\tau)\mathrm{d}x$ by (3.22).

On the other hand, (3.25) implies that

$$F'(t) \geqslant F'(t_0+1) = \int_0^{s(t_0+1)} u^2(x,t_0+1)\mathrm{d}x > 0, \quad t \geqslant t_0+1, \tag{3.27}$$

so that $\lim_{t\to\infty} F(t) = \infty$. We then obtain

$$F(t)F''(t) \geqslant \frac{5}{4}F'^2(t), \quad t \geqslant t_1 \tag{3.28}$$

for some large $t_1 > t_0 + 1$.

Defining $G(t) = F^{-1/4}(t)$ for $t \geqslant t_1$, it follows that

$$\begin{aligned} G'(t) &= -\frac{1}{4}F'(t)F^{-5/4}(t) < 0, \quad t \geqslant t_1, \\ G''(t) &= -\frac{1}{4}F^{-9/4}\left(FF'' - \frac{5}{4}F'^2(t)\right) \leqslant 0, \quad t \geqslant t_1. \end{aligned} \tag{3.29}$$

This implies that G is concave, decreasing, and positive for $t \geqslant t_1$, which is impossible. The contradiction shows that $T_{\max} < \infty$, which gives the blowup result. □

Remark 3.5. The above theorem shows that the solution of the free boundary problem (1.2) blows up if the death rate (b) is sufficiently small and the initial datum (u_0) is sufficiently large.

4. Global Fast Solution and Slow Solution

In this section, we study the asymptotic behavior of the global solutions of (1.2). We first give the following existence of fast solution.

Theorem 4.1 (fast solution). *Let u be a solution of problem* (1.2). *If u_0 is small in the following sense:*

$$\|u_0\|_\infty \leqslant \frac{d}{32}\min\left\{\frac{1}{Kas_0^2}, \frac{1}{\mu}\right\}, \tag{4.1}$$

then $T_{\max} = \infty$. Moreover, $s_\infty < \infty$ and there exist real numbers $C, \beta > 0$ depending on u_0 such that

$$\|u(t)\|_\infty \leqslant Ce^{-\beta t}, \quad t \geqslant 0. \tag{4.2}$$

Proof. It suffices to construct a suitable global supersolution. Inspired by [30], we define

$$\begin{aligned} \vartheta(t) &= 2s_0\left(2 - e^{-\gamma t}\right), \quad t \geqslant 0, \qquad V(y) = 1 - y^2, \quad 0 \leqslant y \leqslant 1, \\ v(x,t) &= \varepsilon e^{-\beta t} V\left(\frac{x}{\vartheta(t)}\right), \quad 0 \leqslant x \leqslant \vartheta(t),\ t \geqslant 0, \end{aligned} \tag{4.3}$$

where $\gamma,\ \beta$ and $\varepsilon > 0$ to be chosen later.

An easy computation yields

$$\begin{aligned} v_t - dv_{xx} - v(Kav - b) &= \varepsilon e^{-\beta t}\left[-\beta V - x\vartheta'\vartheta^{-2}V' - d\vartheta^{-2}V'' - V\left(Ka\varepsilon e^{-\beta t}V - b\right)\right] \\ &\geqslant \varepsilon e^{-\beta t}V\left[-\beta + \frac{d}{8s_0^2} - Ka\varepsilon\right], \end{aligned} \tag{4.4}$$

for all $t > 0$ and $0 < x < \vartheta(t)$.

On the other hand, we have $\vartheta'(t) = 2\gamma s_0 e^{-\gamma t} > 0$ and $-v_x(\vartheta(t), t) = 2\varepsilon e^{-\beta t}/\vartheta(t)$. Setting $\gamma = \beta = d/16s_0^2$, and $\varepsilon \leqslant \varepsilon_0 = \min\{d/16Kas_0^2, d/16\mu\}$, it follows that

$$\begin{gathered} v_t - dv_{xx} - v(Kav - b) \geqslant 0, \quad 0 < x < \vartheta(t),\ t > 0, \\ \vartheta'(t) > -\mu v_x(\vartheta(t), t), \quad t > 0, \\ v(\vartheta(t), t) = v_x(0, \mathrm{t}) = 0, \quad t > 0, \\ \vartheta(0) = 2s_0 > s_0. \end{gathered} \tag{4.5}$$

Assume that $\|u_0\|_\infty \leqslant \min\{d/32Kas_0^2,\ d/32\mu\}$ and choose $\varepsilon = 2\|u_0\|_\infty$, we also get $u_0(x) < v(x, 0)$ for $0 \leqslant x \leqslant s_0$.

By using the maximum principle, one then sees that $s(t) < \vartheta(t)$ and $u(x,t) < v(x,t)$ for $0 \leqslant x \leqslant s(t)$, as long as u exists. In particular, it follows from the continuation property (3.1) that u exists globally. The proof is complete. □

Remark 4.2. The above result shows that the free boundary converges to a finite limit and that the solution $u(t)$ decays uniformly to 0 at an exponential rate. Compared to the case (see Theorem 4.5), the free boundary grows up to infinity and the decay rate of the solution is at most polynomial, the former solution is therefore called fast solution.

Before we give the existence result of slow solution, we need the following uniform a priori estimate for all global solutions of problem (1.2).

Proposition 4.3. *Let u be a solution of the problem* (1.2) *with $T_{\max} = \infty$. Then there is a constant $C = C(\|u_0\|_{C^{1+\alpha}}, s_0, 1/s_0) > 0$, such that*

$$\sup_{t\geqslant 0}\|u(x,t)\|_{L^\infty(0,s(t))} \leqslant C, \tag{4.6}$$

where C remains bounded for $\|u_0\|_{C^{1+\alpha}}$, s_0, and $1/s_0$ bounded.

Proof. First from the local theory for problem (1.2), for each $M > 1$ there exists $\sigma > 0$ such that, if $\|u_0\|_{C^{1+\alpha}} < M$ and $1/M < s_0 < M$, then $\|u(x,t)\|_{L^\infty} < 2M$ on $[0, \sigma]$.

Assume that the result is false. Then there exists a $M > 0$ and a sequence of global solutions (u_n, s_n) of (1.2), such that

$$\frac{1}{M} < s_n(0) < M, \qquad \|u_n(x,0)\|_{C^{1+\alpha}[0,s_0]} < M,$$
$$\sup_{t\geqslant 0}\|u_n(x,t)\|_{L^\infty(0,s_n(t))} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{4.7}$$

For all large n there exist $t_n \geqslant \sigma$ and $x_n \in [0, s_n(t_n))$ such that

$$\sup_{t\geqslant 0}\|u_n(x,t)\|_{L^\infty(0,s_n(t))} = u_n(x_n,t_n) \triangleq \varrho_n. \tag{4.8}$$

We define $\lambda_n = \varrho_n^{-1/2}$, then it is evident to see that $\lambda_n \to 0$ as $n \to \infty$. We extend $u_n(\cdot,t)$ by 0 on $(s_n(t), \infty)$ and define the rescaled function

$$v_n(y,\tau) = \lambda_n^2 u_n\left(x_n + \lambda_n y, t_n + \lambda_n^2 \tau\right), \tag{4.9}$$

for $(y,\tau) \in \widetilde{D}_n = \{(y,\tau) : -\lambda_n^{-1}x_n \leqslant y < \infty \text{ and } -\lambda_n^{-2}t_n \leqslant \tau \leqslant 0\}$. Also, we denote

$$y_1 = -\lambda_n^{-1}x_n, \qquad y_2(\tau) = \lambda_n^{-1}\left(s\left(t_n + \lambda_n^2\tau\right) - x_n\right),$$
$$D_n = \left\{(y,\tau) : \ y_1 \leqslant y < y_2(\tau) \text{ and} - \lambda_n^{-2}t_n \leqslant \tau \leqslant 0\right\}, \tag{4.10}$$

which corresponds to the domain $\{x < s(t)\}$. The function v_n satisfies $v_n(0,0) = 1,\ 0 \leqslant v_n \leqslant 1$ and

$$\partial_\tau v_n - d\partial_y^2 v_n = v_n(Kav_n - b_n), \quad (y,\tau) \in D_n, \tag{4.11}$$

where $b_n = \lambda_n^2 b$. Note that $b_n \to 0$ as $n \to +\infty$. Similarly as Lemma 2.1 in [27], there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\{v_{n_k}\}$ converges in $L^p_{\text{loc}}([0,+\infty) \times (-\infty, 0])$ to a function $w(y,t) \in L^p_{\text{loc}}([0,+\infty)\times(-\infty,0])$ and $\{v_{n_k}(y,0)\}$ converges in $C_{\text{loc}}([0,+\infty))$ to a function $z(y) \in C([0,+\infty))$, which satisfies $z(0) = 1$. Moreover, similarly as Lemmas 2.2 and 2.3 in [27], $w_t = 0$ in $D'((0,+\infty) \times (-\infty,0))$ and there is a function $w(y) \geqslant 0$ which is bounded, continuous on $[0,\infty)$ and satisfies that $-w_{yy} = Kaw^2$, hence w is concave. Therefore $w \equiv 0$, which leads to a contradiction to the fact that $w(0) = z(0) = 1$. This completes the proof of Proposition 4.3. □

The above proposition shows that all global solutions are uniformly bounded and the coming result implies that all global solutions decay uniformly to 0.

Proposition 4.4. *Let u be a solution of the problem* (1.2) *with $T_{\max} = \infty$. Then it holds that*

$$\lim_{t\to+\infty}\|u(x,t)\|_{L^\infty(0,s(t))} = 0. \tag{4.12}$$

Proof. Assume that $l := \limsup_{t\to+\infty}\|u(x,t)\|_{L^\infty(0,s(t))} > 0$ by contradiction. It follows from Proposition 4.3 that $l < +\infty$. Let $t_0 > 0$ be such that $\sup_{[t_0,+\infty)}\|u(x,t)\|_{L^\infty(0,s(t))} \leqslant (3/2)l$. Then there exists a sequence $t_n \to +\infty$ such that $\|u(x,t_n)\|_{L^\infty(0,s(t_n))} \geqslant (3/4)l$.

Now pick $x_n \in [0, s_n(t_n))$ such that

$$\|u(x,t_n)\|_{L^\infty(0,s(t_n))} = u(x_n,t_n). \tag{4.13}$$

As above, we define $\lambda_n = u^{-1/2}(x_n,t_n)$ and then $0 < \lambda_n \leqslant (3l/4)^{-1/2}$. We extend $u(\cdot,t)$ by 0 on $(s(t),\infty)$ and define the rescaled function

$$v_n(y,\tau) = \lambda_n^2 u\left(x_n + \lambda_n y, t_n + \lambda_n^2\tau\right) \tag{4.14}$$

for $(y,\tau) \in \widetilde{D}_n = \{(y,\tau) : -\lambda_n^{-1}x_n \leqslant y < \infty \text{ and } \lambda_n^{-2}(t_0 - t_n) \leqslant \tau \leqslant 0\}$. Also, we denote

$$\begin{gathered} y_1 = -\lambda_n^{-1}x_n, \qquad y_2(\tau) = \lambda_n^{-1}\left(s\left(t_n + \lambda_n^2\tau\right) - x_n\right), \\ D_n = \left\{(y,\tau) : y_1 \leqslant y < y_2(\tau) \text{ and } \lambda_n^{-2}(t_0 - t_n) \leqslant \tau \leqslant 0\right\}. \end{gathered} \tag{4.15}$$

Therefore the function v_n satisfies $v_n(0,0) = 1,\ 0 \leqslant v_n \leqslant 2, \lim_{y\to+\infty}v_n(y,\tau) = 0$ and

$$\partial_\tau v_n - d\partial_y^2 v_n = v_n(Kav_n - b_n), \quad (y,\tau) \in D_n, \tag{4.16}$$

where $b_n = \lambda_n^2 b$. Note that $b_n \leqslant b(3l/4)^{-1}$, therefore there exists a subsequence $\{b_{n_k}\}$ and $b^* \leqslant b(3l/4)^{-1}$ such that $b_{n_k} \to b^*$ as $k \to +\infty$. Similarly as Lemmas 2.1–2.3 in [27], we have obtained a function $w(y) \geqslant 0$, bounded and continuous on $[0,\infty)$ and satisfies that $-w_{yy} = Kaw^2 - b^*w$. Therefore $w \equiv 0$ or $w \equiv b^*/Ka$. If $w \equiv 0$, this is a contradiction to the fact that $w(0) = 1$ since $v_n(0,0) = 1$. If $w \equiv b^*/Ka$, this is also a contradiction to the fact that $\lim_{y\to+\infty}w(y) = 0$. □

Theorem 4.5 (slow solution). *Let $\phi \in C^1([0,s_0])$ satisfy $\phi \geqslant 0$, $\phi \not\equiv 0$ with $\phi_x(0) = \phi(s_0) = 0$, and b satisfy the same condition as in Theorem 3.4. Then there exists $\lambda > 0$ such that the solution of* (1.2) *with initial data $u_0 = \lambda\phi$ is a global slow solution, which satisfies that $s_\infty = \infty$ and*

$$s(t) = O\left(t^{2/3}\right) \quad \text{as } t \longrightarrow \infty. \tag{4.17}$$

Proof. Denote the solution of (1.2) as $u(u_0;\cdot)$ to emphasize the dependence on the initial function u_0 when necessary. So as to $s(t),\ s_\infty$ and the maximal existence time $T_{\max}$.

Similarly as in [27], define

$$\Sigma = \left\{\lambda > 0; T_{\max}(\lambda\phi) = \infty \text{ and } s_\infty(\lambda\phi) < \infty\right\}. \tag{4.18}$$

According to the Theorem 4.1, $\Sigma \neq \emptyset$ since that the solution is global if λ is sufficiently small.

If $b < \pi d^3/8(2ds_0 + \mu|u_0|_1)^2$, then for λ sufficiently large, we have $E(\lambda\phi) < 0$, which implies $T_{\max}(\lambda\phi) < \infty$. Therefore Σ is bounded.

Let $\lambda^* = \sup \Sigma \in (0, \infty)$, $v(x,t) = u(\lambda^*\phi; x, t), \sigma(t) = s(\lambda^*\phi; t)$ and $\tau = T_{\max}(\lambda^*\phi)$.

We first claim that $\tau = \infty$. In fact, by continuous dependence, when $\lambda \to \lambda^*$, we know that $u(\lambda\phi; x, t)$ approaches $v(x,t)$ in $L^\infty(0,\infty)$ and $s(\lambda\phi; t) \to \sigma(t)$ for each fixed $t \in [0,\tau)$. Since $T_{\max}(\lambda\phi) = \infty$ for all $\lambda \in (0, \lambda^*)$, it follows from Proposition 4.3 that $\|v(x,t)\|_{L^\infty(0,\sigma(t))} \leqslant C$ for all $t \in [0,\tau)$. Therefore, $\tau = \infty$ since nonglobal solutions must satisfy $\limsup_{t\to T}\|u(x,t)\|_{L^\infty(0,s(t))} = \infty$.

Next we claim that $\sigma_\infty = \infty$. Assume $\sigma_\infty < \infty$ for contradiction. If follows from Proposition 4.4 that $\|v(x,t)\|_{L^\infty(0,\sigma(t))} \to 0$ as $t \to \infty$, which implies

$$\|v(x,t_0)\|_{L^\infty(0,\sigma(t_0))} < \min\left\{\frac{d}{32Ka\sigma^2(t_0)}, \frac{d}{32\mu}\right\} \tag{4.19}$$

for some large t_0. By continuous dependence, we have

$$\left\|u(\lambda\phi; x, t_0)\right\|_{L^\infty(0,h(t_0))} \leqslant \min\left\{\frac{d}{32Kas^2(t_0)}, \frac{d}{32\mu}\right\} \tag{4.20}$$

for $\lambda > \lambda^*$ sufficiently close to λ^*. But this implies that $s_\infty(\lambda\phi) < \infty$ by Theorem 4.1, which is a contradiction with the definition of λ^*.

On the other hand, as a consequence of the blowup result of Theorem 3.4, we know that $T_{\max} = \infty$ implies $E(t) \geqslant 0$ for all $t \geqslant 0$, hence by using (3.4) we have

$$\int_0^\infty \int_0^{s(t)} u_t^2(x,t)dx\,dt + \frac{d}{2\mu^2}\int_0^\infty s'^3(t)dt \leqslant E(0) < \infty. \tag{4.21}$$

The estimate (4.17) follows from Hölder's inequality and (4.21), by writing $s(t) - s_0 = \int_0^t s'(\tau)d\tau \leqslant (\int_0^t (s'(\tau))^3 d\tau)^{1/3} t^{2/3}$. The proof is complete. □

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