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*Research Article*

# Intuitionistic Fuzzy Stability of Functional Equations Associated with Inner Product Spaces

**Zhihua Wang[1] and Themistocles M. Rassias[2]**

[1] *School of Science, Hubei University of Technology, Wuhan, Hubei 430068, China*
[2] *Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece*

Correspondence should be addressed to Themistocles M. Rassias, trassias@math.ntua.gr

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In intuitionistic fuzzy normed spaces, we investigate some stability results for the functional equation $\sum_{i=1}^{n} f(x_i - (1/n)\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf((1/n)\sum_{i=1}^{n} x_i)$ which is said to be a functional equation associated with inner products space.

## 1. Introduction and Preliminaries

The aim of this article is to prove an intuitionistic fuzzy version of the Hyers-Ulam-Rassias stability for the functional equation:

$$\sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right), \tag{1.1}$$

which is said to be a functional equation associated with inner product spaces. It was shown by Rassias [1] that the norm defined over a real vector space $X$ is induced by an inner product if and only if for a fixed integer $n \geq 2$ it follows

$$\sum_{i=1}^{n} \left\| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n\left\| \frac{1}{n}\sum_{i=1}^{n} x_i \right\|^2, \tag{1.2}$$

for all $x_i, \dots, x_n \in X$. Interesting new results concerning functional equations associated with inner product spaces have recently been obtained by Park et al. [2, 3] and Najati and Rassias [4] as well as for the fuzzy stability of a functional equation associated with inner product spaces [5].

Stability problem of a functional equation was first posed by Ulam [6] which was answered by Hyers [7] on approximately additive mappings and then generalized by Aoki [8] and Rassias [9] for additive mappings and linear mappings, respectively. Later there have been proved several new results on stability of various classes of functional equations in the Hyers-Ulam sense (cf. the following books and papers [10–18] and the references cited therein), as well as various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations (cf. [19–22]). Furthermore some stability results concerning Jensen, cubic, mixed-type additive and cubic functional equations were investigated (cf. [23–26]) in the spirit of intuitionistic fuzzy normed spaces, while the idea of intuitionistic fuzzy normed space was introduced in [27] and further studied in [28–35].

In this section, we recall some notations and basic definitions used in this paper as follows.

*Definition 1.1* (cf. [36]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous $t$-norm if it satisfies the following conditions:

(a) $*$ is commutative and associative, (b) $*$ is continuous,

(c) $a * 1 = a$ for all $a \in [0,1]$,

(d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

*Definition 1.2* (cf. [36]). A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous $t$-conorm if it satisfies the following conditions:

(a$'$) $\diamond$ is commutative and associative, (b$'$) $\diamond$ is continuous,

(c$'$) $a \diamond 0 = a$ for all $a \in [0,1]$,

(d$'$) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

Using the notions of continuous $t$-norm and continuous $t$-conorm, Saadati and Park [27] have recently introduced the concept of intuitionistic fuzzy normed spaces as follows.

*Definition 1.3.* The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if $X$ is a vector space, $*$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm, and $\mu, \nu$ are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$,

(i) $\mu(x,t) + \nu(x,t) \le 1$,

(ii) $\mu(x,t) > 0$,

(iii) $\mu(x,t) = 1$ if and only if $x = 0$,

(iv) $\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ for each $\alpha \neq 0$,

(v) $\mu(x,t) * \mu(y,s) \le \mu(x+y, t+s)$,

(vi) $\mu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous,

(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t \to 0} \mu(x,t) = 0$,

(viii) $\nu(x,t) < 1$, (ix) $\nu(x,t) = 0$ if and only if $x = 0$,

(x) $\nu(\alpha x, t) = \nu(x, t/|\alpha|)$ for each $\alpha \neq 0$, (xi) $\nu(x, t) \diamond \nu(y, s) \geq \nu(x + y, t + s)$,

(xii) $\nu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous,

(xiii) $\lim_{t\to\infty}\nu(x, t) = 0$ and $\lim_{t\to 0}\nu(x, t) = 1$.

In this case $(\mu, \nu)$ is called an intuitionistic fuzzy norm.

*Example 1.4* (cf. [37]). Let $(X, \|\cdot\|)$ be a normed space, $a*b = ab$, and $a \diamond b = \min(a+b, 1)$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $t > 0$ and $k = 1, 2$, consider

$$\mu_k(x,t) = \begin{cases} \dfrac{t}{t+k\|x\|}, & \text{if} \quad t > 0, \\ 0, & \text{if} \quad t \leq 0, \end{cases} \qquad \nu_k(x,t) = \begin{cases} \dfrac{k\|x\|}{t+k\|x\|}, & \text{if} \quad t > 0, \\ 0, & \text{if} \quad t \leq 0. \end{cases} \tag{1.3}$$

Then $(X, \mu, \nu, *, \diamond)$ is an IFNS.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [27].

Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $\{x_k\}$ is said to be intuitionistic fuzzy convergent to $x \in X$ if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x, t) > 1 - \varepsilon$ and $\nu(x_k - x, t) < \varepsilon$ for all $k \geq k_0$. In this case we write $(\mu, \nu) - \lim x_k = x$. The sequence $\{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_\ell, t) > 1 - \varepsilon$ and $\nu(x_k - x_\ell, t) < \varepsilon$ for all $k, \ell \geq k_0$. $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$.

## 2. Intuitionistic Fuzzy Stability

Throughout this section, assume that $X$, $(Z, \mu', \nu')$, and $(Y, \mu, \nu)$ are linear space, IFNS, and intuitionistic fuzzy Banach space, respectively. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$\Delta f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right). \tag{2.1}$$

We begin with the Hyers-Ulam-Rassias type theorem in IFNS for the functional (1.1) which is said to be a functional equation associated with inner product spaces.

**Theorem 2.1.** *Let $\varphi : X \to Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number $\alpha$ with $0 < |\alpha| < 4$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned} &\mu(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n) \geq \mu'(\varphi(x_1), t_1) * \cdots * \mu'(\varphi(x_n), t_n), \\ &\nu(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n) \leq \nu'(\varphi(x_1), t_1) \diamond \cdots \diamond \nu'(\varphi(x_n), t_n), \end{aligned} \tag{2.2}$$

*for all $x_1, \ldots, x_n \in X$ and all $t_1, \ldots, t_n > 0$. Then there exists a unique quadratic function $Q : X \to Y$ such that*

$$\mu(Q(x) - f(x), t) \geq \mu_1''\left(x, \frac{(4-|\alpha|)t}{8}\right), \qquad \nu(Q(x) - f(x), t) \leq \nu_1''\left(x, \frac{(4-|\alpha|)t}{8}\right), \tag{2.3}$$

*for all $x \in X$ and $t > 0$, where*

$$\begin{aligned} \mu_1''(x,t) &:= \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right), \\ \nu_1''(x,t) &:= \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right). \end{aligned} \tag{2.4}$$

*Proof.* Put $x_1 = nx_1$, $x_i = nx_2$ $(i = 2, \ldots, n)$, $t_i = t$ $(i = 1, \ldots, n)$ in (2.2), and, using the evenness of $f$, we obtain

$$\begin{aligned} &\mu(nf(x_1 + (n-1)x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - f(nx_1) - (n-1)f(nx_2), nt) \\ &\qquad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_2), t), \\ &\nu(nf(x_1 + (n-1)x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - f(nx_1) - (n-1)f(nx_2), nt) \\ &\qquad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_2), t), \end{aligned} \tag{2.5}$$

for all $x_1, x_2 \in X$ and $t > 0$. Interchanging $x_1$ with $x_2$ in (2.5) and using the evenness of $f$, we get

$$\begin{aligned} &\mu(nf((n-1)x_1 + x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - (n-1)f(nx_1) - f(nx_2), nt) \\ &\qquad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_2), t), \\ &\nu(nf((n-1)x_1 + x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - (n-1)f(nx_1) - f(nx_2), nt) \\ &\qquad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_2), t), \end{aligned} \tag{2.6}$$

for all $x_1, x_2 \in X$ and $t > 0$. It follows from (2.5) and (2.6) that

$$\begin{aligned}
&\mu(nf((n-1)x_1+x_2)+nf(x_1+(n-1)x_2)+2f((n-1)(x_1-x_2))\\
&\quad +2(n-1)f(x_1-x_2)-nf(nx_1)-nf(nx_2),2nt)\\
&\qquad \geq \mu'(\varphi(nx_1),t)*\mu'(\varphi(nx_2),t),\\
&\nu(nf((n-1)x_1+x_2)+nf(x_1+(n-1)x_2)+2f((n-1)(x_1-x_2))\\
&\quad +2(n-1)f(x_1-x_2)-nf(nx_1)-nf(nx_2),2nt)\\
&\qquad \leq \nu'(\varphi(\mathrm{n}x_1),t)\diamond\nu'(\varphi(nx_2),t),
\end{aligned} \tag{2.7}$$

for all $x_1, x_2 \in X$ and $t > 0$. Putting $x_1 = nx_1$, $x_2 = -nx_2$, $x_i = 0$ $(i = 3, \ldots, n)$, $t_i = t$ $(i = 1, \ldots, n)$ in (2.2) and using the evenness of $f$, we obtain

$$\begin{aligned}
&\mu(f((n-1)x_1+x_2)+f(x_1+(n-1)x_2)+2(n-1)f(x_1-x_2)-f(nx_1)-f(nx_2),nt)\\
&\qquad \geq \mu'(\varphi(nx_1),t)*\mu'(\varphi(-nx_2),t)*\mu'(\varphi(0),t),\\
&\nu(f((n-1)x_1+x_2)+f(x_1+(n-1)x_2)+2(n-1)f(x_1-x_2)-f(nx_1)-f(nx_2),nt)\\
&\qquad \leq \nu'(\varphi(nx_1),t)\diamond\nu'(\varphi(-nx_2),t)\diamond\nu'(\varphi(0),t),
\end{aligned} \tag{2.8}$$

for all $x_1, x_2 \in X$ and $t > 0$. Hence, we obtain from (2.7) and (2.8) that

$$\begin{aligned}
&\mu\left(f((n-1)(x_1-x_2))-(n-1)^2f(x_1-x_2),\frac{n^2+2n}{2}t\right)\\
&\qquad \geq \mu'(\varphi(nx_1),t)*\mu'(\varphi(nx_2),t)*\mu'(\varphi(-nx_2),t)*\mu'(\varphi(0),t),\\
&\nu\left(f((n-1)(x_1-x_2))-(n-1)^2f(x_1-x_2),\frac{n^2+2n}{2}t\right)\\
&\qquad \leq \nu'(\varphi(nx_1),t)\diamond\nu'(\varphi(nx_2),t)\diamond\nu'(\varphi(-nx_2),t)\diamond\nu'(\varphi(0),t),
\end{aligned} \tag{2.9}$$

for all $x_1, x_2 \in X$ and $t > 0$. So

$$\begin{aligned}
&\mu\left(f((n-1)x)-(n-1)^2f(x),\frac{n^2+2n}{2}t\right)\geq \mu'(\varphi(nx),t)*\mu'(\varphi(0),t),\\
&\nu\left(f((n-1)x)-(n-1)^2f(x),\frac{n^2+2n}{2}t\right)\leq \nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t),
\end{aligned} \tag{2.10}$$

for all $x \in X$ and $t > 0$. Putting $x_1 = nx,\ x_i = 0\ (i = 2, \dots, n),\ t_i = t\ (i = 1, \dots, n)$ in (2.2), we obtain

$$\begin{aligned} &\mu(f(nx) - f((n-1)x) - (2n-1)f(x), nt) \geq \mu'(\varphi(nx), t) * \mu'(\varphi(0), t), \\ &\nu(f(nx) - f((n-1)x) - (2n-1)f(x), nt) \leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi(0), t), \end{aligned} \tag{2.11}$$

for all $x \in X$ and $t > 0$. It follows from (2.10) and (2.11) that

$$\begin{aligned} &\mu\left(f(nx) - n^2 f(x), \frac{n^2 + 4n}{2} t\right) \geq \mu'(\varphi(nx), t) * \mu'(\varphi(0), t), \\ &\nu\left(f(nx) - n^2 f(x), \frac{n^2 + 4n}{2} t\right) \leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi(0), t), \end{aligned} \tag{2.12}$$

for all $x \in X$ and $t > 0$. Letting $x_2 = -(n-1)x_1$ in (2.8) and replacing $x_1$ by $x/n$ in the obtained inequality, we get

$$\begin{aligned} &\mu(f((n-1)x) - f((n-2)x) - (2n-3)f(x), nt) \geq \mu'(\varphi(x), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(0), t), \\ &\nu(f((n-1)x) - f((n-2)x) - (2n-3)f(x), nt) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(0), t), \end{aligned} \tag{2.13}$$

for all $x \in X$ and $t > 0$. It follows from (2.10), (2.11), (2.12), and (2.13) that

$$\begin{aligned} &\mu\left(f((n-2)x) - (n-1)^2 f(x), \frac{n^2 + 4n}{2} t\right) \\ &\quad \geq \mu'(\varphi(nx), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(0), t), \\ &\nu\left(f((n-2)x) - (n-1)^2 f(x), \frac{n^2 + 4n}{2} t\right) \\ &\quad \leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(0), t), \end{aligned} \tag{2.14}$$

for all $x \in X$ and $t > 0$. Applying (2.12) and (2.14), we obtain

$$\begin{aligned} &\mu\left(f(nx) - f((n-2)x) - 4(n-1)f(x), \left(n^2 + 4n\right)t\right) \\ &\quad \geq \mu'(\varphi(nx), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(0), t), \\ &\nu\left(f(nx) - f((n-2)x) - 4(n-1)f(x), \left(n^2 + 4n\right)t\right) \\ &\quad \leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(0), t), \end{aligned} \tag{2.15}$$

for all $x \in X$ and $t > 0$. Setting $x_1 = x_2 = nx,\ x_i = 0\ (i = 3, \dots, n),\ t_i = t\ (i = 1, \dots, n)$ in (2.2), we obtain

$$\begin{aligned} &\mu\Big(f((n-2)x) + (n-1)f(2x) - f(nx), \frac{n}{2}t\Big) \ge \mu'(\varphi(nx), t) * \mu'(\varphi(0), t), \\ &\nu\Big(f((n-2)x) + (n-1)f(2x) - f(nx), \frac{n}{2}t\Big) \le \nu'(\varphi(nx), t) \diamond \nu'(\varphi(0), t), \end{aligned} \tag{2.16}$$

for all $x \in X$ and $t > 0$. It follows from (2.15) and (2.16) that

$$\begin{aligned} &\mu\left(f(2x) - 4f(x), \frac{2n^2+9n}{2n-2}t\right) \\ &\quad \ge \mu'(\varphi(nx), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(0), t), \\ &\nu\left(f(2x) - 4f(x), \frac{2n^2+9n}{2n-2}t\right) \\ &\quad \le \nu'(\varphi(nx), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(0), t). \end{aligned} \tag{2.17}$$

It follows that

$$\begin{aligned} \mu\Big(f(x) - 4^{-1}f(2x), t\Big) &\ge \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right), \\ \nu\Big(f(x) - 4^{-1}f(2x), t\Big) &\le \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right). \end{aligned} \tag{2.18}$$

Define

$$\begin{aligned} \mu_1''(x, t) &:= \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right), \\ \nu_1''(x, t) &:= \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right). \end{aligned} \tag{2.19}$$

Then, by our assumption,

$$\mu_1''(2x,t) = \mu_1''\left(x, \frac{t}{\alpha}\right), \qquad \nu_1''(2x,t) = \nu_1''\left(x, \frac{t}{\alpha}\right). \tag{2.20}$$

Replacing $x$ by $2^n x$ in (2.18) and applying (2.20), we get

$$\begin{aligned}
\mu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}, \frac{\alpha^n t}{4^n}\right) &= \mu\left(f(2^n x) - \frac{f(2^{n+1}x)}{4}, \alpha^n t\right) \ge \mu_1''(2^n x, \alpha^n t) = \mu_1''(x,t),\\
\nu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}, \frac{\alpha^n t}{4^n}\right) &= \nu\left(f(2^n x) - \frac{f(2^{n+1}x)}{4}, \alpha^n t\right) \le \nu_1''(2^n x, \alpha^n t) = \nu_1''(x,t).
\end{aligned} \tag{2.21}$$

Thus for each $n > m$, we have

$$\begin{aligned}
\mu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right) &= \mu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1}x)}{4^{k+1}}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)\\
&\ge \prod_{k=m}^{n-1}\mu\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1}x)}{4^{k+1}}, \frac{\alpha^k t}{4^k}\right) \ge \mu_1''(x,t),\\
\nu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right) &= \nu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1}x)}{4^{k+1}}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)\\
&\le \coprod_{k=m}^{n-1}\nu\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1}x)}{4^{k+1}}, \frac{\alpha^k t}{4^k}\right) \le \nu_1''(x,t),
\end{aligned} \tag{2.22}$$

where $\prod_{j=1}^{n} a_j = a_1 * a_2 * \cdots * a_n$, $\coprod_{j=1}^{n} a_j = a_1 \diamond a_2 \diamond \cdots \diamond a_n$. Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty}\mu_1''(x,t) = 1$ and $\lim_{t\to\infty}\nu_1''(x,t) = 0$, there exists some $t_0 > 0$ such that $\mu_1''(x,t_0) > 1-\varepsilon$ and $\nu_1''(x,t_0) < \varepsilon$. Since $\sum_{k=0}^{\infty}(\alpha^k t_0/4^k) < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1}(\alpha^k t_0/4^k) < \delta$ for each $n > m \ge n_0$. It follows that

$$\begin{aligned}
\mu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \delta\right) &\ge \mu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1}\frac{\alpha^k t_0}{4^k}\right) \ge \mu_1''(x,t_0) > 1-\varepsilon,\\
\nu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \delta\right) &\le \nu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1}\frac{\alpha^k t_0}{4^k}\right) \le \nu_1''(x,t_0) < \varepsilon,
\end{aligned} \tag{2.23}$$

for all $t > t_0$. This shows that the sequence $\{f(2^n x)/4^n\}$ is Cauchy in $(Y,\mu,\nu)$. Since $(Y,\mu,\nu)$ is intuitionistic fuzzy Banach space, $\{f(2^n x)/4^n\}$ converges to some point $Q(x) \in Y$. Thus, we can define a mapping $Q(x) : X \to Y$ such that $Q(x) := (\mu,\nu) - \lim_{n\to\infty}(f(2^n x)/4^n)$. Moreover, if we put $m = 0$ in (2.22), we get

$$\mu\left(\frac{f(2^n x)}{4^n} - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{4^k}\right) \ge \mu_1''(x,t), \qquad \nu\left(\frac{f(2^n x)}{4^n} - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{4^k}\right) \le \nu_1''(x,t). \tag{2.24}$$

Thus,

$$\mu\left(\frac{f(2^n x)}{4^n} - f(x), t\right) \ge \mu_1''\left(x, \frac{t}{\sum_{k=0}^{n-1}(\alpha/4)^k}\right), \qquad \nu\left(\frac{f(2^n x)}{4^n} - f(x), t\right) \le \nu_1''\left(x, \frac{t}{\sum_{k=0}^{n-1}(\alpha/4)^k}\right). \tag{2.25}$$

Now, we will show that $Q$ is quadratic. Setting $x_i = 2^m x_i$ $(i = 1,\ldots,n)$ and $t_i = (t/n)$ $(i = 1,\ldots,n)$ in (2.2), we obtain

$$\begin{aligned}
\mu\left(\frac{\Delta f(2^m x_1,\ldots,2^m x_n)}{4^m}, t\right) &\ge \mu'\left(\varphi(2^m x_1), 4^m\frac{t}{n}\right) * \cdots * \mu'\left(\varphi(2^m x_n), 4^m\frac{t}{n}\right) \\
&= \mu'\left(\varphi(x_1), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) * \cdots * \mu'\left(\varphi(x_n), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right), \\
\nu\left(\frac{\Delta f(2^m x_1,\ldots,2^m x_n)}{4^m}, t\right) &\le \nu'\left(\varphi(2^m x_1), 4^m\frac{t}{n}\right) \diamond \cdots \diamond \nu'\left(\varphi(2^m x_n), 4^m\frac{t}{n}\right) \\
&= \nu'\left(\varphi(x_1), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) \diamond \cdots \diamond \nu'\left(\varphi(x_n), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right),
\end{aligned} \tag{2.26}$$

for all $x_1,\ldots,x_n \in X$ and $t > 0$. Letting $n \to \infty$ in (2.26), we obtain

$$\mu(\Delta Q(x_1,\ldots,x_n), t) = 1, \qquad \nu(\Delta Q(x_1,\ldots,x_n), t) = 0, \tag{2.27}$$

for all $x_1,\ldots,x_n \in X$ and all $t > 0$. This means that $Q$ satisfies the functional (1.1) and so it is quadratic (see Lemma 2.2 of [4]).

Next, we approximate the difference between $f$ and $Q$ in intuitionistic fuzzy sense. By (2.25), we have

$$\begin{aligned}
\mu(Q(x) - f(x), t) &\ge \mu\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right) * \mu\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right) \\
&\ge \mu_1''\left(x, \frac{t}{2\sum_{k=0}^{\infty}(\alpha/4)^k}\right) = \mu_1''\left(x, \frac{(4-\alpha)t}{8}\right), \\
\nu(Q(x) - f(x), t) &\le \nu\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right) \diamond \nu\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right) \\
&\le \nu_1''\left(x, \frac{t}{2\sum_{k=0}^{\infty}(\alpha/4)^k}\right) = \nu_1''\left(x, \frac{(4-\alpha)t}{8}\right),
\end{aligned} \tag{2.28}$$

for every $x \in X$, $t > 0$ and large enough $n$. To prove the uniqueness of $Q$, assume that $Q'$ is another quadratic mapping from $X$ to $Y$, which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$,

$$\begin{aligned}\mu(Q(x)-Q'(x),t) &\geq \mu\left(Q(x)-f(x),\frac{t}{2}\right) * \mu\left(Q'(x)-f(x),\frac{t}{2}\right) \geq \mu_1''\left(x,\frac{(4-\alpha)t}{16}\right),\\ \nu(Q(x)-Q'(x),t) &\leq \nu\left(Q(x)-f(x),\frac{t}{2}\right) \diamond \nu\left(Q'(x)-f(x),\frac{t}{2}\right) \leq \nu_1''\left(x,\frac{(4-\alpha)t}{16}\right).\end{aligned} \tag{2.29}$$

Since $Q$ and $Q'$ are quadratic, we have

$$\begin{aligned}\mu(Q(x)-Q'(x),t) &= \mu(Q(2^n x)-Q'(2^n x),4^n t) \geq \mu_1''\left(x,\frac{(4/\alpha)^n(4-\alpha)t}{16}\right),\\ \nu(Q(x)-Q'(x),t) &= \nu(Q(2^n x)-Q'(2^n x),4^n t) \leq \nu_1''\left(x,\frac{(4/\alpha)^n(4-\alpha)t}{16}\right),\end{aligned} \tag{2.30}$$

for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $0 < \alpha < 4$ and $\lim_{n\to\infty}(4/\alpha)^n = \infty$, we get

$$\lim_{n\to\infty}\mu_1''\left(x,\frac{(4/\alpha)^n(4-\alpha)t}{16}\right) = 1, \qquad \lim_{n\to\infty}\nu_1''\left(x,\frac{(4/\alpha)^n(4-\alpha)t}{16}\right) = 0. \tag{2.31}$$

Therefore $\mu(Q(x)-Q'(x),t) = 1$ and $\nu(Q(x)-Q'(x),t) = 0$ for all $x \in X$ and $t > 0$. Hence, $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. □

**Theorem 2.2.** *Let $\varphi : X \to Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number $\alpha$ with $0 < |\alpha| < 2$. Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$\begin{aligned}\mu(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) &\geq \mu'(\varphi(x_1),t_1) * \cdots * \mu'(\varphi(x_n),t_n),\\ \nu(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) &\leq \nu'(\varphi(x_1),t_1) \diamond \cdots \diamond \nu'(\varphi(x_n),t_n),\end{aligned} \tag{2.32}$$

*for all $x_1,\ldots,x_n \in X$ and all $t_1,\ldots,t_n > 0$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\mu(A(x)-f(x),t) \geq \mu_2''\left(x,\frac{(2-|\alpha|)t}{4}\right), \qquad \nu(A(x)-f(x),t) \leq \nu_2''\left(x,\frac{(2-|\alpha|)t}{4}\right), \tag{2.33}$$

*for all $x \in X$ and $t > 0$, where*

$$\begin{aligned}\mu_2''(x,t) &:= \mu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(x), \frac{4}{n^2+4n}t\right)\\&\quad * \mu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(0), \frac{4}{n^2+4n}t\right),\\\nu_2''(x,t) &:= \nu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2+4n}t\right)\\&\quad \diamond \nu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2+4n}t\right).\end{aligned} \tag{2.34}$$

*Proof.* Put $x_1 = nx_1$, $x_i = nx_1'$ $(i = 2, \dots, n)$, $t_i = t$ $(i = 1, \dots, n)$ in (2.32) and using the oddness of $f$, we obtain

$$\begin{aligned}&\mu\left(nf\left(x_1 + (n-1)x_1'\right) + f\left((n-1)\left(x_1 - x_1'\right)\right) - (n-1)f\left(x_1 - x_1'\right) - f(nx_1) - (n-1)f\left(nx_1'\right), nt\right)\\&\qquad \geq \mu'\left(\varphi(nx_1), t\right) * \mu'\left(\varphi\left(nx_1'\right), t\right),\\&\nu\left(nf\left(x_1 + (n-1)x_1'\right) + f\left((n-1)\left(x_1 - x_1'\right)\right) - (n-1)f\left(x_1 - x_1'\right) - f(nx_1) - (n-1)f\left(nx_1'\right), nt\right)\\&\qquad \leq \nu'\left(\varphi(nx_1), t\right) \diamond \nu'\left(\varphi\left(nx_1'\right), t\right),\end{aligned} \tag{2.35}$$

for all $x_1, x_1' \in X$ and $t > 0$. Interchanging $x_1$ with $x_1'$ in (2.35) and using the oddness of $f$, we get

$$\begin{aligned}&\mu\left(nf\left((n-1)x_1 + x_1'\right) - f\left((n-1)\left(x_1 - x_1'\right)\right) + (n-1)f\left(x_1 - x_1'\right) - (n-1)f(nx_1) - f\left(nx_1'\right), nt\right)\\&\qquad \geq \mu'\left(\varphi(nx_1), t\right) * \mu'\left(\varphi\left(nx_1'\right), t\right),\\&\nu\left(nf\left((n-1)x_1 + x_1'\right) - f\left((n-1)\left(x_1 - x_1'\right)\right) + (n-1)f\left(x_1 - x_1'\right) - (n-1)f(nx_1) - f\left(nx_1'\right), nt\right)\\&\qquad \leq \nu'\left(\varphi(nx_1), t\right) \diamond \nu'\left(\varphi\left(nx_1'\right), t\right),\end{aligned} \tag{2.36}$$

for all $x_1, x_1' \in X$ and $t > 0$. It follows from (2.35) and (2.36) that

$$\begin{aligned}&\mu\left(nf\left(x_1 + (n-1)x_1'\right) - nf\left((n-1)x_1 + x_1'\right) + 2f\left((n-1)\left(x_1 - x_1'\right)\right)\right.\\&\quad \left.-2(n-1)f\left(x_1 - x_1'\right) + (n-2)f(nx_1) - (n-2)f\left(nx_1'\right), 2nt\right)\\&\qquad \geq \mu'\left(\varphi(nx_1), t\right) * \mu'\left(\varphi\left(nx_1'\right), t\right),\\&\nu\left(nf\left(x_1 + (n-1)x_1'\right) - nf\left((n-1)x_1 + x_1'\right) + 2f\left((n-1)\left(x_1 - x_1'\right)\right)\right.\\&\quad \left.-2(n-1)f\left(x_1 - x_1'\right) + (n-2)f(nx_1) - (n-2)f\left(nx_1'\right), 2nt\right)\\&\qquad \leq \nu'\left(\varphi(nx_1), t\right) \diamond \nu'\left(\varphi\left(nx_1'\right), t\right),\end{aligned} \tag{2.37}$$

for all $x_1, x_1' \in X$ and $t > 0$. Setting $x_1 = nx_1,\ x_2 = -nx_1',\ x_i = 0\ (i = 3, \ldots, n),\ t_i = t\ (i = 1, \ldots, n)$ in (2.32) and using the oddness of $f$, we get

$$\begin{aligned}
&\mu(f((n-1)x_1 + x_1') - f(x_1 + (n-1)x_1') + 2f(x_1 - x_1') - f(nx_1) + f(nx_1'), nt) \\
&\qquad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(-nx_1'), t) * \mu'(\varphi(0), t), \\
&\nu(f((n-1)x_1 + x_1') - f(x_1 + (n-1)x_1') + 2f(x_1 - x_1') - f(nx_1) + f(nx_1'), nt) \\
&\qquad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(-nx_1'), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.38}$$

for all $x_1, x_1' \in X$ and $t > 0$. It follows from (2.37) and (2.38) that

$$\begin{aligned}
&\mu\left(f((n-1)(x_1 - x_1')) + f(x_1 - x_1') - f(nx_1) + f(nx_1'), \frac{n^2 + 2n}{2}t\right) \\
&\qquad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_1'), t) * \mu'(\varphi(-nx_1'), t) * \mu'(\varphi(0), t), \\
&\nu\left(f((n-1)(x_1 - x_1')) + f(x_1 - x_1') - f(nx_1) + f(nx_1'), \frac{n^2 + 2n}{2}t\right) \\
&\qquad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_1'), t) \diamond \nu'(\varphi(-nx_1'), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.39}$$

for all $x_1, x_1' \in X$ and $t > 0$. Putting $x_1 = n(x_1 - x_1'),\ x_i = 0\ (i = 2, \ldots, n),\ t_i = t\ (i = 1, \ldots, n)$ in (2.32), we obtain

$$\begin{aligned}
&\mu(f(n(x_1 - x_1')) - f((n-1)(x_1 - x_1')) - f(x_1 - x_1'), nt) \geq \mu'(\varphi(n(x_1 - x_1')), t) * \mu'(\varphi(0), t), \\
&\nu(f(n(x_1 - x_1')) - f((n-1)(x_1 - x_1')) - f(x_1 - x_1'), nt) \leq \nu'(\varphi(n(x_1 - x_1')), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.40}$$

for all $x_1, x_1' \in X$ and $t > 0$. It follows from (2.39) and (2.40) that

$$\begin{aligned}
&\mu\left(f(n(x_1 - x_1')) - f(nx_1) + f(nx_1'), \frac{n^2 + 4n}{2}t\right) \\
&\qquad \geq \mu'(\varphi(n(x_1 - x_1')), t) * \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_1'), t) * \mu'(\varphi(-nx_1'), t) * \mu'(\varphi(0), t), \\
&\nu\left(f(n(x_1 - x_1')) - f(nx_1) + f(nx_1'), \frac{n^2 + 4n}{2}t\right) \\
&\qquad \leq \nu'(\varphi(n(x_1 - x_1')), t) \diamond \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_1'), t) \diamond \nu'(\varphi(-nx_1'), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.41}$$

for all $x_1, x_1' \in X$ and $t > 0$. Replacing $x_1$ and $x_1'$ by $x/n$ and $-x/n$ in (2.41); respectively, we have

$$\begin{aligned} &\mu\left(f(2x) - 2f(x), \frac{n^2+4n}{2}t\right) \\ &\quad \geq \mu'(\varphi(2x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(-x), t) * \mu'(\varphi(0), t), \\ &\nu\left(f(2x) - 2f(x), \frac{n^2+4n}{2}t\right) \\ &\quad \leq \nu'(\varphi(2x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(-x), t) \diamond \nu'(\varphi(0), t). \end{aligned} \tag{2.42}$$

It follows that

$$\begin{aligned} \mu\left(f(x) - 2^{-1}f(2x), t\right) &\geq \mu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(x), \frac{4}{n^2+4n}t\right) \\ &\quad * \mu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(0), \frac{4}{n^2+4n}t\right), \\ \nu\left(f(x) - 2^{-1}f(2x), t\right) &\leq \nu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2+4n}t\right) \\ &\quad \diamond \nu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2+4n}t\right). \end{aligned} \tag{2.43}$$

Define

$$\begin{aligned} \mu_2''(x,t) &:= \mu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(x), \frac{4}{n^2+4n}t\right) \\ &\quad * \mu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(0), \frac{4}{n^2+4n}t\right), \\ \nu_2''(x,t) &:= \nu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2+4n}t\right) \\ &\quad \diamond \nu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2+4n}t\right). \end{aligned} \tag{2.44}$$

Then by the assumption

$$\mu_2''(2x,t) = \mu_2''\left(x, \frac{t}{\alpha}\right), \qquad \nu_2''(2x,t) = \nu_2''\left(x, \frac{t}{\alpha}\right). \tag{2.45}$$

Replacing $x$ by $2^n x$ in (2.43) and using (2.45), we obtain

$$\begin{aligned}
\mu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}},\frac{\alpha^n \mathrm{t}}{2^n}\right) &= \mu\left(f(2^n x)-\frac{f(2^{n+1}x)}{2},\alpha^n t\right) \\
&\geq \mu_2''(2^n x,\alpha^n t)=\mu_2''(x,t), \\
\nu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}},\frac{\alpha^n t}{2^n}\right) &= \nu\left(f(2^n x)-\frac{f(2^{n+1}x)}{2},\alpha^n t\right) \\
&\leq \nu_2''(2^n x,\alpha^n t)=\nu_2''(x,t).
\end{aligned}\tag{2.46}$$

Thus, for each $n > m$, we have

$$\begin{aligned}
\mu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right) &= \mu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right) \\
&\geq \prod_{k=m}^{n-1}\mu\left(\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\frac{\alpha^k t}{2^k}\right)\geq \mu_2''(x,t), \\
\nu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right) &= \nu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right) \\
&\leq \coprod_{k=m}^{n-1}\nu\left(\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\frac{\alpha^k t}{2^k}\right)\leq \nu_2''(x,t),
\end{aligned}\tag{2.47}$$

where $\prod_{j=1}^{n} a_j = a_1 * a_2 * \cdots * a_n$, $\coprod_{j=1}^{n} a_j = a_1 \diamond a_2 \diamond \cdots \diamond a_n$. Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty}\mu_2''(x,t) = 1$ and $\lim_{t\to\infty}\nu_2''(x,t) = 0$, there exists some $t_0 > 0$ such that $\mu_2''(x,t_0) > 1-\varepsilon$ and $\nu_2''(x,t_0) < \varepsilon$. Since $\sum_{k=0}^{\infty}\alpha^k t_0/2^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1}\alpha^k t_0/2^k < \delta$ for each $n > m \geq n_0$. It follows that

$$\begin{aligned}
\mu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\delta\right) &\geq \mu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{2^k}\right)\geq \mu_2''(x,t_0) > 1-\varepsilon, \\
\nu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\delta\right) &\leq \nu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{2^k}\right)\leq \nu_2''(x,t_0) < \varepsilon,
\end{aligned}\tag{2.48}$$

for all $t > t_0$. This shows that the sequence $\{f(2^n x)/2^n\}$ is Cauchy in $(Y,\mu,\nu)$. Since $(Y,\mu,\nu)$ is intuitionistic fuzzy Banach space, $\{f(2^n x)/4^n\}$ converges to some point $A(x) \in Y$. Thus, we can define a mapping $A(x) : X \to Y$ such that $A(x) := (\mu,\nu)\ -\lim_{n\to\infty} f(2^n x)/2^n$. Moreover, if we put $m = 0$ in (2.47), we get

$$\mu\left(\frac{f(2^n x)}{2^n}-f(x),\sum_{k=0}^{n-1}\frac{\alpha^k t}{2^k}\right)\geq \mu_2''(x,t),\qquad \nu\left(\frac{f(2^n x)}{2^n}-f(x),\sum_{k=0}^{n-1}\frac{\alpha^k t}{2^k}\right)\leq \nu_2''(x,t).\tag{2.49}$$

Thus,

$$\mu\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq \mu_2''\left(x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/2)^k}\right), \qquad \nu\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \leq \nu_2''\left(x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/2)^k}\right). \tag{2.50}$$

Next we will show that $A$ is additive. Putting $x_i = 2^m x_i (i = 1, \dots, n)$ and $t_i = (t/n)\,(i = 1, \dots, n)$ in (2.32), we obtain

$$\begin{aligned}
\mu\left(\frac{\Delta f(2^m x_1, \dots, 2^m x_n)}{2^m}, t\right) &\geq \mu'\left(\varphi(2^m x_1), 2^m \frac{t}{n}\right) * \cdots * \mu'\left(\varphi(2^m x_n), 2^m \frac{t}{n}\right) \\
&= \mu'\left(\varphi(x_2), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right) * \cdots * \mu'\left(\varphi(x_n), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right), \\
\nu\left(\frac{\Delta f(2^m x_1, \dots, 2^m x_n)}{2^m}, t\right) &\leq \nu'\left(\varphi(2^m x_1), 2^m \frac{t}{n}\right) \diamond \cdots \diamond \nu'\left(\varphi(2^m x_n), 2^m \frac{t}{n}\right) \\
&= \nu'\left(\varphi(x_2), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right) \diamond \cdots \diamond \nu'\left(\varphi(x_n), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right),
\end{aligned} \tag{2.51}$$

for all $x_1, \dots, x_n \in X$ and $t > 0$. Letting $n \to \infty$ in (2.51), we obtain

$$\mu(\Delta A(x_1, \dots, x_n), t) = 1, \qquad \nu(\Delta A(x_1, \dots, x_n), t) = 0, \tag{2.52}$$

for all $x_1, \dots, x_n \in X$ and all $t > 0$. This means that $A$ satisfies the functional (1.1), and so it is additive (see Lemma 2.1 of [4]).

Now, we approximate the difference between $f$ and $A$ in intuitionistic fuzzy sense. For every $x \in X$, $t > 0$, and sufficiently large $n$, by (2.50), we have

$$\begin{aligned}
\mu(A(x) - f(x), t) &\geq \mu\left(A(x) - \frac{f(2^n x)}{2^n}, \frac{t}{2}\right) * \mu\left(\frac{f(2^n x)}{2^n} - f(x), \frac{t}{2}\right) \\
&\geq \mu_2''\left(x, \frac{t}{2\sum_{k=0}^{\infty} (\alpha/2)^k}\right) = \mu_2''\left(x, \frac{(2-\alpha)t}{4}\right), \\
\nu(A(x) - f(x), t) &\leq \nu\left(A(x) - \frac{f(2^n x)}{2^n}, \frac{t}{2}\right) \diamond \nu\left(\frac{f(2^n x)}{2^n} - f(x), \frac{t}{2}\right) \\
&\leq \nu_2''\left(x, \frac{t}{2\sum_{k=0}^{\infty} (\alpha/2)^k}\right) = \nu_2''\left(x, \frac{(2-\alpha)t}{4}\right).
\end{aligned} \tag{2.53}$$

To prove the uniqueness of $A$, assume that $A'$ is another additive mapping from $X$ to $Y$, which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$,

$$\begin{aligned}
&\mu(A(x) - A'(x), t) \geq \mu\left(A(x) - f(x), \frac{t}{2}\right) * \mu\left(A'(x) - f(x), \frac{t}{2}\right) \geq \mu_2''\left(x, \frac{(2-\alpha)t}{8}\right), \\
&\nu(A(x) - A'(x), t) \leq \nu\left(A(x) - f(x), \frac{t}{2}\right) \diamond \nu\left(A'(x) - f(x), \frac{t}{2}\right) \leq \nu_2''\left(x, \frac{(2-\alpha)t}{8}\right).
\end{aligned} \tag{2.54}$$

Therefore, by the additivity of $A$ and $A'$, we have

$$\begin{aligned}
&\mu(A(x) - A'(x), t) = \mu(A(2^n x) - A'(2^n x), 2^n t) \geq \mu_2''\left(x, \frac{(2/\alpha)^n (2-\alpha)t}{8}\right), \\
&\nu(A(x) - A'(x), t) = \nu(A(2^n x) - A'(2^n x), 2^n t) \leq \nu_2''\left(x, \frac{(2/\alpha)^n (2-\alpha)t}{8}\right),
\end{aligned} \tag{2.55}$$

for all $x \in X,\ t > 0$ and $n \in \mathbb{N}$. Since $0 < \alpha < 2$ and $\lim_{n\to\infty}(2/\alpha)^n = \infty$, we get

$$\lim_{n\to\infty} \mu_2''\left(x, \frac{(2/\alpha)^n (2-\alpha)t}{8}\right) = 1, \qquad \lim_{n\to\infty} \nu_2''\left(x, \frac{(4/\alpha)^n (2-\alpha)t}{8}\right) = 0. \tag{2.56}$$

Therefore, $\mu(A(x) - A'(x), t) = 1$ and $\nu(A(x) - A'(x), t) = 0$ for all $x \in X$ and $t > 0$. Hence, $A(x) = A'(x)$ for all $x \in X$. This completes the proof of the theorem. □

**Theorem 2.3.** *Let $\varphi : X \to Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number $\alpha$ with $0 < |\alpha| < 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}
&\mu(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n) \geq \mu'(\varphi(x_1), t_1) * \cdots * \mu'(\varphi(x_n), t_n), \\
&\nu(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n) \leq \nu'(\varphi(x_1), t_1) \diamond \cdots \diamond \nu'(\varphi(x_n), t_n),
\end{aligned} \tag{2.57}$$

*for all $x_1, \ldots, x_n \in X$ and all $t_1, \ldots, t_n > 0$. Then there exists a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that*

$$\begin{aligned}
&\mu(Q(x) - A(x) - f(x), t) \geq M_1''\left(x, \frac{(4-|\alpha|)t}{16}\right) * \widetilde{M}_1''\left(x, \frac{(2-|\alpha|)t}{8}\right), \\
&\nu(Q(x) - A(x) - f(x), t) \leq M_2''\left(x, \frac{(4-|\alpha|)t}{16}\right) \diamond \widetilde{M}_2''\left(x, \frac{(2-|\alpha|)t}{8}\right),
\end{aligned} \tag{2.58}$$

*for all $x \in X$ and $t > 0$, where*

$$\begin{aligned}
M_1''(x,t) &:= \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \\
&\quad * \mu'\left(\varphi(-nx), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(-(n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\
&\quad * \mu'\left(\varphi(-x), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right), \\
\widetilde{M}_1''(x,t) &:= \mu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) \\
&\quad * \mu'\left(\varphi(-2x), \frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(0), \frac{4}{n^2+4n}t\right), \\
M_2''(x,t) &:= \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \\
&\quad \diamond \nu'\left(\varphi(-nx), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(-(n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\
&\quad \diamond \nu'\left(\varphi(-x), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right), \\
\widetilde{M}_2''(x,t) &:= \nu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) \\
&\quad \diamond \nu'\left(\varphi(-2x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2+4n}t\right).
\end{aligned} \tag{2.59}$$

*Proof.* Passing to the even part $f_e$ and odd part $f_o$ of $f$, we deduce from (2.57) that

$$\begin{aligned}
\mu(\Delta f_e(x_1,\ldots,x_n), t_1+\cdots+t_n) &\geq \mu'(\varphi(x_1),t_1) * \mu'(\varphi(-x_1),t_1) \\
&\quad * \cdots * \mu'(\varphi(x_n),t_n) * \mu'(\varphi(-x_n),t_n), \\
\nu(\Delta f_e(x_1,\ldots,x_n), t_1+\cdots+t_n) &\leq \nu'(\varphi(x_1),t_1) \diamond \nu'(\varphi(-x_1),t_1) \\
&\quad \diamond \cdots \diamond \nu'(\varphi(x_n),t_n) \diamond \nu'(\varphi(-x_n),t_n).
\end{aligned} \tag{2.60}$$

On the other hand,

$$\begin{aligned}
\mu(\Delta f_o(x_1,\ldots,x_n), t_1+\cdots+t_n) &\geq \mu'(\varphi(x_1),t_1) * \mu'(\varphi(-x_1),t_1) \\
&\quad * \cdots * \mu'(\varphi(x_n),t_n) * \mu'(\varphi(-x_n),t_n), \\
\nu(\Delta f_o(x_1,\ldots,x_n), t_1+\cdots+t_n) &\leq \nu'(\varphi(x_1),t_1) \diamond \nu'(\varphi(-x_1),t_1) \\
&\quad \diamond \cdots \diamond \nu'(\varphi(x_n),t_n) \diamond \nu'(\varphi(-x_n),t_n).
\end{aligned} \tag{2.61}$$

Applying the proofs of Theorems 2.1 and 2.2, we get a unique quadratic function $Q$ and a unique additive function $A$ satisfying

$$\mu(Q(x) - f_e(x), t) \geq M_1''\left(x, \frac{(4-|\alpha|)t}{8}\right), \qquad \nu(Q(x) - f_e(x), t) \leq M_2''\left(x, \frac{(4-|\alpha|)t}{8}\right). \tag{2.62}$$

Also,

$$\mu(A(x) - f_o(x),\ t) \geq \widetilde{M}_1''\left(x, \frac{(2-|\alpha|)t}{4}\right), \qquad \nu(A(x) - f_o(x), t) \leq \widetilde{M}_2''\left(x, \frac{(2-|\alpha|)t}{4}\right). \tag{2.63}$$

Therefore,

$$\begin{aligned} \mu(Q(x) - A(x) - f(x), t) &\geq M_1''\left(x, \frac{(4-|\alpha|)t}{16}\right) * \widetilde{M}_1''\left(x, \frac{(2-|\alpha|)t}{8}\right), \\ \nu(Q(x) - A(x) - f(x), t) &\leq M_2''\left(x, \frac{(4-|\alpha|)t}{16}\right) \diamond \widetilde{M}_2''\left(x, \frac{(2-|\alpha|)t}{8}\right). \end{aligned} \tag{2.64}$$

This completes the proof of the theorem. □

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