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Research Article

Two-Point Oscillation for a Class of Second-Order Damped Linear Differential Equations

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Using the comparison theorem, the two-point oscillation for linear differential equation with damping term $y'' + (f(x)/(x-x^2)^{\alpha})y' + (g(x)/(x-x^2)^{\beta})y = 0$ is considered, where $\alpha, \beta > 0; f(x), g(x) > 0$, and $f(x), g(x) \in C(\overline{I}), I = (0,1)$. Results are obtained that $0 < \alpha < 3/2, \beta > 3$ or $\alpha > 3/2, \beta > 2\alpha$ imply the two-point oscillation of the equation.

1. Introduction

Under the solution $y(x)$ of a differential equation appearing in the paper, we mean a function $y = y(x)$ such that $y \in C^2(I)$. Here, we allow that $y \notin C(\overline{I})$.

Recently, two-point oscillation of the differential equations caused the concern of many scholars ([1, 2]). In paper [1], Pašić and Wong construct the equation:

$$y'' + \left[\frac{1}{2}S(q')(x) + (q')^2(x)\right]y = 0, \quad x \in I, \tag{1.1}$$

where $I = (0,1)$, $q \in C^3(I)$, $|q(0+)| = |q(1-)| = +\infty$, $|q'(0+)| = |q'(1-)| = +\infty$, $q'(x) < 0$, $x \in I$, $S(q') \in C(I)$, $S(q')(x) = (q'''(x)/q'(x)) - (3/2)[q''(x)/q'(x)]^2$, they study the following equation:

$$y'' + \frac{c(x)}{(x-x^2)^{\sigma}}y = 0, \quad x \in I, \tag{1.2}$$

by comparison theorem (where $c(x) > 0$, $c(x) \in C(\overline{I})$), and they obtain that when $\sigma > 2$, (1.2) is two-point oscillatory.

In this paper, we construct the following equation with damping:

$$(p'(x)y')' - 2p''(x)y' + (p'(x))^3 y = 0, \quad x \in I, \tag{1.3}$$

where $p(x) \in C^2(I)$ and

$$|p(x)(0+)| = |p(x)(1-)| = \infty. \tag{1.4}$$

we study the two-point oscillation of the following damped equation by comparison theorem

$$y'' + \frac{f(x)}{(x - x^2)^{\alpha}} y' + \frac{g(x)}{(x - x^2)^{\beta}} y = 0, \tag{1.5}$$

where $x \in I, \alpha, \beta > 0$; $f(x), g(x) > 0$, $f(x), g(x) \in C(\overline{I})$, the result we obtained is new, and it continues the results obtained in [1].

2. Two-Point Oscillation of (1.3)

Definition 2.1. A function $y = y(x)$, $y(x) \in C(I)$ is said to be two-point oscillation on the interval I, if there exist a decreasing sequence $a_{(k)} \in I$ and an increasing sequence $b_{(k)} \in I$ of consecutive zeros of $y(x)$ such that $a_{(k)} \searrow 0$ and $b_{(k)} \nearrow 1$.

Definition 2.2. A linear differential equation is said to be two-point oscillation on I if all its nontrivial solutions $y = y(x)$, $y(x) \in C^2(I)$ are two-point oscillatory on I.

By Sturm separation theorem, all nontrivial solutions of a linear differential equation are two-point oscillatory if there is a nontrivial solution is two-point oscillatory on I.

We know that $y_1(x) = \cos p(x)$, $y_2(x) = \sin p(x)$ are two linearly independent solutions of (1.3), so the general solution of (1.3) can be expressed as

$$y(x) = c_1 \cos p(x) + c_2 \sin p(x). \tag{2.1}$$

Because of $|p(x)(0+)| = |p(x)(1-)| = \infty$, the function $y(x)$ is two-point oscillatory on I, then (1.3) is two-point oscillatory on I.

Example 2.3. Let $p(x) = -\ln\ln(1/x)$, $x \in I$, then $p'(x) = 1/(x\ln(1/x))$, $p''(x) = (1 - \ln(1/x))/(x\ln(1/x))^2$, $p(x)$ satisfies the condition (1.4), so the following equation:

$$\left(\frac{1}{x\ln(1/x)} y'\right)' - 2\frac{1 - \ln(1/x)}{(x\ln(1/x))^2} y' + \left(\frac{1}{x\ln(1/x)}\right)^3 y = 0 \tag{2.2}$$

is two-point oscillatory on I.

Example 2.4. Let $p(x) = -(1-2x)/(x-x^2)^{\varepsilon}$, $x \in I$, where $\varepsilon > 0$. Then,

$$p'(x) = \frac{2(x-x^2) + \varepsilon(1-2x)^2}{(x-x^2)^{(\varepsilon+1)}},$$
$$p''(x) = -\frac{(\varepsilon+1)(1-2x)(x-x^2)^{\varepsilon}\left[2(x-x^2)+\varepsilon(1-2x)^2\right] - (2-4\varepsilon)(1-2x)(x-x^2)^{\varepsilon+1}}{(x-x^2)^{2\varepsilon+2}}, \tag{2.3}$$

when $x \to 0$, $p(x) \to -\infty$; when $x \to 1$, $p(x) \to -\infty$, which satisfies the condition (1.4); $p'(x) \in C(I)$, $p''(x) \in C(I)$. Substituting $p'(x)$ and $p''(x)$ into (1.3), we obtain that the following equation:

$$\begin{aligned}&\left(\frac{2(x-x^2)+\varepsilon(1-2x)^2}{(x-x^2)^{(\varepsilon+1)}}y'\right)' \\ &\quad -2\frac{(2-4\varepsilon)(1-2x)(x-x^2)^{\varepsilon+1} - (\varepsilon+1)(1-2x)(x-x^2)^{\varepsilon}\left[2(x-x^2)+\varepsilon(1-2x)^2\right]}{(x-x^2)^{2\varepsilon+2}}y' \\ &\quad +\left(\frac{2(x-x^2)+\varepsilon(1-2x)^2}{(x-x^2)^{(\varepsilon+1)}}\right)^3 y = 0\end{aligned} \tag{2.4}$$

is two-point oscillatory on I.

3. A New Comparison Theorem

Theorem 3.1. *Suppose that the second order differential equations*

$$(p_1(x)y'(x))' + r_1(x)y'(x) + q_1(x)y(x) = 0, \tag{3.1}$$

$$(p_2(x)z'(x))' + r_2(x)z'(x) + q_2(x)z(x) = 0, \tag{3.2}$$

satisfy the existence and uniqueness theorem on I, and one of the following conditions holds:

(1) *when $0 < p_2 < p_1$ and $r_1 \neq r_2$,*

$$q_2 \geq q_1 + \frac{r_2^2}{4p_2} + \frac{(r_1-r_2)^2}{4(p_1-p_2)}, \tag{3.3}$$

(2) *when $0 < p_1 = p_2 = p$ and $r_1 \neq r_2$, there exists an $n \in \mathbb{R}^+$, which satisfies*

$$q_2 \geq (1+n)q_1 + \frac{((1+n)r_1 - r_2)^2 + r_2{}^2}{4p}, \tag{3.4}$$

then (3.2) *has at least one zero point between two consecutive zero point* α, β $(\alpha < \beta)$ *of any nontrivial solution* $y(x)$ *of* (3.1).

Proof. (1) We suppose that $z(x)$ has no zero point on $[\alpha, \beta]$ when $0 < p_2 < p_1$. Without loss of generality, let $y(x) \geq 0$, $z(x) > 0$, $x \in [\alpha, \beta]$, then we have

$$
\begin{aligned}
&\frac{d}{dx}\left[\frac{y}{z}(zp_1y' - yp_2z')\right] \\
&\quad = \frac{y}{z}\left[z(p_1y')' + p_1z'y' - y(p_2z')' - p_2y'z'\right] + \frac{y'z - z'y}{z^2}(p_1y'z - p_2yz') \\
&\quad = \frac{y}{z}\left[z(-q_1y - r_1y') - y(-q_2z - r_2z') + (p_1 - p_2)y'z'\right] + \frac{y'z - z'y}{z^2}(p_1y'z - p_2yz') \\
&\quad = (q_2 - q_1)y^2 + r_2\frac{y^2z'}{z} - r_1yy' + (p_1 - p_2)\frac{yy'z'}{z} + p_1(y')^2 - p_1\frac{yy'z'}{z} \\
&\qquad - p_2\frac{yy'zz'}{z^2} + p_2\frac{(yz')^2}{z^2} \\
&\quad = (q_2 - q_1)y^2 + r_2\frac{y^2z'}{z} - r_1yy' + (p_1 - p_2)(y')^2 + p_2\left(y' - \frac{yz'}{z}\right)^2 \\
&\quad = (q_2 - q_1)y^2 + \left(\sqrt{p_1 - p_2}\,y'\right)^2 - (r_1 - r_2)yy' - r_2y\left(y' - \frac{yz'}{z}\right) + \left[\sqrt{p_2}\left(y' - \frac{yz'}{z}\right)\right]^2 \\
&\qquad + \left(\frac{r_1 - r_2}{2\sqrt{p_1 - p_2}}y\right)^2 + \left(\frac{r_2y}{2\sqrt{p_2}}\right)^2 - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)}y^2 - \frac{r_2^2}{4p_2}y^2 \\
&\quad = \left[\sqrt{p_1 - p_2}\,y' - \frac{r_1 - r_2}{2\sqrt{p_1 - p_2}}y\right]^2 + \left[\frac{r_2y}{2\sqrt{p_2}} - \sqrt{p_2}\left(y' - \frac{yz'}{z}\right)\right]^2 \\
&\qquad + \left[q_2 - q_1 - \frac{r_2^2}{4p_2} - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)}\right]y^2.
\end{aligned} \tag{3.5}
$$

Integrating the above equation from α to β, we obtain

$$
\begin{aligned}
0 &= \int_\alpha^\beta \left[\sqrt{p_1 - p_2}\,y' - \frac{r_1 - r_2}{2\sqrt{p_1 - p_2}}y\right]^2 dx \\
&\quad + \int_\alpha^\beta \left[\frac{r_2y}{2\sqrt{p_2}} - \sqrt{p_2}\left(y' - \frac{yz'}{z}\right)\right]^2 dx \\
&\quad + \int_\alpha^\beta \left[q_2 - q_1 - \frac{r_2^2}{4p_2} - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)}\right] y^2 dx,
\end{aligned} \tag{3.6}
$$

that is,

$$0 \geq \int_{\alpha}^{\beta}\left[q_2 - q_1 - \frac{r_2^2}{4p_2} - \frac{(r_1 - r_2)^2}{4(p_1 - p_2)}\right]y^2 dx. \tag{3.7}$$

From previous equality and assumption (3.3), we obtain the next equalities:

$$\sqrt{p_1 - p_2}\,y' = \frac{r_1 - r_2}{2\sqrt{p_1 - p_2}}y, \tag{3.8}$$

$$\frac{r_2 y}{2\sqrt{p_2}} = \sqrt{p_2}\left(y' - \frac{yz'}{z}\right). \tag{3.9}$$

By (3.8), we obtain $y'/y = (r_1 - r_2)/2(p_1 - p_2)$, $y = e^{\int((r_1 - r_2)/2(p_1 - p_2))dx}$. By (3.9), we obtain $(y'/y) - (z'/z) = (r_2/2p_2)$. In summary, we obtain $z = ye^{-\int(r_2/2p_2)dx}$, that is, $z(\alpha) = z(\beta) = 0$, which contradicts with the assumption.

(2) We suppose that $z(x)$ has no zero point on $[\alpha, \beta]$ when $0 < p_1 = p_2 = p$. Without loss of generality, let $y(x) \geq 0$, $z(x) > 0$, $x \in [\alpha, \beta]$, for all $n \in \mathbb{R}^+$, then

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{y}{z}(zpy' - ypz') + nypy'\right] \\
&\quad = \frac{y}{z}\left(z(py')' - y(pz')'\right) + p\frac{y'z - z'y}{z^2}(y'z - yz') + p(y')^2 + ny(-q_1 y - r_1 y') \\
&\quad = \frac{y}{z}\left[z(-q_1 y - r_1 y') - y(-q_2 z - r_2 z')\right] + p\frac{(y'z - z'y)^2}{z^2} + p(y')^2 + ny(-q_1 y - r_1 y') \\
&\quad = \left[(q_2 - (1+n)q_1)y^2 + r_2\frac{y^2 z'}{z} - (1+n)r_1 yy'\right] + p\frac{(y'z - z'y)^2}{z^2} + p(y')^2 \\
&\quad = \left[(q_2 - (1+n)q_1)y^2 - ((1+n)r_1 - r_2)yy' - r_2 y\left(y' - \frac{yz'}{z}\right)\right] + p\frac{(y'z - z'y)^2}{z^2} + p(y')^2 \\
&\quad = (q_2 - (1+n)q_1)y^2 - ((1+n)r_1 - r_2)yy' - r_2 y\left(y' - \frac{yz'}{z}\right) \\
&\qquad + p\left(y' - \frac{yz'}{z}\right)^2 + p(y')^2 + \frac{r_2^2}{4p}y^2 + \frac{(2r_1 - r_2)^2}{4p}y^2 - \frac{((1+n)r_1 - r_2)^2}{4p}y^2 - \frac{r_2^2}{4p}y^2 \\
&\quad = \left[\frac{r_2 y}{2\sqrt{p}} - \sqrt{p}\left(y' - \frac{yz'}{z}\right)\right]^2 + \left[\frac{((1+n)r_1 - r_2)y}{2\sqrt{p}} - y'\sqrt{p}\right]^2 \\
&\qquad + \left[q_2 - (1+n)q_1 - \frac{(2r_1 - r_2)^2 + r_2{}^2}{4p}\right]y^2.
\end{aligned} \tag{3.10}$$

Integrating the above equation from α to β, we obtain

$$0 \geq \int_{\alpha}^{\beta} \left[q_2 - (1+n)q_1 - \frac{((1+n)r_1 - r_2)^2 + {r_2}^2}{4p} \right] y^2 dx. \tag{3.11}$$

We can find the contradiction similarly; here, we delete the details. This completes the proof. □

When $z(x) = w(z(x))$ is a nonlinear term, where $w(z) : R \to R$ is a continuous function and $zw(z) > 0$ for $z \neq 0$, Zhuang and Wu established some comparison theorems if $w'(z) \geq K > 0$ holds in [3]. The condition of Corollary 2.2 in [3] is identical with (3.3) when $w'(z)$ is smooth and $K = 1$, but there's no condition about the situation of $p_1 = p_2 = p$. We put "$nyp'y$" added to Picone identity, which solve the problem of the vacuousness of (3.3) when $p_1 = p_2 = p$. Then, we obtain (3.4) and establish the integrated comparison theorem of second order damped linear differential equations.

We can easily obtain the following corollaries by Theorem 3.1.

Corollary 3.2. *Suppose* (3.1), (3.2) *satisfy the existence and uniqueness theorem on* I. *If* (3.1) *is two-point oscillatory on* I, $r_1 \neq r_2$ *and satisfies one of the following conditions:*

(1) *when* $0 < p_2 < p_1$, *the following condition is satisfied on* I,

$$q_2 \geq q_1 + \frac{r_2^2}{4p_2} + \frac{(r_2 - r_1)^2}{4(p_1 - p_2)}, \tag{3.12}$$

(2) *when* $0 < p_1 = p_2 = p$, *the following condition is satisfied on* I,

$$q_2 \geq (1+n)q_1 + \frac{((1+n)r_1 - r_2)^2 + {r_2}^2}{4p}, \tag{3.13}$$

then (3.2) *is two-point oscillatory on* I.

Corollary 3.3. *Consider the second order equation* (1.3) *and the following equation:*

$$(A(x)y')' + B(x)y' + C(x)y = 0, \tag{3.14}$$

where $x \in I$, $p(x)$ *satisfies condition* (1.4), $B(x) \neq -2p''(x)$. *Suppose they satisfy the existence and uniqueness theorem on* I. *When* $x \to 0+$ *and* $x \to 1-$, *if* $0 < A(x) < p'(x)$ *is satisfied on* I, *and*

$$C(x) \geq (p'(x))^3 + \frac{B(x)^2}{4A(x)} + \frac{(B(x) + 2p''(x))^2}{4(p'(x) - A(x))}, \tag{3.15}$$

then (3.14) *is two-point oscillatory on* I.

Remark 3.4. The two-point oscillation of (1.2) is studied by comparison theorem and two-point oscillatory equation in [1]. When $p_1(x) = p_2(x) = 1$ and $r_1(x) = r_2(x) = 0$, Theorem 3.1 reduces to Theorem 2.1 in [1].

As an application of Corollary 3.3, we discuss the two-point oscillation of (1.5). Since Example 2.4 is the known two-point oscillatory equation, that is $p(x) = -(1-2x)/(x-x^2)^{\varepsilon}$, $x \in I$, where $\varepsilon > 0$, $p'(x) \sim (x-x^2)^{-(\varepsilon+1)}$ as $x \to 0+$ or $x \to 1-$; $p''(x) \sim (x-x^2)^{-(\varepsilon+2)}$ as $x \to 0+$ or $x \to 1-$. For (1.5), $A(x) = 1$, $B(x) = f(x)/(x-x^2)^{\alpha}$, $C(x) = g(x)/(x-x^2)^{\beta}$. Because of $f(x) > 0$, $f(x) \in C(\overline{I})$, there exists $M > 0$ such that $f(x) < M$ for all $x \in I$. Therefore,

$$\begin{aligned} B(x) &= \frac{f(x)}{(x-x^2)^{\alpha}} \sim \left(x-x^2\right)^{-\alpha}, \quad \text{as } x \longrightarrow 0+ \text{ or } x \longrightarrow 1-, \\ C(x) &= \frac{g(x)}{(x-x^2)^{\beta}} \sim \left(x-x^2\right)^{-\beta}, \quad \text{as } x \longrightarrow 0+ \text{ or } x \longrightarrow 1-, \end{aligned} \tag{3.16}$$

Thus, when $x \to 0+$ and $x \to 1-$, $p'(x) \to +\infty$, $A(x) \equiv 1$, $-2p''(x) \neq B(x)$,

$$\begin{aligned} \frac{(B(x)+2p''(x))^2}{4(p'(x)-A(x))} &\sim \frac{\left[(x-x^2)^{-\alpha} + (x-x^2)^{-\varepsilon-2}\right]^2}{(x-x^2)^{-(\varepsilon+1)}} \\ &\sim \left(x-x^2\right)^{-(3+\varepsilon)} + \left(x-x^2\right)^{-(2\alpha-\varepsilon-1)} + \left(x-x^2\right)^{-(\alpha+1)}, \\ &\qquad \text{as } x \to 0+ \text{ or } x \to 1-, \\ \frac{B(x)^2}{4A(x)} &\sim \left(x-x^2\right)^{-2\alpha}, \quad \text{as } x \longrightarrow 0+ \text{ or } x \longrightarrow 1-, \\ (p')^3 &= \left[\frac{2(x-x^2)+\varepsilon(1-2x)^2}{(x-x^2)^{(\varepsilon+1)}}\right]^3 \sim \left(x-x^2\right)^{-3(\varepsilon+1)}, \quad \text{as } x \longrightarrow 0+ \text{ or } x \longrightarrow 1-. \end{aligned} \tag{3.17}$$

By (3.17), the following condition need to be satisfied if (3.15) holds,

$$\left(x-x^2\right)^{-\beta} \geq \left(x-x^2\right)^{-3(\varepsilon+1)} + \left(x-x^2\right)^{-2\alpha} + \left(x-x^2\right)^{-(3+\varepsilon)} + \left(x-x^2\right)^{-(2\alpha-\varepsilon-1)} + \left(x-x^2\right)^{-(\alpha+1)}, \quad \text{as } x \longrightarrow 0+ \text{ or } x \longrightarrow 1-, \tag{3.18}$$

that is,

$$\beta > \max\{3(\varepsilon+1), 2\alpha, \alpha+1\}. \tag{3.19}$$

In summary, let $\varepsilon \to 0$, then,

when $0 < \alpha < 3/2$, $3 + 3\varepsilon > \max\{2\alpha, \alpha + 1\}$, condition (3.19) holds with $\beta > 3$, (1.5) is two-point oscillatory on I in this case,

when $\alpha > 3/2$, $2\alpha > \max\{3 + 3\varepsilon, \alpha + 1\}$, condition (3.19) holds with $\beta > 2\alpha$, (1.5) is two-point oscillatory on I in this case.

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