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Research Article

Approximation of Analytic Functions by Chebyshev Functions

Soon-Mo Jung[1] and Themistocles M. Rassias[2]

[1] *Mathematics Section, College of Science and Technology, Hongik University, Jochiwon 339-701, Republic of Korea*
[2] *Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece*

Correspondence should be addressed to Soon-Mo Jung, smjung@hongik.ac.kr

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We solve the inhomogeneous Chebyshev's differential equation and apply this result for approximating analytic functions by the Chebyshev functions.

1. Introduction

Let X be a normed space over a scalar field $\mathbb{K}$, and let $I \subset \mathbb{R}$ be an open interval, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Assume that $a_0, a_1, \ldots, a_n : I \to \mathbb{K}$, and $g : I \to X$ are given continuous functions and that $y : I \to X$ is an n times continuously differentiable function satisfying the inequality

$$\left\| a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + g(t) \right\| \le \varepsilon \tag{1.1}$$

for all $t \in I$ and for a given $\varepsilon > 0$. If there exists an n times continuously differentiable function $y_0 : I \to X$ satisfying

$$a_n(t)y_0^{(n)}(t) + a_{n-1}(t)y_0^{(n-1)}(t) + \cdots + a_1(t)y_0'(t) + a_0(t)y_0(t) + g(t) = 0 \tag{1.2}$$

and $\|y(t) - y_0(t)\| \le K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon\to 0}K(\varepsilon) = 0$, then we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [1–7].

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [8, 9]). Here, we will introduce a result of Alsina and Ger [10]. They proved that if a differentiable function $f : I \to \mathbb{R}$ satisfies the inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a constant c such that $|f(t) - ce^t| \leq 3\varepsilon$ for any $t \in I$. Their result was generalized by Takahasi et al. Indeed, it was proved in [11] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [12, 13]).

Moreover, Miura et al. [14] investigated the Hyers-Ulam stability of nth order linear differential equation with complex coefficients. They [15] also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function.

Jung also proved the Hyers-Ulam stability of various linear differential equations of first order [16–19]. Moreover, he applied the power series method to the study of the Hyers-Ulam stability of Legendre's differential equation (see [20, 21]). Recently, Jung and Kim tried to prove the Hyers-Ulam stability of the Chebyshev's differential equation

$$\left(1 - x^2\right)y''(x) - xy'(x) + n^2 y(x) = 0 \tag{1.3}$$

for all $x \in (-1, 1)$. However, the obtained theorem unfortunately does not describe the Hyers-Ulam stability of the Chebyshev's differential equation in a strict sense (see [22]).

In Section 2 of this paper, by using the ideas from [20–26], we investigate the general solution of the inhomogeneous Chebyshev's differential equation of the form

$$\left(1 - x^2\right)y''(x) - xy'(x) + n^2 y(x) = \sum_{m=0}^{\infty} a_m x^m, \tag{1.4}$$

where n is a given positive integer. Section 3 will be devoted to the investigation of the Hyers-Ulam stability and an approximation property of the Chebyshev functions.

2. Inhomogeneous Chebyshev's Equation

Every solution of the Chebyshev's differential equation (1.3) is called a Chebyshev function. The Chebyshev's differential equation has regular singular points at -1, 1, and ∞, and it plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting certain symmetries.

In this section, we set $c_0 = c_1 = 0$ and define, for all $m \in \mathbb{N}$,

$$\begin{aligned} c_{2m} &= \frac{1}{2m}\sum_{i=0}^{m-1} \frac{a_{2i}}{2i+1} \prod_{j=i+1}^{m-1} \frac{(2j)^2 - n^2}{2j(2j+1)}, \\ c_{2m+1} &= \frac{1}{2m+1}\sum_{i=0}^{m-1} \frac{a_{2i+1}}{2i+2} \prod_{j=i+1}^{m-1} \frac{(2j+1)^2 - n^2}{(2j+1)(2j+2)}, \end{aligned} \tag{2.1}$$

where we refer to (1.4) for the a_m's and we follow the convention $\prod_{j=m}^{m-1}[\cdots] = 1$. We can easily check that c_m's satisfy the following relation:

$$(m+2)(m+1)c_{m+2} - \left(m^2 - n^2\right)c_m = a_m \tag{2.2}$$

for any $m \in \{0, 1, 2, \ldots\}$.

Theorem 2.1. *Assume that n is a positive integer and the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$. Let $\rho_0 = \min\{1, \rho\}$. Then, every solution $y : (-\rho_0, \rho_0) \to \mathbb{C}$ of the Chebyshev's differential equation* (1.4) *can be expressed by*

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \tag{2.3}$$

where $y_h(x)$ is a Chebyshev function and the c_m's are given in (2.1).

Proof. It is not difficult to see that, if $j \in \mathbb{N}$ and $|(2j)^2 - n^2| > 2j(2j+1)$, then

$$j < \frac{-1 + \sqrt{1 + 8n^2}}{8} < \frac{\sqrt{8n^2}}{8} \quad (\text{for } 2j < n). \tag{2.4}$$

Hence, we have $1 \le j \le n_e$ with $n_e = [n/\sqrt{8}]$. If $m > n_e$, then it follows from (2.1) that

$$\begin{aligned}
|c_{2m}| &\le \frac{1}{2m}\sum_{i=0}^{n_e-1}\frac{|a_{2i}|}{2i+1}\left(\prod_{j=i+1}^{n_e}\frac{\left|(2j)^2 - n^2\right|}{2j(2j+1)}\right)\left(\prod_{j=n_e+1}^{m-1}\frac{\left|(2j)^2 - n^2\right|}{2j(2j+1)}\right) \\
&\quad + \frac{1}{2m}\sum_{i=n_e}^{m-1}\frac{|a_{2i}|}{2i+1}\prod_{j=i+1}^{m-1}\frac{\left|(2j)^2 - n^2\right|}{2j(2j+1)} \\
&\le \frac{1}{2m}\sum_{i=0}^{n_e-1}\frac{|a_{2i}|}{2i+1}\prod_{j=i+1}^{n_e}\frac{n^2-4}{2j(2j+1)} + \frac{1}{2m}\sum_{i=n_e}^{m-1}\frac{|a_{2i}|}{2i+1} \\
&\le \frac{1}{2m}\sum_{i=0}^{n_e-1}\frac{(2i)!\left(n^2-4\right)^{n_e-i}}{(2n_e+1)!}|a_{2i}| + \frac{1}{2m}\sum_{i=n_e}^{m-1}\frac{|a_{2i}|}{2n_e+1} \\
&\le \frac{\max_{0\le i\le n_e}((2i)!/(2n_e+1)!)\left(n^2-4\right)^{n_e-i}}{2m}\sum_{i=0}^{m-1}|a_{2i}|.
\end{aligned} \tag{2.5}$$

We now suppose $1 \leq m \leq n_e$. Then it holds true that $n \geq 3$, and we have

$$\begin{aligned}
|c_{2m}| &\leq \frac{1}{2m}\sum_{i=0}^{m-1}\frac{|a_{2i}|}{2i+1}\prod_{j=i+1}^{m-1}\frac{\left|(2j)^2-n^2\right|}{2j(2j+1)} \\
&\leq \frac{1}{2m}\sum_{i=0}^{m-1}\frac{|a_{2i}|}{2i+1}\prod_{j=i+1}^{m-1}\frac{n^2-4}{2j(2j+1)} \\
&= \frac{1}{2m}\sum_{i=0}^{m-1}\frac{(2i)!\left(n^2-4\right)^{m-1-i}}{(2m-1)!}|a_{2i}| \\
&\leq \frac{\max_{0\leq i\leq m-1}((2i)!/(2m-1)!)\left(n^2-4\right)^{m-1-i}}{2m}\sum_{i=0}^{m-1}|a_{2i}|.
\end{aligned} \tag{2.6}$$

Hence, we conclude from the above two inequalities that

$$|c_{2m}| \leq \frac{M_e}{2m}\sum_{i=0}^{m-1}|a_{2i}| \tag{2.7}$$

for all $m \in \mathbb{N}$, where we set

$$M_e = \max_{0\leq i\leq \ell\leq n_e}\frac{(2i)!}{(2\ell+1)!}\left(n^2-4\right)^{\ell-i}. \tag{2.8}$$

On the other hand, if $j \in \mathbb{N}$ and $|(2j+1)^2-n^2| > (2j+1)(2j+2)$, then

$$j < \frac{\sqrt{8n^2+1}-5}{8} < \frac{\sqrt{8n^2}-4}{8} < \frac{n}{2}-\frac{1}{2} \quad (\text{for } 2j+1<n). \tag{2.9}$$

Hence, we get $1 \leq j \leq n_o$ with $n_o = [\mathrm{n}/\sqrt{8}-1/2]$. If $m > n_o$, then it follows from (2.1) that

$$\begin{aligned}
|c_{2m+1}| &\leq \frac{1}{2m+1}\sum_{i=0}^{n_o-1}\frac{|a_{2i+1}|}{2i+2}\left(\prod_{j=i+1}^{n_o}\frac{\left|(2j+1)^2-n^2\right|}{(2j+1)(2j+2)}\right)\left(\prod_{j=n_o+1}^{m-1}\frac{\left|(2j+1)^2-n^2\right|}{(2j+1)(2j+2)}\right) \\
&\quad + \frac{1}{2m+1}\sum_{i=n_o}^{m-1}\frac{|a_{2i+1}|}{2i+2}\prod_{j=i+1}^{m-1}\frac{\left|(2j+1)^2-n^2\right|}{(2j+1)(2j+2)} \\
&\leq \frac{1}{2m+1}\sum_{i=0}^{n_o-1}\frac{|a_{2i+1}|}{2i+2}\prod_{j=i+1}^{n_o}\frac{n^2-9}{(2j+1)(2j+2)} + \frac{1}{2m+1}\sum_{i=n_o}^{m-1}\frac{|a_{2i+1}|}{2i+2} \\
&\leq \frac{1}{2m+1}\sum_{i=0}^{n_o-1}\frac{(2i+1)!\left(n^2-9\right)^{n_o-i}}{(2n_o+2)!}|a_{2i+1}| + \frac{1}{2m+1}\sum_{i=n_o}^{m-1}\frac{|a_{2i+1}|}{2n_o+2} \\
&\leq \frac{\max_{0\leq i\leq n_o}((2i+1)!/(2n_o+2)!)\left(n^2-9\right)^{n_o-i}}{2m+1}\sum_{i=0}^{m-1}|a_{2i+1}|.
\end{aligned} \tag{2.10}$$

If $1 \le m \le n_o$, then we have $n \ge 5$, and it follows from (2.1) that

$$|c_{2m+1}| \le \frac{1}{2m+1}\sum_{i=0}^{m-1}\frac{|a_{2i+1}|}{2i+2}\prod_{j=i+1}^{m-1}\frac{\left|(2j+1)^2-n^2\right|}{(2j+1)(2j+2)} \le \frac{1}{2m+1}\sum_{i=0}^{m-1}\frac{|a_{2i+1}|}{2i+2}\prod_{j=i+1}^{m-1}\frac{n^2-9}{(2j+1)(2j+2)} \tag{2.11}$$

since $j < n_o$ and hence $2j+1 < 2n/\sqrt{8} < n$. Furthermore, we have

$$|c_{2m+1}| \le \frac{1}{2m+1}\sum_{i=0}^{m-1}\frac{(2i+1)!\left(n^2-9\right)^{m-1-i}}{(2m)!}|a_{2i+1}| \le \frac{\max_{0\le i\le m-1}((2i+1)!/(2m)!)\left(n^2-9\right)^{m-1-i}}{2m+1}\sum_{i=0}^{m-1}|a_{2i+1}|. \tag{2.12}$$

Thus, we may conclude from the last two inequalities that

$$|c_{2m+1}| \le \frac{M_o}{2m+1}\sum_{i=0}^{m-1}|a_{2i+1}| \tag{2.13}$$

for any $m \in \mathbb{N}$, where

$$M_o = \max_{0\le i\le \ell\le n_o}\frac{(2i+1)!}{(2\ell+2)!}\left(n^2-9\right)^{\ell-i}. \tag{2.14}$$

Let ρ_1 be an arbitrary positive number less than ρ_0. Then it follows from (2.7) and (2.13) that

$$\begin{aligned}
\left|\sum_{m=2}^{\infty}c_m x^m\right| &\le \sum_{m=1}^{\infty}|c_{2m}||x|^{2m} + \sum_{m=1}^{\infty}|c_{2m+1}||x|^{2m+1} \\
&\le M_e\sum_{m=1}^{\infty}\frac{|x|^{2m}}{2m}\sum_{i=0}^{m-1}|a_{2i}| + M_o\sum_{m=1}^{\infty}\frac{|x|^{2m+1}}{2m+1}\sum_{i=0}^{m-1}|a_{2i+1}| \\
&= M_e|a_0|\left(\frac{|x|^2}{2}+\frac{|x|^4}{4}+\frac{|x|^6}{6}+\frac{|x|^8}{8}+\frac{|x|^{10}}{10}+\cdots\right) \\
&\quad + M_e|a_2||x|^2\left(\frac{|x|^2}{4}+\frac{|x|^4}{6}+\frac{|x|^6}{8}+\frac{|x|^8}{10}+\frac{|x|^{10}}{12}+\cdots\right)
\end{aligned}$$

$$
\begin{aligned}
&+ M_e|a_4||x|^4\left(\frac{|x|^2}{6}+\frac{|x|^4}{8}+\frac{|x|^6}{10}+\frac{|x|^8}{12}+\frac{|x|^{10}}{14}+\cdots\right)\\
&+\cdots\\
&+ M_o|a_1||x|\left(\frac{|x|^2}{3}+\frac{|x|^4}{5}+\frac{|x|^6}{7}+\frac{|x|^8}{9}+\frac{|x|^{10}}{11}+\cdots\right)\\
&+ M_o|a_3||x|^3\left(\frac{|x|^2}{5}+\frac{|x|^4}{7}+\frac{|x|^6}{9}+\frac{|x|^8}{11}+\frac{|x|^{10}}{13}+\cdots\right)\\
&+ M_o|a_5||x|^5\left(\frac{|x|^2}{7}+\frac{|x|^4}{9}+\frac{|x|^6}{11}+\frac{|x|^8}{13}+\frac{|x|^{10}}{15}+\cdots\right)\\
&+\cdots\\
&= M_e\sum_{m=0}^{\infty}|a_{2m}||x|^{2m}\sum_{i=1}^{\infty}\frac{|x|^{2i}}{2(m+i)}+M_o\sum_{m=0}^{\infty}|a_{2m+1}||x|^{2m+1}\sum_{i=1}^{\infty}\frac{|x|^{2i}}{2(m+i)+1}
\end{aligned}
\tag{2.15}
$$

for any $x \in [-\rho_1, \rho_1]$.

Because of $0 < \rho_1 < \rho_0 \le 1$, we obtain

$$
\sum_{i=1}^{\infty}\frac{|x|^{2i}}{2(m+i)} \le \frac{1}{2m+2}\frac{|x|^2}{1-|x|^2}, \qquad \sum_{i=1}^{\infty}\frac{|x|^{2i}}{2(m+i)+1} \le \frac{1}{2m+3}\frac{|x|^2}{1-|x|^2}
\tag{2.16}
$$

for all $x \in [-\rho_1, \rho_1]$. Thus, we have

$$
\begin{aligned}
\left|\sum_{m=2}^{\infty}c_m x^m\right| &\le M_e\sum_{m=0}^{\infty}\frac{\left|a_{2m}x^{2m}\right|}{2m+2}\frac{|x|^2}{1-|x|^2}+M_o\sum_{m=0}^{\infty}\frac{\left|a_{2m+1}x^{2m+1}\right|}{2m+3}\frac{|x|^2}{1-|x|^2}\\
&\le M_e\frac{|x|^2}{1-|x|^2}\sum_{m=0}^{\infty}\frac{|a_m x^m|}{m+2}
\end{aligned}
\tag{2.17}
$$

for all $x \in [-\rho_1, \rho_1]$. Since ρ_1 is arbitrarily given with $0 < \rho_1 < \rho_0$, inequality (2.17) holds true for all $x \in (-\rho_0, \rho_0)$. Moreover, the power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on $(-\rho, \rho)$. Hence, we conclude that

$$
\left|\sum_{m=2}^{\infty}c_m x^m\right| < \infty
\tag{2.18}
$$

for all $x \in (-\rho_0, \rho_0)$. That is, the power series $\sum_{m=2}^{\infty} c_m x^m$ is convergent for each $x \in (-\rho_0, \rho_0)$.

We will now prove that $\sum_{m=2}^{\infty} c_m x^m$ satisfies the inhomogeneous Chebyshev's differential equation (1.4) for all $x \in (-\rho_0, \rho_0)$. If we substitute $\sum_{m=2}^{\infty} c_m x^m = \sum_{m=1}^{\infty} c_{2m} x^{2m} + \sum_{m=1}^{\infty} c_{2m+1} x^{2m+1}$ for $y(x)$ in (1.4), then it follows from (2.2) that

$$\begin{aligned}
&\left(1-x^2\right)y''(x) - xy'(x) + n^2 y(x) \\
&\quad= \sum_{m=0}^{\infty}(2m+2)(2m+1)c_{2m+2}x^{2m} + \sum_{m=0}^{\infty}(2m+3)(2m+2)c_{2m+3}x^{2m+1} \\
&\qquad - \sum_{m=1}^{\infty}2m(2m-1)c_{2m}x^{2m} - \sum_{m=1}^{\infty}(2m+1)(2m)c_{2m+1}x^{2m+1} \\
&\qquad - \sum_{m=1}^{\infty}2mc_{2m}x^{2m} - \sum_{m=1}^{\infty}(2m+1)c_{2m+1}x^{2m+1} \\
&\qquad + \sum_{m=1}^{\infty}n^2c_{2m}x^{2m} + \sum_{m=1}^{\infty}n^2c_{2m+1}x^{2m+1} \\
&\quad= 2c_2 + 6c_3x + \sum_{m=1}^{\infty}\left[(2m+2)(2m+1)c_{2m+2} + \left(n^2-(2m)^2\right)c_{2m}\right]x^{2m} \\
&\qquad + \sum_{m=1}^{\infty}\left[(2m+3)(2m+2)c_{2m+3} + \left(n^2-(2m+1)^2\right)c_{2m+1}\right]x^{2m+1} \\
&\quad= 2c_2 + 6c_3x + \sum_{m=1}^{\infty}a_{2m}x^{2m} + \sum_{m=1}^{\infty}a_{2m+1}x^{2m+1} \\
&\quad= \sum_{m=0}^{\infty}a_mx^m
\end{aligned} \tag{2.19}$$

for all $x \in (-\rho_0, \rho_0)$. That is, $\sum_{m=2}^{\infty} c_m x^m$ is a particular solution of the inhomogeneous Chebyshev's differential equation (1.4), and hence every solution $y : (-\rho_0, \rho_0) \to \mathbb{C}$ of (1.4) can be expressed by

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \tag{2.20}$$

where $y_h(x)$ is a Chebyshev function. □

3. Approximate Chebyshev Differential Equation

In this section, let $K \geq 0$ and $\rho > 0$ be constants. We denote by $\mathcal{C}_K$ the set of all functions $y : (-\rho, \rho) \to \mathbb{C}$ with the following properties:

(a) $y(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(b) $\sum_{m=0}^{\infty} |a_m x^m| \le K|\sum_{m=0}^{\infty} a_m x^m|$ for any $x \in (-\rho, \rho)$, where

$$a_m = (m+2)(m+1)b_{m+2} - \left(m^2 - n^2\right)b_m \tag{3.1}$$

for all $m \in \mathbb{N}_0$ and set $b_0 = b_1 = 0$.

We now investigate the (local) Hyers-Ulam stability problem of the Chebyshev differential equation. More precisely, we try to answer the question, whether there exists a Chebyshev function near any approximate Chebyshev function.

Theorem 3.1. *Let n be a positive integer, and assume that a function $y \in C_K$ satisfies the differential inequality*

$$\left|\left(1 - x^2\right)y''(x) - xy'(x) + n^2 y(x)\right| \le \varepsilon \tag{3.2}$$

for all $x \in (-\rho, \rho)$ and for some $\varepsilon > 0$. Let $\rho_0 = \min\{1, \rho\}$. Then there exists a Chebyshev function $y_h : (-\rho_0, \rho_0) \to \mathbb{C}$ such that

$$\left|y(x) - y_h(x)\right| \le \frac{KM_e\varepsilon}{2}\frac{x^2}{1 - x^2} \tag{3.3}$$

for all $x \in (-\rho_0, \rho_0)$, where the constant M_e is defined in (2.8).

Proof. It follows from (a) and (b) that

$$\left(1 - x^2\right)y''(x) - xy'(x) + n^2 y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{3.4}$$

for all $x \in (-\rho, \rho)$ (*cf.* (2.19)). Moreover, by using (b) and (3.2), we get

$$\sum_{m=0}^{\infty} |a_m x^m| \le K\left|\sum_{m=0}^{\infty} a_m x^m\right| \le K\varepsilon \tag{3.5}$$

for any $x \in (-\rho, \rho)$.

According to Theorem 2.1 and (3.4), $y(x)$ can be written as $y_h(x) + \sum_{m=2}^{\infty} c_m x^m$ for all $x \in (-\rho_0, \rho_0)$, where y_h is some Chebyshev function and c_m's are given in (2.1). It moreover follows from (2.17) and (3.5) that

$$\left|y(x) - y_h(x)\right| = \left|\sum_{m=2}^{\infty} c_m x^m\right| \le M_e \frac{x^2}{1 - x^2}\frac{K}{2}\varepsilon \tag{3.6}$$

for all $x \in (-\rho_0, \rho_0)$. □

If ρ is assumed to be less than 1, then $\rho_0 = \rho < 1$ and Theorem 3.1 implies the Hyers-Ulam stability of the Chebyshev's differential equation (1.3).

Table 1

n	n_e	n_o	M_e	M_o
1	0	−1	1	$-\infty$
2	0	0	1	1/2
3	1	0	1	1/2
4	1	0	2	1/2
5	1	1	7/2	2/3
6	2	1	128/15	9/8

Remark 3.2. We give some values for n_e, n_o, M_e, and M_o in Table 1.

Corollary 3.3. *Let n be a positive integer, and assume that a function $y \in C_K$ satisfies the differential inequality* (3.2) *for all $x \in (-\rho, \rho)$ and for some $\varepsilon > 0$. Let $\rho_0 = \min\{1, \rho\}$. Then there exists a Chebyshev function $y_h : (-\rho_0, \rho_0) \to \mathbb{C}$ such that*

$$\left|y(x) - y_h(x)\right| = O\left(x^2\right) \tag{3.7}$$

as $x \to 0$.

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