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Research Article

Solvability of a Second Order Nonlinear Neutral Delay Difference Equation

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This paper studies the second-order nonlinear neutral delay difference equation $\Delta[a_n\Delta(x_n + b_n x_{n-\tau}) + f(n, x_{f_{1n}}, \ldots, x_{f_{kn}})] + g(n, x_{g_{1n}}, \ldots, x_{g_{kn}}) = c_n$, $n \geq n_0$. By means of the Krasnoselskii and Schauder fixed point theorem and some new techniques, we get the existence results of uncountably many bounded nonoscillatory, positive, and negative solutions for the equation, respectively. Ten examples are given to illustrate the results presented in this paper.

1. Introduction

We are concerned with the second-order nonlinear neutral delay difference equation of the form

$$\Delta\left[a_n\Delta(x_n + b_n x_{n-\tau}) + f\left(n, x_{f_{1n}}, \ldots, x_{f_{kn}}\right)\right] + g\left(n, x_{g_{1n}}, \ldots, x_{g_{kn}}\right) = c_n, \quad n \geq n_0, \tag{1.1}$$

where $\tau, k \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$, $\{b_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{c_n\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $a_n \neq 0$ for each $n \in \mathbb{N}_{n_0}$, $f, g \in C(\mathbb{N}_{n_0} \times \mathbb{R}^k, \mathbb{R})$, and $f_l, g_l : \mathbb{N}_{n_0} \to \mathbb{Z}$ with

$$\lim_{n\to\infty} f_{ln} = \lim_{n\to\infty} g_{ln} = +\infty, \ l \in \{1, 2, \ldots, k\}. \tag{1.2}$$

Note that a few special cases of (1.1) were studied in [1–9]. In particular, González and Jiménez-Melado [3] used a fixed-point theorem derived from the theory of measures of noncompactness to investigate the existence of solutions for the second-order difference equation

$$\Delta(q_n \Delta x_n) + f_n(x_n) = 0, \quad n \geq 0. \tag{1.3}$$

By applying the Leray-Schauder nonlinear alternative theorem for condensing operators, Agarwal et al. [1] studied the existence of a nonoscillatory solution for the second-order neutral delay difference equation

$$\Delta(a_n \Delta(x_n + p x_{n-\tau})) + F(n+1, x_{n+1-\sigma}) = 0, \quad n \geq 0, \tag{1.4}$$

where $p \in \mathbb{R} \setminus \{\pm 1\}$. Using the Banach contraction principle, Cheng [5] discussed the existence of a positive solution for the second-order neutral delay difference equation with positive and negative coefficients

$$\Delta^2(x_n + p x_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0, \tag{1.5}$$

where $p \in \mathbb{R} \setminus \{-1\}$, Liu et al. [6] and Liu et al. [7] extended the results due to cheng [5] and got the existence of uncountably many bounded nonoscillatory solutions for (1.1) and the second-order nonlinear neutral delay difference equation

$$\Delta[a_n \Delta(x_n + b x_{n-\tau})] + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0, \tag{1.6}$$

with respect to $b \in \mathbb{R}$, where f is Lipschitz continuous, respectively.

The purpose of this paper is to establish the existence results of uncountably many bounded nonoscillatory, positive, and negative solutions, respectively, for (1.1) by using the Krasnoselskii fixed point theorem, Schauder fixed point theorem, and a few new techniques. The results obtained in this paper improve essentially the corresponding results in [5–7] by removing the Lipschitz continuity condition. Ten nontrivial examples are given to reveal the superiority and applications of our results.

2. Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$, and $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ stand for the sets of all integers, positive integers, and nonnegative integers, respectively,

$$\begin{gathered} \mathbb{N}_{n_0} = \{n : n \in \mathbb{N}_0 \text{ with } n \geq n_0\}, \quad n_0 \in \mathbb{N}_0, \\ \beta = \min\{n_0 - \tau, \ \inf\{f_{ln}, g_{ln} : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\}\}, \\ \mathbb{Z}_\beta = \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\}. \end{gathered} \tag{2.1}$$

Let l_β^∞ denote the Banach space of all bounded sequences in $\mathbb{Z}_\beta$ with norm

$$\|x\| = \sup_{n\in\mathbb{Z}_\beta}|x_n| \quad \text{for } x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in l_\beta^\infty,$$
$$B(d,D) = \left\{x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in l_\beta^\infty : \|x-d\| \le D\right\} \quad \text{for } d = \{d\}_{n\in\mathbb{Z}_\beta} \in l_\beta^\infty,\ D>0 \tag{2.2}$$

represent the closed ball centered at d and with radius D in l_β^∞.

By a solution of (1.1), we mean a sequence $\{x_n\}_{n\in Z_\beta}$ with a positive integer $T \ge n_0 + \tau + |\beta|$ such that (1.1) is satisfied for all $n \ge T$. As is customary, a solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

Lemma 2.1 ([2]). *A bounded, uniformly Cauchy subset Y of l_β^∞ is relatively compact.*

Lemma 2.2 (Krasnoselskii fixed point theorem [10]). *Let Y be a nonempty bounded closed convex subset of a Banach space X and $S, G : Y \to X$ mappings such that $Sx + Gy \in Y$ for every pair $x, y \in Y$. If S is a contraction and G is completely continuous, then*

$$Sx + Gx = x \tag{2.3}$$

has a solution in Y.

Lemma 2.3 (Schauder fixed point theorem [10]). *Let Y be a nonempty closed convex subset of a Banach space X and $S : Y \to Y$ a continuous mapping such that $S(Y)$ is a relatively compact subset of X. Then, S has a fixed point in Y.*

Lemma 2.4. *Let $\tau \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$ and $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ be a nonnegative sequence. Then,*

$$\sum_{i=0}^{\infty}\sum_{s=n_0+i\tau}^{\infty} a_s < +\infty \Longleftrightarrow \sum_{s=n_0}^{\infty} s a_s < +\infty . \tag{2.4}$$

Moreover, if $\sum_{i=0}^{\infty}\sum_{s=n_0+i\tau}^{\infty} a_s < +\infty$, then

$$\sum_{i=0}^{\infty}\sum_{s=n_0+i\tau}^{\infty} a_s \le \sum_{s=n_0}^{\infty}\left(1+\frac{s}{\tau}\right)a_s < +\infty. \tag{2.5}$$

Proof. For each $t \in \mathbb{R}$, let $[t]$ stand for the largest integer not exceeding t. It follows that

$$\begin{aligned}\sum_{i=0}^{\infty}\sum_{s=n_0+i\tau}^{\infty} a_s &= \sum_{s=n_0}^{\infty} a_s + \sum_{s=n_0+\tau}^{\infty} a_s + \sum_{s=n_0+2\tau}^{\infty} a_s + \cdots \\ &= \sum_{s=n_0}^{\infty}\left(1+\left[\frac{s-n_0}{\tau}\right]\right)a_s \le \sum_{s=n_0}^{\infty}\left(1+\frac{s}{\tau}\right)a_s,\end{aligned} \tag{2.6}$$

$$\lim_{s\to\infty}\frac{1+[s-n_0/\tau]}{s/\tau} = 1. \tag{2.7}$$

Combining (2.6) and (2.7), we infer that (2.4) holds. Assume that $\sum_{i=0}^{\infty}\sum_{s=n_0+i\tau}^{\infty} a_s < +\infty$. In view of (2.4), we get that $\sum_{s=n_0}^{\infty} s a_s < +\infty$, which gives that $\sum_{s=n_0}^{\infty} a_s < +\infty$. It follows that

$$\sum_{s=n_0}^{\infty}\left(1+\frac{s}{\tau}\right)a_s < +\infty. \tag{2.8}$$

This completes the proof. □

3. Existence of Uncountably Many Bounded Positive Solutions

Now, we use the Krasnoselskii fixed point theorem to prove the existence of uncountably many bounded nonoscillatory, positive, and negative solutions of (1.1) under various conditions relative to the sequence $\{b_n\}_{n\in\mathbb{N}_\beta} \subset \mathbb{R}$.

Theorem 3.1. *Assume that there exist* $n_1 \in \mathbb{N}_{n_0}$, $d \in \mathbb{R}$, $D, b \in \mathbb{R}^+ \setminus \{0\}$ *and two nonnegative sequences* $\{F_n\}_{n\in\mathbb{N}_{n_0}}$ *and* $\{G_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying*

$$1 < \frac{|d|}{D} < \frac{1-b}{b}, \quad |b_n| \le b, \ \forall n \ge n_1, \tag{3.1}$$

$$\left|f(n,u_1,\ldots,u_k)\right| \le F_n, \quad \left|g(n,u_1,\ldots,u_k)\right| \le G_n, \ \forall (n,u_l) \in \mathbb{N}_{n_0}\times[d-D,d+D], \ 1\le l\le k, \tag{3.2}$$

$$\sum_{i=n_0+1}^{\infty}\frac{1}{|a_i|}\max\left\{F_i, \sum_{j=n_0}^{i-1}\max\{G_j,|c_j|\}\right\} < +\infty. \tag{3.3}$$

Then, (1.1) *has uncountably many bounded nonoscillatory solutions in* $B(d,D)$.

Proof. Let $L \in (d-(1-b)D+b|d|, d+(1-b)D-b|d|)$. It follows from (3.3) that there exists $T \ge 1+n_0+n_1+\tau+|\beta|$ satisfying

$$\sum_{i=T}^{\infty}\frac{1}{|a_i|}\left[F_i + \sum_{j=n_0}^{i-1}\left(G_j+|c_j|\right)\right] \le (1-b)D - b|d| - |L-d|. \tag{3.4}$$

Define two mappings S_L and $G_L : B(d,D) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} L - b_n x_{n-\tau}, & n \geq T, \\ S_L x_T, & \beta \leq n < T, \end{cases} \tag{3.5}$$

$$G_L x_n = \begin{cases} \displaystyle\sum_{i=n}^{\infty} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) - c_j \right] \right\}, & n \geq T, \\ G_L x_T, & \beta \leq n < T, \end{cases} \tag{3.6}$$

for any $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in B(d,D)$.

Now, we assert that

$$S_L x + G_L y \in B(d,D), \quad \forall x, y \in B(d,D), \tag{3.7}$$

$$\left\| S_L x - S_L y \right\| \leq b \left\| x - y \right\|, \quad \forall x, y \in B(d,D), \tag{3.8}$$

$$\|G_L x\| \leq D, \quad \forall x \in B(d,D). \tag{3.9}$$

It follows from (3.1), (3.2), and (3.4)–(3.6) that for any $x = \{x_n\}_{n \in \mathbb{Z}_\beta}, y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in B(d,D)$, and $n \geq T$,

$$\begin{aligned} \left| S_L x_n + G_L y_n - d \right| &= \Bigg| L - d - b_n x_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{a_i} \\ &\quad \times \left\{ f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, y_{g_{1j}}, \dots, y_{g_{kj}}\right) - c_j \right] \right\} \Bigg| \\ &\leq |L - d| + b(|d| + D) + \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + \left| c_j \right| \right) \right] \\ &\leq |L - d| + b(|d| + D) + (1 - b)D - b|d| - |L - d| = D, \\ \left| S_L x_n - S_L y_n \right| &= \left| b_n \left(x_{n-\tau} - y_{n-\tau} \right) \right| \leq b \left\| x - y \right\|, \\ |G_L x_n| &\leq \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + \left| c_j \right| \right) \right] \leq (1 - b)D - b|d| - |L - d| \leq D, \end{aligned} \tag{3.10}$$

which imply that (3.7)–(3.9) hold.

Next, we prove that G_L is continuous and $G_L(B(d,D))$ is uniformly Cauchy. It follows from (3.3) that for each $\varepsilon > 0$, there exists $M > T$ satisfying

$$\sum_{i=M}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + \left| c_j \right| \right) \right] < \frac{\varepsilon}{4}. \tag{3.11}$$

Let $x^\nu = \{x_n^\nu\}_{n\in\mathbb{Z}_\beta}$ and $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,D)$ satisfy that

$$\lim_{\nu\to\infty} x^\nu = x. \tag{3.12}$$

In view of (3.12) and the continuity of f and g, we know that there exists $V \in \mathbb{N}$ such that

$$\sum_{i=T}^{M-1} \frac{1}{|a_i|} \left[\left| f\left(i, x_{f_{1i}}^\nu, \dots, x_{f_{ki}}^\nu\right) - f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) \right| + \sum_{j=n_0}^{i-1} \left| g\left(j, x_{g_{1j}}^\nu, \dots, x_{g_{kj}}^\nu\right) - g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) \right| \right] < \frac{\varepsilon}{2}, \quad \forall \nu \ge V. \tag{3.13}$$

Combining (3.6), (3.11), and (3.13), we obtain that

$$\begin{aligned} \|G_L x^\nu - G_L x\| &\le \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left\{ \left| f\left(i, x_{f_{1i}}^\nu, \dots, x_{f_{ki}}^\nu\right) - f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) \right| + \sum_{j=n_0}^{i-1} \left| g\left(j, x_{g_{1j}}^\nu, \dots, x_{g_{kj}}^\nu\right) - g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) \right| \right\} \\ &\le \sum_{i=T}^{M-1} \frac{1}{|a_i|} \left\{ \left| f\left(i, x_{f_{1i}}^\nu, \dots, x_{f_{ki}}^\nu\right) - f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) \right| + \sum_{j=n_0}^{i-1} \left| g\left(j, x_{g_{1j}}^\nu, \dots, x_{g_{kj}}^\nu\right) - g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) \right| \right\} \\ &\quad + 2\sum_{i=M}^{\infty} \frac{1}{|a_i|} \left(F_i + \sum_{j=n_0}^{i-1} G_j \right) < \varepsilon, \quad \forall \nu \ge V, \end{aligned} \tag{3.14}$$

which means that G_L is continuous in $B(d,D)$.

In view of (3.6) and (3.11), we obtain that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,D)$ and $t, h \ge M$

$$\begin{aligned} |G_L x_t - G_L x_h| &\le \left| \sum_{i=t}^{\infty} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) - c_j \right] \right\} \right| \\ &\quad + \left| \sum_{i=h}^{\infty} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) - c_j \right] \right\} \right| \\ &\le 2\sum_{i=M}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j|\right) \right] < \varepsilon, \end{aligned} \tag{3.15}$$

which implies that $G_L(B(d,D))$ is uniformly Cauchy, which together with (3.9) and Lemma 2.1 yields that $G_L(B(d,D))$ is relatively compact. Consequently, G_L is completely continuous in $B(d,D)$. Thus, (3.7), (3.8), and Lemma 2.2 ensure that the mapping $S_L + G_L$ has a fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,D)$, which together with (3.5) and (3.6) implies that

$$x_n = L - b_n x_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{a_i} \left\{ f(i, x_{f_{1i}}, \ldots, x_{f_{ki}}) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j \right] \right\}, \quad n \geq T, \tag{3.16}$$

which yields that

$$\Delta\left[a_n \Delta(x_n + b_n x_{n-\tau}) + f(n, x_{f_{1n}}, \ldots, x_{f_{kn}})\right] + g(n, x_{g_{1n}}, \ldots, x_{g_{kn}}) = c_n, \quad n \geq T. \tag{3.17}$$

That is, $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$ is a bounded nonoscillatory solution of (1.1) in $B(d,D)$.

Let $L_1, L_2 \in (d - (1-b)D + b|d|,\ d + (1-b)D - b|d|)$, and $L_1 \neq L_2$. Similarly, we can prove that for each $l \in \{1,2\}$, there exist a constant $T_l \geq 1 + n_0 + n_1 + \tau + |\beta|$ and two mappings S_{L_l} and $G_{L_l} : B(d,D) \to l_\beta^\infty$ satisfying (3.4)–(3.6), where T, L, S_L, and G_L are replaced by T_l, L_l, S_{L_l}, and G_{L_l}, respectively, and $S_{L_l} + G_{L_l}$ has a fixed point $z^l \in B(d,D)$, which is a bounded nonoscillatory solution of (1.1); that is,

$$z_n^l = L_l - b_n z_{n-\tau}^l + \sum_{i=n}^{\infty} \frac{1}{a_i} \left\{ f\left(i, z_{f_{1i}}^l, \ldots, z_{f_{ki}}^l\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, z_{g_{1j}}^l, \ldots, z_{g_{kj}}^l\right) - c_j \right] \right\}, \qquad \forall n \geq T_l, \quad l \in \{1,2\}. \tag{3.18}$$

Note that (3.3) implies that there exists $T_3 > \max\{T_1, T_2\}$ satisfying

$$\sum_{i=T_3}^{\infty} \frac{1}{|a_i|} \left(F_i + \sum_{j=n_0}^{i-1} G_j \right) < \frac{|L_1 - L_2|}{4}. \tag{3.19}$$

Using (3.2), (3.18), and (3.19), we get that for any $n \geq T_3$,

$$\begin{aligned} &\left| z_n^1 - z_n^2 + b_n\left(z_{n-\tau}^1 - z_{n-\tau}^2\right) \right| \\ &\quad = \left| L_1 - L_2 + \sum_{i=n}^{\infty} \frac{1}{a_i} \left\{ \left[f\left(j, z_{f_{1j}}^1, \ldots, z_{f_{kj}}^1\right) - f\left(j, z_{f_{1j}}^2, \ldots, z_{f_{kj}}^2\right) \right] \right.\right. \\ &\qquad \left.\left. + \sum_{j=n_0}^{i-1} \left[g\left(j, z_{g_{1j}}^1, \ldots, z_{g_{kj}}^1\right) - g\left(j, z_{g_{1j}}^2, \ldots, z_{g_{kj}}^2\right) \right] \right\} \right| \end{aligned}$$

$$\begin{aligned}
&\geq |L_1 - L_2| - 2\sum_{i=T_3}^{\infty} \frac{1}{|a_i|}\left(F_i + \sum_{j=n_0}^{i-1} G_j\right) \\
&> \frac{|L_1 - L_2|}{2} \\
&> 0,
\end{aligned} \tag{3.20}$$

that is, $z^1 \neq z^2$. Therefore, (1.1) possesses uncountably many bounded nonoscillatory solutions in $B(d, D)$. This completes the proof. □

Theorem 3.2. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}$, $d \in \mathbb{R}$, $D, b \in \mathbb{R}^+ \setminus \{0\}$, and two nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{G_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.2), (3.3), *and*

$$\frac{|d|}{D} < \frac{1-b}{b}, \quad |b_n| \leq b, \quad \forall n \geq n_1. \tag{3.21}$$

Then, (1.1) *has uncountably many bounded solutions in $B(d, D)$.*

The proof of Theorem 3.2 is analogous to that of Theorem 3.1 and hence is omitted.

Theorem 3.3. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}$, $d \in \mathbb{R}$, $D, b_*, b^* \in \mathbb{R}^+ \setminus \{0\}$, and two nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{G_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.2), (3.3), *and*

$$\frac{|d|}{D} < \frac{b_* - 1}{b^* + 1}, \quad 1 < b_* \leq |b_n| \leq b^*, \quad \forall n \geq n_1. \tag{3.22}$$

Then, (1.1) *has uncountably many bounded solutions in $B(d, D)$.*

Proof. Let $L \in (-(b_* - 1)D + (b^* + 1)|d|,\ (b_* - 1)D - (b^* + 1)|d|)$. It follows from (3.3) that there exists $T \geq 1 + n_0 + n_1 + \tau + |\beta|$ satisfying

$$\sum_{i=T}^{\infty} \frac{1}{|a_i|}\left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j|\right)\right] \leq (b_* - 1)D - (b^* + 1)|d| - |L|. \tag{3.23}$$

Define two mappings S_L and $G_L : B(d, D) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} \dfrac{L}{b_{n+\tau}} - \dfrac{x_{n+\tau}}{b_{n+\tau}}, & n \geq T, \\ S_L x_T, & \beta \leq n < T, \end{cases} \tag{3.24}$$

$$G_L x_n = \begin{cases} \dfrac{1}{b_{n+\tau}} \displaystyle\sum_{i=n+\tau}^{\infty} \dfrac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j \right] \right\}, & n \geq T, \\ G_L x_T, & \beta \leq n < T, \end{cases} \tag{3.25}$$

for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$.

Now, we assert that (3.7), (3.9), and the below

$$\left\| S_L x - S_L y \right\| \leq \frac{1}{b_*} \|x - y\|, \quad \forall x, y \in B(d, D) \tag{3.26}$$

hold. It follows from (3.2) and (3.22)–(3.25) that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$, $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$, and $n \geq T$,

$$\begin{aligned} &\left| S_L x_n + G_L y_n - d \right| \\ &\quad = \left| \frac{L}{b_{n+\tau}} - d - \frac{x_{n+\tau}}{b_{n+\tau}} + \frac{1}{b_{n+\tau}} \sum_{i=n+\tau}^{\infty} \frac{1}{a_i} \right. \\ &\qquad \left. \times \left\{ f\left(i, y_{f_{1i}}, \ldots, y_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, y_{g_{1j}}, \ldots, y_{g_{kj}}\right) - c_j \right] \right\} \right| \\ &\quad \leq \frac{1}{b_*} |L - b_{n+\tau} d| + \frac{|d| + D}{b_*} + \frac{1}{b_*} \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j| \right) \right] \\ &\quad \leq \frac{1}{b_*} (|L| + b^* |d|) + \frac{|d| + D}{b_*} + \frac{1}{b_*} [(b_* - 1)D - (b^* + 1)|d| - |L|] \leq D, \end{aligned}$$

$$\left| S_L x_n - S_L y_n \right| = \left| \frac{1}{b_{n+\tau}} \left(x_{n+\tau} - y_{n+\tau} \right) \right| \leq \frac{1}{b_*} \|x - y\|,$$

$$|G_L x_n| \leq \frac{1}{b_*} \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j| \right) \right] \leq \frac{1}{b_*} [(b_* - 1)D - (b^* + 1)|d| - |L|] \leq D, \tag{3.27}$$

which imply (3.7), (3.9), and (3.26).

Next, we show that G_L is continuous and $G_L(B(d, D))$ is uniformly Cauchy. It follows from (3.3) that for each $\varepsilon > 0$, there exists $M > T$ satisfying (3.11). Let $x^\nu = \{x_n^\nu\}_{n\in\mathbb{Z}_\beta}$ and

$x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$ with (3.12). It follows from (3.12) and the continuity of f and g that there exists $V \in \mathbb{N}$ satisfying (3.13). In light of (3.11), (3.13), and (3.25), we deduce that

$$\begin{aligned}
\|G_L x^\nu - G_L x\| &\le \frac{1}{b_*}\sum_{i=T+\tau}^{\infty}\frac{1}{|a_i|}\Bigg[\left|f\left(i, x^\nu_{f_{1i}}, \ldots, x^\nu_{f_{ki}}\right) - f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right)\right| \\
&\quad + \sum_{j=n_0}^{i-1}\left|g\left(j, x^\nu_{g_{1j}}, \ldots, x^\nu_{g_{kj}}\right) - g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right)\right|\Bigg] \\
&\le \frac{1}{b_*}\sum_{i=T}^{M-1}\frac{1}{|a_i|}\Bigg[\left|f\left(i, x^\nu_{f_{1i}}, \ldots, x^\nu_{f_{ki}}\right) - f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right)\right| \\
&\quad + \sum_{j=n_0}^{i-1}\left|g\left(j, x^\nu_{g_{1j}}, \ldots, x^\nu_{g_{kj}}\right) - g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right)\right|\Bigg] + \frac{2}{b_*}\sum_{i=M}^{\infty}\frac{1}{|a_i|}\left(F_i + \sum_{j=n_0}^{i-1}G_j\right) \\
&< \frac{\varepsilon}{b_*}, \quad \forall \nu \ge V,
\end{aligned} \tag{3.28}$$

which yields that G_L is continuous in $B(d, D)$.

Using (3.1) and (3.25), we get that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$ and $t, h \ge M$

$$\begin{aligned}
|G_L x_t - G_L x_h| &\le \left|\frac{1}{b_{t+\tau}}\sum_{i=t+\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1}\left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j\right]\right\}\right| \\
&\quad + \left|\frac{1}{b_{h+\tau}}\sum_{i=h+\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1}\left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j\right]\right\}\right| \\
&\le \frac{2}{b_*}\sum_{i=M}^{\infty}\frac{1}{|a_i|}\left[F_i + \sum_{j=n_0}^{i-1}\left(G_j + |c_j|\right)\right] \\
&< \frac{\varepsilon}{b_*},
\end{aligned} \tag{3.29}$$

which means that $G_L(B(d, D))$ is uniformly Cauchy, which together with (3.9) and Lemma 2.1 yields that $G_L(B(d, D))$ is relatively compact. Consequently, G_L is completely continuous in $B(d, D)$. Thus, (3.22), (3.26), and Lemma 2.2 ensure that the mapping $S_L + G_L$ has a fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$; that is,

$$x_n = \frac{L}{b_{n+\tau}} - \frac{x_{n+\tau}}{b_{n+\tau}} + \frac{1}{b_{n+\tau}}\sum_{i=n+\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1}\left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j\right]\right\}, \quad n \ge T, \tag{3.30}$$

which gives that

$$\Delta\left[a_n\Delta(x_n+b_nx_{n-\tau})+f(n,x_{f_{1n}},\dots,x_{f_{kn}})\right]+g(n,x_{g_{1n}},\dots,x_{g_{kn}})=c_n,\quad n\ge T+\tau. \tag{3.31}$$

That is, $x=\{x_n\}_{n\in\mathbb{Z}_\beta}$ is a bounded solution of (1.1) in $B(d,D)$.

Let $L_1,L_2\in(-(b_*-1)D+(b^*+1)|d|,(b_*-1)D-(b^*+1)|d|)$ and $L_1\ne L_2$. Similarly, we conclude that for each $l\in\{1,2\}$, there exist a constant $T_l\ge 1+n_0+n_1+\tau+|\beta|$ and two mappings S_{L_l} and $G_{L_l}:B(d,D)\to l_\beta^\infty$ satisfying (3.23)–(3.25), where T,L,S_L, and G_L are replaced by T_l,L_l,S_{L_l}, and G_{L_l}, respectively, and $S_{L_l}+G_{L_l}$ has a fixed point $z^l\in B(d,D)$, which is a bounded solution of (1.1); that is,

$$z_n^l=\frac{L_l}{b_{n+\tau}}-\frac{z_{n+\tau}^l}{b_{n+\tau}}+\frac{1}{b_{n+\tau}}\sum_{i=n+\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i,z_{f_{1i}}^l,\dots,z_{f_{ki}}^l\right)+\sum_{j=n_0}^{i-1}\left[g\left(j,z_{g_{1j}}^l,\dots,z_{g_{kj}}^l\right)-c_j\right]\right\}, \tag{3.32}$$

for all $n\ge T_l$ and $l\in\{1,2\}$. Note that (3.3) implies that there exists $T_3>\max\{T_1,T_2\}$ satisfying (3.19). By means of (3.2), (3.19), and (3.32), we infer that for any $n\ge T_3$,

$$\begin{aligned}
&\left|z_n^1-z_n^2+\frac{z_{n+\tau}^l-z_{n+\tau}^2}{b_{n+\tau}}\right|\\
&\quad=\left|\frac{L_1-L_2}{b_{n+\tau}}+\frac{1}{b_{n+\tau}}\sum_{i=n+\tau}^{\infty}\frac{1}{a_i}\left\{\left[f\left(j,z_{f_{1j}}^1,\dots,z_{f_{kj}}^1\right)-f\left(j,z_{f_{1j}}^2,\dots,z_{f_{kj}}^2\right)\right]\right.\right.\\
&\qquad\left.\left.+\sum_{j=n_0}^{i-1}\left[g\left(j,z_{g_{1j}}^1,\dots,z_{g_{kj}}^1\right)-g\left(j,z_{g_{1j}}^2,\dots,z_{g_{kj}}^2\right)\right]\right\}\right|\\
&\quad\ge\frac{|L_1-L_2|}{b_*}-\frac{2}{b_*}\sum_{i=T_3}^{\infty}\frac{1}{|a_i|}\left(F_i+\sum_{j=n_0}^{i-1}G_j\right)\\
&\quad>\frac{|L_1-L_2|}{2b_*}\\
&\quad>0,
\end{aligned} \tag{3.33}$$

that is, $z^1\ne z^2$. Therefore, (1.1) possesses uncountably many bounded solutions in $B(d,D)$. This completes the proof. □

Similar to the proofs of Theorems 3.1 and 3.3, we have the following results.

Theorem 3.4. *Assume that there exist $n_1\in\mathbb{N}_{n_0}$, $d\in\mathbb{R}$, $D,b_*,b^*\in\mathbb{R}^+\setminus\{0\}$ and two nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{G_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.2), (3.3), *and*

$$|d| > D, \quad \left(b_*^2 b^* + b_* b^{*2} - b^{*2} - b_*^2\right) D > \left(b^{*2} - b_*^2 - b_*^2 b^* + b_* b^{*2}\right)|d|, \qquad 1 < b_* \le b_n \le b^*, \quad \forall n \ge n_1. \tag{3.34}$$

Then, (1.1) *has uncountably many bounded nonoscillatory solutions in* $B(d, D)$.

Theorem 3.5. *Assume that there exist* $n_1 \in \mathbb{N}_{n_0}$, $d, D \in \mathbb{R}^+ \setminus \{0\}$, $b_*, b^* \in \mathbb{R}_-$ *and two nonnegative sequences* $\{F_n\}_{n \in \mathbb{N}_{n_0}}$ *and* $\{G_n\}_{n \in \mathbb{N}_{n_0}}$ *satisfying* (3.2), (3.3), *and*

$$d > D, \quad D(2 + b^* + b_*) < d(b_* - b^*), \quad b_* \le b_n \le b^* < -1, \quad \forall n \ge n_1. \tag{3.35}$$

Then, (1.1) *has uncountably many bounded positive solutions in* $B(d, D)$.

Theorem 3.6. *Assume that there exist* $n_1 \in \mathbb{N}_{n_0}$, $D \in \mathbb{R}^+ \setminus \{0\}$, $d, b_* \in \mathbb{R}_- \setminus \{0\}$, $b^* \in \mathbb{R}_-$ *and two nonnegative sequences* $\{F_n\}_{n \in \mathbb{N}_{n_0}}$ *and* $\{G_n\}_{n \in \mathbb{N}_{n_0}}$ *satisfying* (3.2), (3.3), *and*

$$-d > D, \quad d(b^* - b_*) > D(2 + b_* + b^*), \quad b_* \le b_n \le b^*, \quad \forall n \ge n_1. \tag{3.36}$$

Then, (1.1) *has uncountably many bounded negative solutions in* $B(d, D)$.

Theorem 3.7. *Assume that there exist* $n_1 \in \mathbb{N}_{n_0}$, $d \in \mathbb{R} \setminus \{0\}$, $b^*, D \in \mathbb{R}^+ \setminus \{0\}$ *and two negative sequences* $\{F_n\}_{n \in \mathbb{N}_{n_0}}$ *and* $\{G_n\}_{n \in \mathbb{N}_{n_0}}$ *satisfying* (3.2), (3.3), *and*

$$1 < \frac{|d|}{D} < \frac{2 - b^*}{b^*}, \quad 0 \le b_n \le b^*, \quad \forall n \ge n_1. \tag{3.37}$$

Then, (1.1) *has uncountably many bounded nonoscillatory solutions in* $B(d, D)$.

Theorem 3.8. *Assume that there exist* $n_1 \in \mathbb{N}_{n_0}$, $d \in \mathbb{R} \setminus \{0\}$, $b_* \in \mathbb{R}_- \setminus \{0\}$, $D \in \mathbb{R}^+ \setminus \{0\}$ *and two negative sequences* $\{F_n\}_{n \in \mathbb{N}_{n_0}}$ *and* $\{G_n\}_{n \in \mathbb{N}_{n_0}}$ *satisfying* (3.2), (3.3), *and*

$$1 < \frac{|d|}{D} < \frac{b_* + 2}{-b_*}, \quad b_* \le b_n \le 0, \quad \forall n \ge n_1. \tag{3.38}$$

Then, (1.1) *has uncountably many bounded nonoscillatory solutions in* $B(d, D)$.

Next, we investigate the existence of uncountably bounded nonoscillatory solutions for (1.1) with the help of the Schauder fixed point theorem under the conditions of $b_n = \pm 1$.

Theorem 3.9. *Assume that there exist* $n_1 \in \mathbb{N}_{n_0}$, $d \in \mathbb{R}$, $D \in \mathbb{R}^+ \setminus \{0\}$ *and two nonnegative sequences* $\{F_n\}_{n \in \mathbb{N}_{n_0}}$ *and* $\{G_n\}_{n \in \mathbb{N}_{n_0}}$ *satisfying* (3.2), (3.3), *and*

$$|d| > D, \qquad b_n = 1, \quad \forall n \ge n_1. \tag{3.39}$$

Then, (1.1) *has uncountably many bounded nonoscillatory solutions in* $B(d, D)$.

Proof. Let $L \in (d - D, d + D)$. It follows from (3.3) that there exists $T \geq 1 + n_0 + n_1 + \tau + |\beta|$ satisfying

$$\sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j| \right) \right] \leq D - |L - d|. \tag{3.40}$$

Define a mapping $S_L : B(d, D) \longrightarrow l_\beta^\infty$ by

$$S_L x_n = \begin{cases} L + \displaystyle\sum_{s=1}^{\infty} \sum_{i=n+(2s-1)\tau}^{n+2s\tau-1} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j \right] \right\}, & n \geq T, \\ S_L x_T, & \beta \leq n < T, \end{cases} \tag{3.41}$$

for any $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in B(d, D)$.

Now, we prove that

$$S_L x \in B(d, D), \quad \|S_L x\| \leq |L| + D, \quad \forall x \in B(d, D). \tag{3.42}$$

It follows from (3.2) and (3.39)–(3.41) that for any $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in B(d, D)$ and $n \geq T$,

$$\begin{aligned} |S_L x_n - d| &= \left| L - d + \sum_{s=1}^{\infty} \sum_{i=n+(2s-1)\tau}^{n+2s\tau-1} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j \right] \right\} \right| \\ &\leq |L - d| + \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j| \right) \right] \\ &\leq |L - d| + D - |L - d| \\ &= D, \\ |S_L x_n| &\leq |L| + \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j| \right) \right] \leq |L| + D - |L - d| \leq |L| + D, \end{aligned} \tag{3.43}$$

which imply (3.42).

Next, we prove that S_L is continuous and $S_L(B(d, D))$ is uniformly Cauchy. It follows from (3.3) that for each $\varepsilon > 0$, there exists $M > T$ satisfying (3.11). Let $x^\nu = \{x_n^\nu\}_{n \in \mathbb{Z}_\beta}$ and

$x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$ satisfying (3.12). It follows from (3.12) and the continuity of f and g that there exists $V \in \mathbb{N}$ satisfying (3.13). Combining (3.11), (3.13), and (3.41), we infer that

$$\begin{aligned}
&\|S_L x^{\nu} - S_L x\| \\
&\quad\le \sup_{n\ge T}\left\{\sum_{s=1}^{\infty}\sum_{i=n+(2s-1)\tau}^{n+2s\tau-1}\frac{1}{|a_i|}\left\{\left|f\left(i, x^{\nu}_{f_{1i}}, \dots, x^{\nu}_{f_{ki}}\right) - f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right)\right|\right.\right. \\
&\qquad\qquad\left.\left. + \sum_{j=n_0}^{i-1}\left|g\left(j, x^{\nu}_{g_{1j}}, \dots, x^{\nu}_{g_{kj}}\right) - g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right)\right|\right\}\right\} \\
&\quad\le \sum_{i=T}^{\infty}\frac{1}{|a_i|}\left\{\left|f\left(i, x^{\nu}_{f_{1i}}, \dots, x^{\nu}_{f_{ki}}\right) - f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right)\right|\right. \\
&\qquad\qquad\left. + \sum_{j=n_0}^{i-1}\left|g\left(j, x^{\nu}_{g_{1j}}, \dots, x^{\nu}_{g_{kj}}\right) - g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right)\right|\right\} \\
&\quad\le \sum_{i=T}^{M-1}\frac{1}{|a_i|}\left\{\left|f\left(i, x^{\nu}_{f_{1i}}, \dots, x^{\nu}_{f_{ki}}\right) - f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right)\right|\right. \\
&\qquad\qquad\left. + \sum_{j=n_0}^{i-1}\left|g\left(j, x^{\nu}_{g_{1j}}, \dots, x^{\nu}_{g_{kj}}\right) - g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right)\right|\right\} + 2\sum_{i=M}^{\infty}\frac{1}{|a_i|}\left(F_i + \sum_{j=n_0}^{i-1}G_j\right) \\
&\quad< \varepsilon, \quad \forall \nu \ge V,
\end{aligned} \tag{3.44}$$

which implies that S_L is continuous in $B(d, D)$.

By means of (3.11) and (3.41), we get that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$ and $t, h \ge M$

$$\begin{aligned}
|S_L x_t - S_L x_h| &\le \left|\sum_{s=1}^{\infty}\sum_{i=t+(2s-1)\tau}^{t+2s\tau-1}\frac{1}{a_i}\left\{f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1}\left[g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) - c_j\right]\right\}\right| \\
&\quad + \left|\sum_{s=1}^{\infty}\sum_{i=h+(2s-1)\tau}^{h+2s\tau-1}\frac{1}{a_i}\left\{f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1}\left[g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) - c_j\right]\right\}\right| \\
&\le 2\sum_{i=M}^{\infty}\frac{1}{|a_i|}\left[F_i + \sum_{j=n_0}^{i-1}\left(G_j + |c_j|\right)\right] \\
&< \varepsilon,
\end{aligned} \tag{3.45}$$

which means that $S_L(B(d,D))$ is uniformly Cauchy, which together with (3.42) and Lemma 2.1 yields that $S_L(B(d,D))$ is relatively compact. It follows from Lemma 2.3 that the mapping S_L has a fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,D)$; that is,

$$x_n = L + \sum_{s=1}^{\infty} \sum_{i=n+(2s-1)\tau}^{n+2s\tau-1} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \dots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \dots, x_{g_{kj}}\right) - c_j \right] \right\}, \quad n \geq T, \tag{3.46}$$

which give that

$$\Delta\left[a_n \Delta(x_n + x_{n-\tau}) + f\left(n, x_{f_{1n}}, \dots, x_{f_{kn}}\right)\right] + g\left(n, x_{g_{1n}}, \dots, x_{g_{kn}}\right) = c_n, \quad n \geq T + \tau. \tag{3.47}$$

That is, $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,D)$ is a bounded nonoscillatory solution of (1.1).

Let $L_1, L_2 \in (d-D, d+D)$ and $L_1 \neq L_2$. Similarly, we infer that for each $l \in \{1,2\}$, there exist a constant $T_l \geq 1 + n_0 + n_1 + \tau + |\beta|$ and a mapping $S_{L_l} : B(d,D) \to l_\beta^\infty$ satisfying (3.41), where $L, T,$ and S_L are replaced by $T_l, L_l,$ and S_{L_l}, respectively, and S_{L_l} has a fixed point $z^l \in B(d,D)$, which is a bounded nonoscillatory solution of (1.1); that is,

$$z_n^l = L_l + \sum_{s=1}^{\infty} \sum_{i=n+(2s-1)\tau}^{n+2s\tau-1} \frac{1}{a_i} \left\{ f\left(i, z_{f_{1i}}^l, \dots, z_{f_{ki}}^l\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, z_{g_{1j}}^l, \dots, z_{g_{kj}}^l\right) - c_j \right] \right\}, \quad n \geq T_l, \tag{3.48}$$

for $l \in \{1,2\}$. Note that (3.3) implies that there exists $T_3 > \max\{T_1, T_2\}$ satisfying (3.19). Using (3.2), (3.19), and (3.48), we conclude that for any $n \geq T_3$

$$\begin{aligned}
&\left|z_n^1 - z_n^2\right| \\
&\quad = \Bigg| L_1 - L_2 + \sum_{s=1}^{\infty} \sum_{i=n+(2s-1)\tau}^{n+2s\tau-1} \frac{1}{a_i} \Bigg\{ \left[f\left(i, z_{f_{1i}}^1, \dots, z_{f_{ki}}^1\right) - f\left(i, z_{f_{1i}}^2, \dots, z_{f_{ki}}^2\right) \right] \\
&\qquad + \sum_{j=n_0}^{i-1} \left[g\left(j, z_{g_{1j}}^1, \dots, z_{g_{kj}}^1\right) - g\left(j, z_{g_{1j}}^2, \dots, z_{g_{kj}}^2\right) \right] \Bigg\} \Bigg| \\
&\quad \geq |L_1 - L_2| - 2\sum_{s=1}^{\infty} \sum_{i=n+(2s-1)\tau}^{n+2s\tau-1} \frac{1}{|a_i|} \left(F_i + \sum_{j=n_0}^{i-1} G_j \right) \\
&\quad \geq |L_1 - L_2| - 2\sum_{i=T_3}^{\infty} \frac{1}{|a_i|} \left(F_i + \sum_{j=n_0}^{i-1} G_j \right) \\
&\quad > \frac{|L_1 - L_2|}{2} \\
&\quad > 0,
\end{aligned} \tag{3.49}$$

which gives that $z^1 \neq z^2$. Therefore, (1.1) possesses uncountably many bounded nonoscillatory solutions in $B(d, D)$. This completes the proof. □

Theorem 3.10. *Assume that there exist* $n_1 \in \mathbb{N}_{n_0}$, $d \in \mathbb{R}$, $D \in \mathbb{R}^+ \setminus \{0\}$ *and two nonnegative sequences* $\{F_n\}_{n\in\mathbb{N}_{n_0}}$ *and* $\{G_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (3.2),

$$|d| > D, \quad b_n = -1, \quad \forall n \geq n_1 , \tag{3.50}$$

$$\sum_{s=1}^{\infty} \sum_{i=n_0+s\tau}^{\infty} \frac{1}{|a_i|} \max\left\{ F_i, \sum_{j=n_0}^{i-1} \max\{G_j, |c_j|\} \right\} < +\infty. \tag{3.51}$$

Then, (1.1) *has uncountably many bounded nonoscillatory solutions in* $B(d, D)$.

Proof. Let $L \in (d - D, d + D)$. It follows from (3.41) that there exists $T \geq 1 + n_0 + n_1 + \tau + |\beta|$ satisfying

$$\sum_{s=1}^{\infty} \sum_{i=T+s\tau}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j|\right) \right] \leq D - |L - d|. \tag{3.52}$$

Define a mapping $S_L : B(d, D) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} L - \displaystyle\sum_{s=1}^{\infty} \sum_{i=n+s\tau}^{\infty} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j \right] \right\}, & n \geq T \\ S_L x_T, & \beta \leq n < T, \end{cases} \tag{3.53}$$

for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$. It follows from (3.2), (3.52), and (3.53) that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d, D)$ and $n \geq T$

$$\begin{aligned} |S_L x_n - d| &= \left| L - d - \sum_{s=1}^{\infty} \sum_{i=n+s\tau}^{\infty} \frac{1}{a_i} \left\{ f\left(i, x_{f_{1i}}, \ldots, x_{f_{ki}}\right) + \sum_{j=n_0}^{i-1} \left[g\left(j, x_{g_{1j}}, \ldots, x_{g_{kj}}\right) - c_j \right] \right\} \right| \\ &\leq |L - d| + \sum_{s=1}^{\infty} \sum_{i=T+s\tau}^{\infty} \frac{1}{|a_i|} \left[F_i + \sum_{j=n_0}^{i-1} \left(G_j + |c_j|\right) \right] \\ &\leq |L - d| + D - |L - d| \\ &= D \end{aligned} \tag{3.54}$$

$$|S_L x_n| \leq |L| + \sum_{s=1}^{\infty} \sum_{i=T+s\tau}^{\infty} \frac{1}{|a_i|} \left[\sum_{j=n_0}^{i-1} \left(G_j + |c_j|\right) + F_i \right] \leq |L| + D,$$

which imply (3.42).

Next, we show that S_L is continuous and $S_L(B(d,D))$ is uniformly Cauchy. It follows from (3.51) and Lemma 2.4 that for each $\varepsilon > 0$, there exists $M > 1 + T + \tau$ satisfying

$$\sum_{i=M}^{\infty}\left(1+\frac{i}{\tau}\right)\frac{1}{|a_i|}\left[F_i+\sum_{j=n_0}^{i-1}(G_j+|c_j|)\right]<\frac{\varepsilon}{4}. \tag{3.55}$$

Let $x^{\nu} = \{x_n^{\nu}\}_{n\in\mathbb{Z}_\beta}$ and $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,D)$ satisfying (3.12). By means of (3.12) and the continuity of f and g, we deduce that there exists $V \in \mathbb{N}$ satisfying

$$\begin{aligned}\sum_{i=T+\tau}^{M-1}\left(1+\frac{i}{\tau}\right)\frac{1}{|a_i|}&\left\{\left|f\left(i,x^{\nu}_{f_{1i}},\dots,x^{\nu}_{f_{ki}}\right)-f\left(i,x_{f_{1i}},\dots,x_{f_{ki}}\right)\right|\right.\\&\left.+\sum_{j=n_0}^{i-1}\left|g\left(j,x^{\nu}_{g_{1j}},\dots,x^{\nu}_{g_{kj}}\right)-g\left(j,x_{g_{1j}},\dots,x_{g_{kj}}\right)\right|\right\}<\frac{\varepsilon}{2},\quad \forall \nu\geq V.\end{aligned} \tag{3.56}$$

In light of (3.2), (3.53)–(3.56) and Lemma 2.4, we conclude that

$$\begin{aligned}\|S_Lx^{\nu}-S_Lx\|&\leq\sum_{s=1}^{\infty}\sum_{i=T+s\tau}^{\infty}\frac{1}{|a_i|}\left\{\left|f\left(i,x^{\nu}_{f_{1i}},\dots,x^{\nu}_{f_{ki}}\right)-f\left(i,x_{f_{1i}},\dots,x_{f_{ki}}\right)\right|\right.\\&\qquad\left.+\sum_{j=n_0}^{i-1}\left|g\left(j,x^{\nu}_{g_{1j}},\dots,x^{\nu}_{g_{kj}}\right)-g\left(j,x_{g_{1j}},\dots,x_{g_{kj}}\right)\right|\right\}\\&\leq\sum_{i=T+\tau}^{\infty}\left(1+\frac{i}{\tau}\right)\frac{1}{|a_i|}\left\{\left|f\left(i,x^{\nu}_{f_{1i}},\dots,x^{\nu}_{f_{ki}}\right)-f\left(i,x_{f_{1i}},\dots,x_{f_{ki}}\right)\right|\right.\\&\qquad\left.+\sum_{j=n_0}^{i-1}\left|g\left(j,x^{\nu}_{g_{1j}},\dots,x^{\nu}_{g_{kj}}\right)-g\left(j,x_{g_{1j}},\dots,x_{g_{kj}}\right)\right|\right\}\\&\leq\sum_{i=T+\tau}^{M-1}\left(1+\frac{i}{\tau}\right)\frac{1}{|a_i|}\left\{\left|f\left(i,x^{\nu}_{f_{1i}},\dots,x^{\nu}_{f_{ki}}\right)-f\left(i,x_{f_{1i}},\dots,x_{f_{ki}}\right)\right|\right.\\&\qquad\left.+\sum_{j=n_0}^{i-1}\left|g\left(j,x^{\nu}_{g_{1j}},\dots,x^{\nu}_{g_{kj}}\right)-g\left(j,x_{g_{1j}},\dots,x_{g_{kj}}\right)\right|\right\}\\&\quad+2\sum_{i=M}^{\infty}\left(1+\frac{i}{\tau}\right)\frac{1}{|a_i|}\left(F_i+\sum_{j=n_0}^{i-1}G_j\right)\\&<\varepsilon,\quad\forall\nu\geq V,\end{aligned} \tag{3.57}$$

which implies that S_L is continuous in $B(d,D)$.

By virtue of (3.53), (3.55), and Lemma 2.4, we get that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,D)$ and $t,h \geq M$,

$$\begin{aligned}
|S_L x_t - S_L x_h| &\leq \left|\sum_{s=1}^{\infty}\sum_{i=t+s\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i,x_{f_{1i}},\dots,x_{f_{ki}}\right)+\sum_{j=n_0}^{i-1}\left[g\left(j,x_{g_{1j}},\dots,x_{g_{kj}}\right)-c_j\right]\right\}\right| \\
&\quad+\left|\sum_{s=1}^{\infty}\sum_{i=h+s\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i,x_{f_{1i}},\dots,x_{f_{ki}}\right)+\sum_{j=n_0}^{i-1}\left[g\left(j,x_{g_{1j}},\dots,x_{g_{kj}}\right)-c_j\right]\right\}\right| \\
&\leq 2\sum_{s=1}^{\infty}\sum_{i=M+s\tau}^{\infty}\frac{1}{|a_i|}\left[F_i+\sum_{j=n_0}^{i-1}\left(G_j+|c_j|\right)\right] \\
&\leq 2\sum_{i=M}^{\infty}\left(1+\frac{i}{\tau}\right)\frac{1}{|a_i|}\left[F_i+\sum_{j=n_0}^{i-1}\left(G_j+|c_j|\right)\right] \\
&< \varepsilon,
\end{aligned} \tag{3.58}$$

which means that $S_L(B(d,D))$ is uniformly Cauchy, which together with (3.42) and Lemma 2.1 yields that $S_L(B(d,R))$ is relatively compact. It follows from Lemma 2.3 that the mapping S_L has a fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in B(d,R)$; that is,

$$x_n = L-\sum_{s=1}^{\infty}\sum_{i=n+s\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i,x_{f_{1i}},\dots,x_{f_{ki}}\right)+\sum_{j=n_0}^{i-1}\left[g\left(j,x_{g_{1j}},\dots,x_{g_{kj}}\right)-c_j\right]\right\}, \quad n \geq T, \tag{3.59}$$

which means that

$$\Delta\left[a_n\Delta(x_n-x_{n-\tau})+f\left(n,x_{f_{1n}},\dots,x_{f_{kn}}\right)\right]+g\left(n,x_{g_{1n}},\dots,x_{g_{kn}}\right)=c_n, \quad n \geq T+\tau. \tag{3.60}$$

That is, $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$ is a bounded nonoscillatory solution of (1.1) in $B(d,D)$.

Let $L_1, L_2 \in (d-D, d+D)$ and $L_1 \neq L_2$. Similarly, we conclude that for each $l \in \{1,2\}$, there exist a positive integer $T_l \geq 1+n_0+n_1+\tau+|\beta|$ and a mapping $S_{L_l} : B(d,D) \to l_\beta^\infty$ satisfying (3.53), where $T, L,$ and S_L are replaced by $T_l, L_l,$ and S_{L_l}, respectively, and S_{L_l} has a fixed point $z^l \in B(d,D)$, which is a bounded nonoscillatory solution of (1.1); that is,

$$z_n^l = L_1-\sum_{s=1}^{\infty}\sum_{i=n+s\tau}^{\infty}\frac{1}{a_i}\left\{f\left(i,z_{f_{1i}}^l,\dots,z_{f_{ki}}^l\right)+\sum_{j=n_0}^{i-1}\left[g\left(j,z_{g_{1j}}^l,\dots,z_{g_{kj}}^l\right)-c_j\right]\right\}, \quad n \geq T, \tag{3.61}$$

for $l \in \{1,2\}$. Note that (3.41) implies that there exists $T_3 > \max\{T_1, T_2\}$ satisfying

$$\sum_{s=1}^{\infty}\sum_{i=T_3+s\tau}^{\infty}\frac{1}{|a_i|}\left(F_i+\sum_{j=n_0}^{i-1}G_j\right) < \frac{|L_1-L_2|}{4}, \tag{3.62}$$

which together with (3.2), (3.53), and (3.61) gives that

$$\begin{aligned}
&\left|z_n^1 - z_n^2\right| \\
&= \Bigg| L_1 - L_2 - \sum_{s=1}^{\infty}\sum_{i=n+s\tau}^{\infty}\frac{1}{a_i}\Bigg\{\left[f\left(j, z^1_{f_{1j}}, \dots, z^1_{f_{kj}}\right) - f\left(j, z^2_{f_{1j}}, \dots, z^2_{f_{kj}}\right)\right] \\
&\qquad + \sum_{j=n_0}^{i-1}\left[g\left(j, z^1_{g_{1j}}, \dots, z^1_{g_{kj}}\right) - g\left(j, z^2_{g_{1j}}, \dots, z^2_{g_{kj}}\right)\right]\Bigg\}\Bigg| \\
&\geq |L_1 - L_2| - 2\sum_{s=1}^{\infty}\sum_{i=T_3+s\tau}^{\infty}\frac{1}{|a_i|}\left(F_i + \sum_{j=n_0}^{i-1}G_j\right) \\
&> \frac{|L_1 - L_2|}{2} \\
&> 0, \quad \forall n \geq T_3,
\end{aligned} \tag{3.63}$$

that is, $z^1 \neq z^2$. Therefore, (1.1) possesses uncountably many bounded nonoscillatory solutions in $B(d, D)$. This completes the proof. □

Remark 3.11. Theorems 3.1 and 3.4–3.10 generalize Theorem 1 in [5] and Theorems 2.1–2.7 in [6, 7], respectively. The examples in Section 4 reveal that Theorems 3.1 and 3.4–3.10 extend authentically the corresponding results in [5–7].

4. Examples and Applications

Now, we construct ten examples to explain the advantage and applications of the results presented in Section 3. Note that Theorem 1 in [5] and Theorem 2.1–2.7 in [6, 7] are invalid for Examples 4.1–4.10, respectively.

Example 4.1. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[\left(n^6 \ln n\right)\Delta\left(x_n + \frac{(-1)^n(n-1)}{3n+1}x_{n-\tau}\right) + \frac{5n^3 x_{2n+1}}{n + x^2_{3n-15}}\right] + n^2 x_{n^3+1}x_{2n^2+3} = (n-1)^2, \quad n \geq 2, \tag{4.1}$$

where $n_0 = 2$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 6$, $k = 2$, $d = \pm 6$, $D = 5$, $b = 1/3$, $\beta = \min\{2 - \tau, -9\}$, and

$$\begin{gathered}
a_n = n^6 \ln n, \qquad b_n = \frac{(-1)^n(n-1)}{3n+1}, \qquad c_n = (n-1)^2, \qquad f(n, u, v) = \frac{5un^3}{n + v^2}, \\
f_{1n} = 2n + 1, \qquad f_{2n} = 3n - 15, \qquad g(n, u, v) = uvn^2, \qquad g_{1n} = n^3 + 1, \qquad g_{2n} = 2n^2 + 3, \\
F_n = 55n^2, \qquad G_n = 121n^2, \quad (n, u, v) \in \mathbb{N}_{n_0} \times [d - D, d + D]^2.
\end{gathered} \tag{4.2}$$

It is easy to show that (3.1)–(3.3) hold. It follows from Theorem 3.1 that (4.1) possesses uncountably many bounded nonoscillatory solutions in $B(d, D)$.

Example 4.2. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[n^2(1-2n)^3\Delta\left(x_n+\frac{(-1)^{n-1}(3+\sin n)}{4+\sin n}x_{n-\tau}\right)+3n^2x_{n^2}x_{5n}\right]+\frac{2nx_{2n-9}}{1+n^3|x_{3n^3}|}=n^2,\quad n\geq 2, \tag{4.3}$$

where $n_0 = 2$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 5$, $k = 2$, $d = \pm 2$, $D = 11$, $b = 5/6$, $\beta = \min\{2-\tau, -5\}$, and

$$\begin{gathered} a_n=n^2(1-2n)^3,\qquad b_n=\frac{(-1)^{n-1}(3+\sin n)}{4+\sin n},\qquad c_n=n^2,\qquad f(n,u,v)=3n^2uv,\\ f_{1n}=n^2,\qquad f_{2n}=5n,\qquad g(n,u,v)=\frac{2nu}{1+n^3|v|},\qquad g_{1n}=2n-9,\qquad g_{2n}=3n^3,\\ F_n=507n^2,\qquad G_n=26n,\quad (n,u,v)\in\mathbb{N}_{n_0}\times[d-D,d+D]^2. \end{gathered} \tag{4.4}$$

It is clear that (3.2), (3.3), and (3.21) hold. It follows from Theorem 3.2 that (4.3) possesses uncountably many bounded solutions in $B(d, D)$.

Example 4.3. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[\left(n^8\sin\frac{(-1)^n}{n}\right)\Delta\left(x_n+\frac{(-1)^{n-1}(4n^3+1)}{n^3+2n+2}x_{n-\tau}\right)+\frac{4n^2x_{n+1}^3}{1+nx_{n^2-1}^2}\right]+n^4x_{2n}^2x_{n-3}^3=n^3,\quad n\geq 1, \tag{4.5}$$

where $n_0 = 1$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 10$, $k = 2$, $d = \pm 1$, $D = 5$, $b_* = 3$, $b^* = 4$, $\beta = \min\{1-\tau, -2\}$, and

$$\begin{gathered} a_n=n^8\sin\frac{(-1)^n}{n},\qquad b_n=\frac{(-1)^{n-1}(4n^3+1)}{n^3+2n+2},\qquad c_n=n^3,\qquad f(n,u,v)=\frac{4nu^3}{1+nv^2},\\ f_{1n}=n+1,\qquad f_{2n}=n^2-1,\qquad g(n,u,v)=n^4u^2v^3,\qquad g_{1n}=2n,\qquad g_{2n}=n-3,\\ F_n=864n^2,\qquad G_n=7776n^4,\quad (n,u,v)\in\mathbb{N}_{n_0}\times[d-D,d+D]^2. \end{gathered} \tag{4.6}$$

It is easy to see that (3.2), (3.3), and (3.22) hold. It follows from Theorem 3.3 that (4.5) possesses uncountably many bounded solutions in $B(d, D)$.

Example 4.4. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[n^5\left(\sin\frac{1}{n}\right)^{-1/(\ln n)}\Delta\left(x_n+\frac{3n^2+1}{n^2+n+2}x_{n-\tau}\right)+\frac{nx_{n^2+3}^2}{1+n\cos^2 x_{2n}}\right]+\frac{n^4-x_{3n}}{n^2+x_{n+1}^2}=\frac{(-1)^n}{n-1},\quad n\geq 2, \tag{4.7}$$

where $n_0 = 2$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 5$, $k = 2$, $d = \pm 4$, $D = 3$, $b_* = 2$, $b^* = 3$, $\beta = 2 - \tau$, and

$$a_n = n^5 \left(\sin \frac{1}{n} \right)^{-1/(\ln n)}, \qquad b_n = \frac{3n^2 + 1}{n^2 + n + 2}, \qquad c_n = \frac{(-1)^n}{n-1}, \qquad f(n,u,v) = \frac{nu^2}{1 + n\cos^2 v},$$
$$f_{1n} = n^2 + 3, \qquad f_{2n} = 2n, \qquad g(n,u,v) = \frac{n^4 - u}{n^2 + v^2}, \qquad g_{1n} = 3n, \qquad g_{2n} = n + 1,$$
$$F_n = 49n, \qquad G_n = n^2 + 7, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [d - D, d + D]^2. \tag{4.8}$$

It is easy to show that (3.2), (3.3), and (3.34) hold. It follows from Theorem 3.4 that (4.7) has uncountably many bounded nonoscillatory solutions in $B(d,D)$.

Example 4.5. Consider the second-order nonlinear neutral delay difference equation

$$\Delta \left[n^6 \left(1 + \frac{1}{1+n} \right)^{3n} \Delta \left(x_n - \frac{6 - 2\ln(1+n^2)}{5 + \ln(1+n^2)} x_{n-\tau} \right) + n x_{2n+1} x_{n-9}^4 \right] + n^2 x_{3n} x_{4n}^3 = 2n^2, \quad n \geq 0, \tag{4.9}$$

where $n_0 = 0$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 10$, $k = 2$, $d = 9$, $D = 7$, $b_* = -2$, $b^* = -17/15$, $\beta = \min\{-\tau, -9\}$, and

$$a_n = n^6 \left(1 + \frac{1}{1+n} \right)^{3n}, \qquad b_n = -\frac{6 - 2\ \ln(1+n^2)}{5 + \ln(1+n^2)}, \qquad c_n = 2n^2, \qquad f(n,u,v) = nuv^4,$$
$$f_{1n} = 2n + 1, \qquad f_{2n} = n + 2, \qquad g(n,u,v) = n^2 uv^3, \qquad g_{1n} = 3n, \qquad g_{2n} = 4n,$$
$$F_n = 1048576n, \qquad G_n = 65536n^2, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [d - D, d + D]^2. \tag{4.10}$$

It is easy to show that (3.2), (3.3), and (3.35) hold. It follows from Theorem 3.5 that (4.9) has uncountably many bounded positive solutions in $B(d,D)$.

Example 4.6. Consider the second-order nonlinear neutral delay difference equation

$$\Delta \left[(-1)^{n-1} n^{17} (n-4)^5 \Delta \left(x_n + \frac{3 - 2n^4}{5 + n + n^4} x_{n-\tau} \right) + \frac{n^{18} - n^7 x_{3n-19}^2}{\ln(3 + n^5 |x_{2n}|)} \right] + \frac{n^2 x_{n+5}^2}{1 + x_{2n+3}^2} = n^{15}, \quad n \geq 5, \tag{4.11}$$

where $n_0 = 5$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 5$, $k = 2$, $d = -9$, $D = 7$, $b_* = -2$, $b^* = -\frac{17}{15}$, $\beta = \min\{5-\tau, -4\}$, and

$$a_n = (-1)^{n-1} n^{17} (n-4)^5, \qquad b_n = \frac{3-2n^4}{5+n+n^4}, \qquad c_n = n^{15}, \qquad f(n,u,v) = \frac{n^{18} - n^7 u^2}{\ln(3+n^5|v|)},$$
$$f_{1n} = 3n - 18, \qquad f_{2n} = 2n, \qquad g(n,u,v) = \frac{n^2 u^2}{1+v^2}, \qquad g_{1n} = n+5, \qquad g_{2n} = 2n+3,$$
$$F_n = n^{18} + 256n^7, \qquad G_n = 256n^2, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [d-D, d+D]^2. \tag{4.12}$$

It is easy to show that (3.2), (3.3), and (3.36) hold. It follows from Theorem 3.6 that (4.11) has uncountably many bounded negative solutions in $B(d, D)$.

Example 4.7. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[n^8 \ln\left(\cos\frac{\pi}{n}\right)\Delta\left(x_n + \frac{5n^2-2n+170}{6n^2+n+1}x_{n-\tau}\right) - \frac{n^2 x_{2n}}{n+x_{4n}^2}\right] + \frac{n^2 - x_{2n-3}^3}{1+n|x_{n-6}|} = n^2(2-n), \quad n \geq 3, \tag{4.13}$$

where $n_0 = 3$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 60$, $k = 2$, $d = \pm 7$, $D = 6$, $b^* = 5/6$, $\beta = \min\{3-\tau, -3\}$, and

$$a_n = n^8 \ln\left(\cos\frac{\pi}{n}\right), \qquad b_n = \frac{5n^2-2n+170}{6n^2+n+1}, \qquad c_n = n^2(2-n), \qquad f(n,u,v) = -\frac{n^2 u}{n+v^2},$$
$$f_{1n} = 2n, \qquad f_{2n} = 4n, \qquad g(n,u,v) = \frac{n^2 - u^3}{1+n|v|}, \qquad g_{1n} = 2n-3, \qquad g_{2n} = n-6,$$
$$F_n = 13n, \qquad G_n = 2197 + n^2, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [d-D, d+D]^2. \tag{4.14}$$

It is clear (3.2), (3.3), and (3.37) hold. It follows from Theorem 3.7 that (4.13) has uncountably many bounded nonoscillatory solutions in $B(d, D)$.

Example 4.8. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[n^6 \Delta\left(x_n + \frac{1-8n^4}{3+9n^4}x_{n-\tau}\right) + \frac{n^3 x_{n+1} - (n^2+1)x_{n-3}^2}{1+nx_{n-3}^2}\right] + \frac{n^2 + x_{2n+5}^3}{2+n^2|x_{3n-1}|} = (-1)^n n^3, \quad n \geq 4, \tag{4.15}$$

where $n_0 = 4$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 4$, $k = 2$, $d = \pm 7$, $D = 6$, $b_* = -8/9$, $\beta = \min\{4-\tau, 1\}$, and

$$a_n = n^6, \qquad b_n = \frac{1-8\mathrm{n}^4}{3+9n^4}, \qquad c_n = (-1)^n n^3, \qquad f(n,u,v) = \frac{n^3u-(n^2+1)v^2}{1+nv^2},$$
$$f_{1n} = n+1, \qquad f_{2n} = n-3, \qquad g(n,u,v) = \frac{n^2+u^3}{2+n^2|v|}, \qquad g_{1n} = 2n+5, \qquad g_{2n} = 3n-1,$$
$$F_n = 13n^3 + 169\left(n^2+1\right), \qquad G_n = n^2 + 2197, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [d-D, d+D]^2. \tag{4.16}$$

It is clear (3.2), (3.3), and (3.38) hold. It follows from Theorem 3.8 that (4.15) has uncountably many bounded nonoscillatory solutions in $B(d,D)$.

Example 4.9. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[n^6\left(1+\frac{1}{n}\right)^n \Delta(x_n + x_{n-\tau}) + \frac{(n-1)^2 - nx_{3n+1}}{n^2\ln(3+nx_{6n}^2)}\right] + \frac{1-n^3+nx_{7n}^2}{1+n+n^5\left|x_{3n}^5x_{7n}^3\right|} = \frac{(-1)^n n^2}{n^3+1}, \quad n \ge 1, \tag{4.17}$$

where $n_0 = 1$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 1$, $k = 2$, $d = \pm 6$, $D = 2$, $\beta = 1-\tau$, and

$$a_n = n^6\left(1+\frac{1}{n}\right)^n, \qquad c_n = \frac{(-1)^n n^2}{n^3+1}, \qquad f(n,u,v) = \frac{(n-1)^2 - nu}{n^2\ln(3+nv^2)}, \qquad f_{1n} = 3n+1,$$
$$f_{2n} = 6n, \qquad g(n,u,v) = \frac{1-n^3+nu^2}{1+n+n^5|v^5u^3|}, \qquad g_{1n} = 7n, \qquad g_{2n} = 3n, \qquad F_n = 1+\frac{8}{n},$$
$$G_n = 1+64n+n^3, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [d-D, d+D]^2. \tag{4.18}$$

It is clear (3.2), (3.3), and (3.39) hold. It follows from Theorem 3.9 that (4.17) has uncountably many bounded nonoscillatory solutions in $B(d,D)$.

Example 4.10. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left[n^4(1-2n)(2n-3)^2\Delta(x_n - x_{n-\tau}) + \frac{5nx_{3n^3-n+2}}{2+n^3|x_{5n^5+3}|}\right] + n^2x_{3n^3+4}x_{4n^3-5} = n^2, \quad n \ge 2, \tag{4.19}$$

where $n_0 = 2$ and $\tau \in \mathbb{N}$ are fixed. Let $n_1 = 2$, $k = 2$, $d = \pm 10$, $D = 6$, $\beta = 2-\tau$, and

$$a_n = n^4(1-2n)(2n-3)^2, \qquad c_n = n^2, \qquad f(n,u,v) = \frac{5nu}{2+n^3|v|}, \qquad f_{1n} = 3n^3 - n + 2,$$
$$f_{2n} = 5n^5 + 3, \qquad g(n,u,v) = uvn^2, \qquad g_{1n} = 3n^3 + 4, \qquad g_{2n} = 4n^3 - 5, \qquad F_n = 40n,$$
$$G_n = 256n^2, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [d-D, d+D]^2. \tag{4.20}$$

It is clear (3.2), (3.50), and (3.51) hold. It follows from Theorem 3.10 that (4.19) possesses uncountably bounded nonoscillatory solutions in $B(d, D)$.

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