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Research Article

Certain K_0-Monoid Properties Preserved by Tracial Approximation

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We show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$: (1) pseudocancellation property, (2) weakly divisible, (3) strongly separative, (4) separative, and (5) preminimal.

1. Introduction

The Elliott conjecture asserts that all nuclear, separable C^*-algebras are classified up to isomorphism by an invariant, called the Elliott invariant. A first version of the Elliott conjecture might be said to have begun with the K-theoretical classification of AF-algebras in [1]. Since then, many classes of C^*-algebras have been found to be classified by the Elliott invariant. Among them, one important class is the class of simple unital AH-algebras. A very important axiomatic version of the classification of AH-algebras without dimension growth was given by H. Lin. Instead of assuming inductive limit structure, he started with a certain abstract approximation property and showed that C^*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth. More precisely, Lin introduced the class of tracially approximate interval algebras.

Following the notion of Lin on the tracial approximation by interval algebras, Elliott and Niu in [2] considered tracial approximation by more general C^*-algebras. Let Ω be a class of unital C^*-algebras. Then, the class of C^*-algebras which can be tracially approximated by C^*-algebra in Ω, denoted by $TA\Omega$, is defined as follows. A simple unital C^*-algebra A is said to belong to the class $TA\Omega$, if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element

$a \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \Omega$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in {}_{\varepsilon}B$ for all $x \in F$,

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

The question of the behavior of C^*-algebra properties under passage from a class Ω to the class $TA\Omega$ is interesting and sometimes important. In fact, the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections is determined by traces were used in the proof of the classification theorem in [2], and [3] by Elliott and Niu.

In this paper, we show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$:

(1) pseudocancellation property,

(2) weakly divisible,

(3) strongly separative,

(4) separative,

(5) preminimal.

2. Preliminaries and Definitions

Let a and b be two positive elements in a C^*-algebra A. We write $[a] \leq [b]$ (cf. Definition 3.5.2 in [4]), if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in \mathrm{Her}(a)$, $v^*c, cv \in A$, $vv^* = P_a$, where P_a is the range projection of a in A^{**} and $v^*cv \in \mathrm{Her}(b)$. We write $[a] = [b]$ if $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \leq [b]$, if there are n mutually orthogonal positive elements $b_1, b_2, \ldots, b_n \in \mathrm{Her}(b)$ such that $[a] \leq [b_i]$, $i = 1, 2, \ldots, n$.

Let $0 < \sigma_1 < \sigma_2 \leq 1$ be two positive numbers. Define

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1, & \text{if } t \geq \sigma_2, \\ \dfrac{t - \sigma_1}{\sigma_2 - \sigma_1}, & \text{if } \sigma_1 \leq t \leq \sigma_2, \\ 0, & \text{if } 0 < t \leq \sigma_1. \end{cases} \tag{2.1}$$

Let Ω be a class of unital C^*-algebras. Then, the class of C^*-algebras which can be tracially approximated by C^*-algebras in Ω is denoted by $TA\Omega$.

Definition 2.1 (see [2]). A simple unital C^*-algebra A is said to belong to the class $TA\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \Omega$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in {}_{\varepsilon}B$ for all $x \in F$,

(3) $[1 - p] \leq [a]$.

Definition 2.2 (see [5]). Let Ω be a class of unital C^*-algebras. A unital C^*-algebra A is said to have property (III) if, for any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element a, and any integer $n > 0$, there exist a nonzero projection $p \in A$, and a C^*-subalgebra B of A with $B \in \Omega$ and $1_B = p$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in {}_{\varepsilon}B$ for all $x \in F$, $\|pap\| \geq \|a\| - \varepsilon$,

(3) $n[f_{\sigma_1}^{\sigma_2}((1-p)a(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(pap)]$.

Lemma 2.3 (see [2]). *If the class Ω is closed under tensoring with matrix algebras or closed under taking unital hereditary C^*-subalgebras, then $TA\Omega$ is closed under passing to matrix algebras or unital hereditary C^*-subalgebras.*

Theorem 2.4 (see [5]). *Let Ω be a class of unital C^*-algebras such that Ω is closed under taking unital hereditary C^*-subalgebras and closed taking finite direct sums. Let A be a simple unital C^*-algebra. Then, the following are equivalent:*

(1) *$A \in TA\Omega$,*

(2) *A has property (III).*

Call projections $p, q \in M_\infty(A)$ equivalent, denoted $p \sim q$, when there is a partial isometry $v \in M_\infty(A)$ such that $p = v^*v$, $q = vv^*$. The equivalent classes are denoted by $[\cdot]$, and the set of all these is

$$V(A) := \left\{ [p] \mid p = p^* = p^2 \in M_\infty(A) \right\}. \tag{2.2}$$

Addition in $V(A)$ is defined by

$$[p] + [q] := [\mathrm{diag}(p, q)]. \tag{2.3}$$

$V(A)$ becomes an abelian monoid, and we call $V(A)$ the K_0-monoid of A.

All abelian monoids have a natural preorder, the algebraic ordering, defined as follow: if $x, y \in M$, we write $x \leq y$ if there is a z in M such that $x + z = y$. In the case of $V(A)$, the algebraic ordering is given by Murray-von Neumann subequivalence, that is, $[p] \leq [q]$ if and only if there is a projection $p' \leq q$ such that $p \sim p'$. We also write, as is customary, $p \preceq q$ to mean that p is subequivalent to q.

If $x, y \in M$, we will write $x \leq^* y$ if there is a nonzero element z in M, such that $x + z = y$.

Let us recall that an element u in a monoid M is an order unit provided $u \neq 0$, and, for any x in M, there is $n \in \mathbb{N}$ such that $x \leq nu$.

Let M be an order monoid and $x, y \in M$. We write $x \preceq y$ if and only if there exists an integer $n > 0$ such that $x \leq ny$. We write $x \ll y$ if and only if $x + y = y$.

We say that a monoid M is conical if $x + y = 0$ only when $x = y = 0$. Note that, for any C^*-algebra A, the monoid $V(A)$ is conical.

We say that an order monoid M has the pseudocancellation property when it satisfies the statement that, for any a, b, c, d with $a + c \leq b + c$, there exists $d \ll c$ such that $a \leq b + d$.

Let M be a monoid. An element x in M will be termed weakly divisible if there exist a and b in M such that $x = 2a + 3b$. We say that M is weakly divisible if every element is weakly divisible. We say that M has weak divisible for order units if every unit is weakly divisible.

We say that an order monoid M is said to be strongly separative when it satisfies the statement that, for any $x, y \in M$ such that $2x = x + y$, we have $x = y$.

Definition 2.5 (see [6]). We say that an order monoid M is preminimal when it satisfies both following statements:

(1) $a + d \leq b + d$ for any a, b, c, d with $a + c \leq b + c$ and $c \leq d$,

(2) $a + d = b + d$ for any a, b, c, d with $a + c = b + c$ and $c \leq d$.

Definition 2.6 (see [6]). We say that an order monoid M is separative when it satisfies both following statements:

(1) $a \leq b$ for any a, b, c with $a + c \leq b + c$ and $c \preceq b$,

(2) $a = b$ for any a, b, c with $a + c = b + c$ and $c \preceq a, b$.

3. The Main Results

Theorem 3.1. *Let Ω be a class of unital C^*-algebras such that for any $B \in \Omega$ the K_0-monoid $V(B)$ has the pseudocancellation property. Then, the K_0-monoid $V(A)$ has the pseudocancellation property for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that there exists $d \ll c$ such that $a \leq b + d$ for any $a, b, c, d \in V(A)$ with $a + c \leq b + c$. By Lemma 2.3, we may assume that $a = [p], b = [q], c = [e]$ for some projections $p, q, e \in \mathrm{proj}(A)$. For $F = \{p, q, e\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega, 1_B = r$ such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(2) $rxr \in {}_{\varepsilon}B$ for all $x \in F$.

By (1) and (2), there exist projections $p_1, q_1, e_1 \in B$ and $p_2, q_2, e_2 \in (1 - r)A(1 - r)$ such that

$$\|p - p_1 - p_2\| < \varepsilon, \qquad \|q - q_1 - q_2\| < \varepsilon, \qquad \|e - e_1 - e_2\| < \varepsilon. \tag{3.1}$$

Therefore, we have

$$\begin{gathered} [p] = [p_1] + [p_2], \qquad [q] = [q_1] + [q_2], \qquad [e] = [e_1] + [e_2], \\ [p_1] + [e_1] \leq [q_1] + [e_1], \qquad [p_2] + [e_2] \leq [e_2] + [q_2]. \end{gathered} \tag{3.2}$$

Since $B \in \Omega$ and $V(B)$ has the pseudocancellation property, we may assume that there exists a projection $f \in A$ such that $[f] \ll [e_1]$ and $[p_1] \leq [q_1] + [f]$ in $V(B)$.

For $G = \{p_2, q_2, e_2, f\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $s \in A$ and a C^*-subalgebra $C \subseteq A$ with $C \in \Omega, 1_C = s$ such that

(1′) $\|xs - sx\| < \varepsilon$ for all $x \in G$, (2′) $sxs \in {}_{\varepsilon}C$ for all $x \in G$,

(3′) $[1-s] \le [f]$.

By (1′) and (2′), there exist projections $p_3, q_3, e_3 \in C$ and $p_4, q_4, e_4 \in (1-s)A(1-s)$ such that

$$\|p_2 - p_3 - p_4\| < \varepsilon, \qquad \|q_2 - q_3 - q_4\| < \varepsilon, \qquad \|e_2 - e_3 - e_4\| < \varepsilon. \tag{3.3}$$

Therefore, we have

$$\begin{gathered} [p_2] = [p_3] + [p_4], \qquad [q_2] = [q_3] + [q_4], \qquad [e_2] = [e_3] + [e_4], \\ [p_3] + [e_3] \le [e_3] + [q_3], \qquad [p_4] + [e_4] \le [e_4] + [q_4]. \end{gathered} \tag{3.4}$$

Since $C \in \Omega$ and $V(C)$ has the pseudocancellation property, we may assume that there exists a projection $g \in A$ such that $[g] \ll [e_3]$ and $[p_3] \le [q_3] + [g]$ in $V(C)$.

By (3′), we have $[e_4] \le [1-s] \le [f]$, there exists a partial isometry $v \in A$ such that $vv^* = e_4$, $v^*v \le f$.

Therefore, we have

$$\begin{aligned} [p] &= [p_1] + [p_3] + [p_4] \\ &\le [q_1] + [f] + [q_3] + [g] + [p_4] \\ &\le [q_1] + [f - v^*v] + [vv^*] + [q_3] + [g] + [p_4] \\ &\le [q_1] + [f - v^*v] + [q_4] + [e_4] + [g] + [q_3] \\ &\le [q_1] + [f] + [q_3] + [q_4] + [g] \\ &\le [q] + [f] + [g]. \end{aligned} \tag{3.5}$$

Since $[f] + [g] + [e_1] + [e_3] = [e_1] + [e_3]$, therefore $[f] + [g] + [e] = [e]$, that is, $[f] + [g] \ll [e]$. □

Theorem 3.2. *Let Ω be a class of unital C^*-algebras such that, for any $B \in \Omega$, the K_0-monoid $V(B)$ is weakly divisible. Then, the K_0-monoid $V(A)$ is weakly divisible for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that there exist a and b in $V(A)$ such that $x = 2a + 3b$ for any $x \in V(A)$. By Lemma 2.3, we may assume that $x = [p]$ for some projection $p \in \mathrm{proj}(A)$. For $F = \{p\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = r$ such that

(1) $\|pr - rp\| < \varepsilon$,

(2) $rpr \in {}_{\varepsilon}B$.

By (1) and (2), there exist projections $p_1 \in B$ and $p_2 \in (1-r)A(1-r)$ such that

$$\|p - p_1 - p_2\| < \varepsilon. \tag{3.6}$$

Therefore, we have

$$[p] = [p_1] + [p_2]. \tag{3.7}$$

Since $B \in \Omega$ and $V(B)$ is weakly divisible, we may assume that there exist projections $e, f \in B$ such that $[p_1] = 2[e] + 3[f]$ in $V(B)$.

For $G = \{p_2, e, f\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $s \in A$ and a C^*-subalgebra $C \subseteq A$ with $C \in \Omega$, $1_C = s$ such that

(1′) $\|xs - sx\| < \varepsilon$ for all $x \in G$,

(2′) $sxs \in {}_{\varepsilon}C$ for all $x \in G$,

(3′) $3[1 - s] \leq [e]$.

By (1′) and (2′), there exist projections $p_3 \in C$ and $p_4 \in (1 - s)A(1 - s)$ such that

$$\|p_2 - p_3 - p_4\| < \varepsilon. \tag{3.8}$$

Therefore, we have

$$[p_2] = [p_3] + [p_4]. \tag{3.9}$$

Since $C \in \Omega$ and $V(C)$ is weakly divisible, we may assume that there exist projections $g, h \in B$ such that $[p_3] = 2[g] + 3[h]$ in $V(C)$.

By (3′), we have $3[p_4] \leq [e]$, and there exist a partial isometry $v \in A$ such that $vv^* = p_4$, $v^*v \leq e$.

Therefore, we have

$$\begin{aligned}
[p] &= [p_1] + [p_3] + [p_4] \\
&= 2[e] + 3[f] + 2[g] + 3[h] + [p_4] \\
&= 2[e - v^*v] + 2[v^*v] + [vv^*] + 3[f] + 2[g] + 3[h] \\
&= 2[e - v^*v] + 3[v^*v] + 3[f] + 2[g] + 3[h] \\
&= 2([e - v^*v] + [g]) + 3([v^*v] + [f] + [h]).
\end{aligned} \tag{3.10}$$

□

Theorem 3.3. *Let Ω be a class of unital C^*-algebras such that, for any $B \in \Omega$, the K_0-monoid $V(B)$ is strongly separative. Then, the K_0-monoid $V(A)$ is strongly separative for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that $x = y$ for any $x, y \in V(A)$ with $2x = x + y$. By Lemma 2.3, we may assume that $x = [p]$, $y = [q]$ for some projections $p, q \in \mathrm{proj}(A)$. For $F = \{p, q\}$, any $\varepsilon > 0$, any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, since $A \in TA\Omega$, by Theorem 2.4, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = r$ such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(2) $rxr \in {}_{\varepsilon}B$ for all $x \in F$,

(3) $[f_{\sigma_1}^{\sigma_2}((1 - r)p(1 - r))] \leq [f_{\sigma_3}^{\sigma_4}(rpr)]$.

By (1) and (2), there exist projections $p_1, q_1 \in B$ and $p_2, q_2 \in (1-r)A(1-r)$ such that

$$\|p - p_1 - p_2\| < \varepsilon, \qquad \|q - q_1 - q_2\| < \varepsilon. \tag{3.11}$$

Therefore, we have

$$\begin{aligned} [p] &= [p_1] + [p_2], \qquad & [q] &= [q_1] + [q_2], \\ [p_1] + [p_1] &= [p_1] + [q_1], \qquad & [p_2] + [p_2] &= [p_2] + [q_2]. \end{aligned} \tag{3.12}$$

Since $B \in \Omega$ and $V(B)$ is strongly separative, we have $[p_1] = [q_1]$ in $V(B)$.

By (3), we have $[p_2] \leq [p_1]$, there exists a partial isometry $v \in A$ such that $v^*v = p_2$, $vv^* \leq p_1$. Therefore, we have

$$\begin{aligned} [p] &= [p_1] + [p_2] \\ &= [p_1 - vv^*] + [vv^*] + [p_2] \\ &= [p_1 - vv^*] + [p_2] + [p_2] \\ &= [p_1 - vv^*] + [p_2] + [q_2] \\ &= [p_1] + [q_2] \\ &= [q_1] + [q_2] \\ &= [q]. \end{aligned} \tag{3.13}$$

□

Theorem 3.4. *Let Ω be a class of unital C^*-algebras such that, for any $B \in \Omega$, the K_0-monoid $V(B)$ is separative. Then, the K_0-monoid $V(A)$ is separative for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We prove this theorem by two steps.

Firstly, we need to show that $a \leq b$ for any $a, b, c \in V(A)$ with $a + c \leq b + c$ and $c \precsim b$. By Lemma 2.3, we may assume that $a = [p]$, $b = [q]$, $c = [t]$ for some projections $p, q, t \in \operatorname{proj}(A)$. For $F = \{p, q, t\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = r$ such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(2) $rxr \in_{\varepsilon} B$ for all $x \in F$.

By (1) and (2), there exist projections $p_1, q_1, t_1 \in B$ and $p_2, q_2, t_2 \in (1-r)A(1-r)$ such that

$$\|p - p_1 - p_2\| < \varepsilon, \qquad \|q - q_1 - q_2\| < \varepsilon, \qquad \|t - t_1 - t_2\| < \varepsilon. \tag{3.14}$$

Therefore, we have

$$\begin{gathered}[p]=[p_1]+[p_2], \qquad [q]=[q_1]+[q_2], \qquad [t]=[t_1]+[t_2], \\ [p_1]+[t_1]\le[q_1]+[t_1], \qquad [p_2]+[t_2]\le[t_2]+[q_2], \\ [t_1]\preceq[q_1], \qquad [t_2]\preceq[q_2].\end{gathered} \tag{3.15}$$

Since $B\in\Omega$ and $V(B)$ is separative, we have $[p_1]\le[q_1]$ in $V(B)$.

For $G=\{p_1,p_2,q_2,t_2\}$, any $\varepsilon>0$, since $A\in TA\Omega$, there exist a projection $w\in A$ and a C^*-subalgebra $C\subseteq A$ with $C\in\Omega$, $1_C=w$ such that

(1′) $\|xw-wx\|<\varepsilon$ for all $x\in G$,

(2′) $wxw\in{}_\varepsilon C$ for all $x\in G$,

(3′) $2[1-w]\le[p_1]$.

By (1′) and (2′), there exist projections $p_3,q_3,t_3\in C$ and $p_4,q_4,t_4\in(1-w)A(1-w)$ such that

$$\|p_2-p_3-p_4\|<\varepsilon, \qquad \|q_2-q_3-q_4\|<\varepsilon, \qquad \|t_2-t_3-t_4\|<\varepsilon. \tag{3.16}$$

Therefore, we have

$$\begin{gathered}[p_2]=[p_3]+[p_4], \qquad [q_2]=[q_3]+[q_4], \qquad [t_2]=[t_3]+[t_4], \\ [p_3]+[t_3]\le[t_3]+[q_3], \qquad [p_4]+[t_4]\le[t_4]+[q_4], \\ [t_3]\preceq[q_3], \qquad [t_4]\preceq[q_4].\end{gathered} \tag{3.17}$$

Since $C\in\Omega$ and $V(C)$ is separative, we have $[p_3]\le[q_3]$ in $V(C)$.

By (3′), we have $[t_4]\le[1-w]<^*[p_1]$, there exists a partial isometry $v\in A$ such that $vv^*=t_4$, $v^*v\le p_1$.

Therefore, we have

$$\begin{aligned}[p]&=[p_1]+[p_3]+[p_4]\\ &\le[p_1-v^*v]+[vv^*]+[p_3]+[p_4]\\ &\le[p_1-v^*v]+[q_4]+[p_3]+[t_4]\\ &\le[p_1]+[q_3]+[q_4]\\ &\le[q_1]+[q_3]+[q_4]\\ &=[q].\end{aligned} \tag{3.18}$$

Secondly, with the same methods and technique, we can show that $a=b$ for any $a,b,c\in V(A)$ with $a+c=b+c$ and $c\preceq a,b$. $\square$

Theorem 3.5. *Let Ω be a class of unital C^*-algebras such that, for any $B \in \Omega$, the K_0-monoid $V(B)$ is a preminimal monoid. Then, the K_0-monoid $V(A)$ is a preminimal monoid for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We prove this theorem by two steps.

Firstly, we need to show that $a + d \leq b + d$ for any $a, b, c, d \in V(A)$ with $a + c \leq b + c$ and $c \leq d$. By Lemma 2.3, we may assume that $a = [p]$, $b = [q]$, $c = [e]$, $d = [f]$ for some projections $p, q, e, f \in \mathrm{proj}(A)$. For $F = \{p, q, e, f\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega, 1_B = r$ such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(2) $rxr \in_{\varepsilon} B$ for all $x \in F$.

By (1) and (2), there exist projections $p_1, q_1, e_1, f_1 \in B$ and $p_2, q_2, e_2, f_2 \in (1-r)A(1-r)$ such that

$$\begin{aligned} &\|p - p_1 - p_2\| < \varepsilon, \qquad \|q - q_1 - q_2\| < \varepsilon, \\ &\|e - e_1 - e_2\| < \varepsilon, \qquad \|f - f_1 - f_2\| < \varepsilon. \end{aligned} \tag{3.19}$$

Therefore, we have

$$\begin{aligned} [p] &= [p_1] + [p_2], & [q] &= [q_1] + [q_2], \\ [e] &= [e_1] + [e_2], & [f] &= [f_1] + [f_2], \\ [p_1] + [e_1] &\leq [q_1] + [e_1], & [p_2] + [e_2] &\leq [e_2] + [q_2], \\ [e_1] &\leq [f_1], & [e_2] &\leq [f_2]. \end{aligned} \tag{3.20}$$

Since $B \in \Omega$ and $V(B)$ is preminimal, we have $[p_1] + [f_1] \leq [q_1] + [f_1]$ in $V(B)$.

Since $[e_2] \leq [f_2]$, there exists a partial isometry $v \in A$ such that $vv^* = e_2, v^*v \leq f_2$. Therefore, we have

$$\begin{aligned} [p] + [f] &= [p_1] + [p_2] + [f_1] + [f_2] \\ &= [p_1] + [p_2] + [f_1] + [f_2 - v^*v] + [v^*v] \\ &\leq [p_1] + [q_2] + [f_1] + [f_2 - v^*v] + [e_2] \\ &\leq [p_1] + [q_2] + [f_1] + [f_2] \\ &\leq [q_1] + [q_2] + [f_1] + [f_2] \\ &= [q] + [f]. \end{aligned} \tag{3.21}$$

Secondly, with the same methods and technique, we can show that $a + c = b + c$ for any $a, b, c, d \in V(A)$ with $a + d = b + d$ and $c \leq d$. $\square$

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