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Research Article

Optimal Lower Power Mean Bound for the Convex Combination of Harmonic and Logarithmic Means

Yu-Ming Chu,[1] Shan-Shan Wang,[2] and Cheng Zong[2]

[1] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*
[2] *School of Science, Hangzhou Normal University, Hangzhou 310012, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

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We find the least value $\lambda \in (0,1)$ and the greatest value $p = p(\alpha)$ such that $\alpha H(a,b)+(1-\alpha)L(a,b) > M_p(a,b)$ for $\alpha \in [\lambda, 1)$ and all $a, b > 0$ with $a \neq b$, where $H(a,b)$, $L(a,b)$, and $M_p(a,b)$ are the harmonic, logarithmic, and p-th power means of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$, the p-th power mean $M_p(a,b)$ and logarithmic mean $L(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1.1}$$

$$L(a,b) = \begin{cases} \dfrac{b-a}{\log b - \log a}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively.

It is well known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed a, $b > 0$ with $a \neq b$. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ and $L(a,b)$ can be found in the literature [1–17]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [18–20]. In [18], the authors study a variant of Jensen's functional equation involving L, which appears in a heat conduction problem. A representation of L as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [8]. In [21, 22], it is shown that L can be expressed in terms of Gauss's hypergeometric function ${}_2F_1$. And, in [21], the authors prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \cdots < a_n$ and $0 < b_1 < b_2 < \cdots < b_n$, is positive for all $n \geq 1$.

Let $A(a,b) = 1/2(a+b), I(a,b) = 1/e(b^b/a^a)^{1/(b-a)} (b \neq a), I(a,b) = a\ (b = a), G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, geometric, and harmonic means of two positive numbers a and b, respectively, then it is well known that

$$\begin{aligned} \min\{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) \\ < L(a,b) < I(a,b) < A(a,b) = M_1(a,b) < \max\{a,b\} \end{aligned} \tag{1.3}$$

for all $a, b > 0$ with $a \neq b$.

In [23], Alzer and Janous established the following best possible inequality:

$$M_{\log 2/\log 3}(a,b) < \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) < M_{2/3}(a,b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$.

In [8, 11, 24], the authors presented bounds for L in terms of G and A

$$G^{2/3}(a,b)A^{1/3}(a,b) < L(a,b) < \frac{2}{3}G(a,b) + \frac{1}{3}A(a,b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of L and I. A proof can be found in [25]

$$G^{1/2}(a,b)A^{1/2}(a,b) < L^{1/2}(a,b)I^{1/2}(a,b) < \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) < \frac{1}{2}G(a,b) + \frac{1}{2}A(a,b) \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for L, I, $(LI)^{1/2}$, and $(L+I)/2$ in terms of the power means are proved in [4, 5, 7, 9, 16, 25, 26]:

$$\begin{gathered} M_0(a,b) < L(a,b) < M_{1/3}(a,b), \\ M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) < L^{1/2}(a,b)I^{1/2}(a,b) < M_{1/2}(a,b), \\ \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) < M_{1/2}(a,b) \end{gathered} \tag{1.7}$$

for all $a, b > 0$ with $a \neq b$.

Alzer and Qiu [27] found the sharp bound of $1/2(L(a,b) + I(a,b))$ in terms of the power mean as follows:

$$M_c(a,b) < \frac{1}{2}(L(a,b) + I(a,b)) \tag{1.8}$$

for all $a, b > 0$ with $a \neq b$, with the best possible parameter $c = \log 2/(1 + \log 2)$.

The main purpose of this paper is to find the least value $\lambda \in (0,1)$ and the greatest value $p = p(\alpha)$ such that $\alpha H(a,b) + (1-\alpha)L(a,b) > M_p(a,b)$ for $\alpha \in [\lambda, 1)$ and all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to establish our main result we need three lemmas, which we present in this section.

Lemma 2.1. *Let* $\alpha \in (1/4, 1)$, $p = (1-4\alpha)/3 \in (-1, 0)$, *and* $f(t) = -4\alpha p(p+1)^2(p+2)t^{p-1} + 2(1-\alpha)p^2(1-p^2)t^{p-2} + 2(1-\alpha)p(1-p)^2(2-p)t^{p-3} + 12(1-\alpha)(1-p)$. *Then* $f(t) > 0$ *for* $t \in [1, +\infty)$.

Proof. Simple computations lead to

$$f(1) = \frac{64}{81}(1-\alpha)^2\left(56\alpha^2 + 23\alpha + 11\right) > 0, \tag{2.1}$$

$$\lim_{t \to +\infty} f(t) = 12(1-\alpha)(1-p) = 8(1-\alpha)(1+2\alpha) > 0, \tag{2.2}$$

$$f'(t) = -2p(1-p)t^{p-4} f_1(t), \tag{2.3}$$

where

$$\begin{gathered} f_1(t) = -2\alpha(p+1)^2(p+2)t^2 + (1-\alpha)p(p+1)(2-p)t + (1-\alpha)(1-p)(2-p)(3-p), \\ f_1(1) = \frac{4}{27}(1-\alpha)\left(148\alpha^2 - 11\alpha + 25\right) > 0, \end{gathered} \tag{2.4}$$

$$\lim_{t \to +\infty} f_1(t) = -\infty, \tag{2.5}$$

$$\begin{aligned} f'_1(t) &= -4\alpha(p+1)^2(p+2)t + (1-\alpha)p(p+1)(2-p) \\ &= -\frac{4}{27}(1-\alpha)^2[16\alpha(7-4\alpha)t + (4\alpha-1)(4\alpha+5)] < 0 \end{aligned} \tag{2.6}$$

for $t \in [1, +\infty)$.

Inequality (2.6) implies that $f_1(t)$ is strictly decreasing in $[1, +\infty)$, then from (2.4) and (2.5) we know that $\lambda_1 > 1$ exists such that $f_1(t) > 0$ for $t \in [1, \lambda_1)$ and $f_1(t) < 0$ for $t \in (\lambda_1, +\infty)$. Hence, equation (2.3) leads to the conclusion that $f(t)$ is strictly increasing in $[1, \lambda_1]$ and strictly decreasing in $[\lambda_1, +\infty)$. □

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the piecewise monotonicity of $f(t)$.

Lemma 2.2. *Let* $\alpha \in (1/4, 1)$, $p = (1-4\alpha)/3 \in (-1, 0)$, *and* $g(t) = -(1-\alpha)(p+1)(p+2)^2(p+3)t^p + (p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t^{p-1} + (1-\alpha)p(p^3 - 8p^2 - p + 4)t^{p-2} + (1-\alpha)(1-p)(p^3 + 5p^2 - 14p + 4)t^{p-3} + 4(1-\alpha)(7-4p) - 4p(1-\alpha)t^{-1} + 4\alpha(1+p)t^{-2}$, *then* $g(t) > 0$ *for* $t \in [1, +\infty)$.

Proof. Let $g_1(t) = -(1-\alpha)(p+1)(p+2)^2(p+3)t^3 + (p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t^2 + (1-\alpha)p(p^3 - 8p^2 - p + 4)t + (1-\alpha)(1-p)(p^3 + 5p^2 - 14p + 4) + 4(1-\alpha)(7-4p)t^{3-p} - 4p(1-\alpha)t^{2-p} + 4\alpha(1+p)t^{1-p}$. Then simple computations lead to

$$g(t) = t^{p-3}g_1(t), \tag{2.7}$$

$$g_1(1) = \frac{16}{27}(1-\alpha)\left(80\alpha^2 + 110\alpha - 1\right) > 0, \tag{2.8}$$

$g_1'(t) = -3(1-\alpha)(p+1)(p+2)^2(p+3)t^2 + 2(p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t + (1-\alpha)p(p^3 - 8p^2 - p + 4) + 4(1-\alpha)(7-4p)(3-p)t^{2-p} - 4p(1-\alpha)(2-p)t^{1-p} + 4\alpha(1-p^2)t^{-p}$,

$$g_1'(1) = \frac{32}{27}(1-\alpha)\left(-16\alpha^3 + 38\alpha^2 + 176\alpha - 9\right) > 0, \tag{2.9}$$

$g_1''(t) = -6(1-\alpha)(p+1)(p+2)^2(p+3)t + 2(p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha) + 4(1-\alpha)(7-4p)(3-p)(2-p)t^{1-p} - 4p(1-\alpha)(2-p)(1-p)t^{-p} - 4\alpha p(1-p^2)t^{-p-1}$,

$$\begin{aligned} g_1''(1) &= \frac{8}{81}(1-\alpha)\left(-128\alpha^4 + 896\alpha^3 + 288\alpha^2 + 5294\alpha - 437\right) \\ &> 0, \end{aligned} \tag{2.10}$$

$g_1'''(t) = -6(1-\alpha)(p+1)(p+2)^2(p+3) + 4(1-\alpha)(7-4p)(3-p)(2-p)(1-p)t^{-p} + 4p^2(1-\alpha)(2-p)(1-p)t^{-p-1} + 4\alpha p(1+p)^2(1-p)t^{-p-2}$

$$\begin{aligned} g_1'''(1) &= \frac{8}{81}(1-\alpha)\left(576\alpha^4 + 3872\alpha^3 + 660\alpha^2 + 6612\alpha - 785\right) \\ &> 0, \end{aligned} \tag{2.11}$$

$$g_1^{(4)}(t) = -4p(1-p)t^{-p-3}g_2(t), \tag{2.12}$$

where

$$\begin{gathered} g_2(t) = (1-\alpha)(7-4p)(3-p)(2-p)t^2 + (1-\alpha)p(2-p)(p+1)t + \alpha(p+1)^2(p+2), \\ g_2(1) = \frac{4}{27}(1-\alpha)\left(96\alpha^3 + 232\alpha^2 + 388\alpha + 175\right) > 0, \end{gathered} \tag{2.13}$$

$$\begin{aligned} g_2'(t) &= 2(1-\alpha)(7-4p)(3-p)(2-p)t + (1-\alpha)p(2-p)(p+1) \\ &\geq g_2'(1) = \frac{4}{9}(1-\alpha)(5+4\alpha)\left(12\alpha^2 + 31\alpha + 23\right) > 0 \end{aligned} \tag{2.14}$$

for $t \in [1, +\infty)$. □

From (2.13) and (2.14), we clearly see that $g_2(t) > 0$ for $t \in [1, +\infty)$, then (2.12) leads to the conclusion that $g_1'''(t)$ is strictly in $[1, +\infty)$.

Therefore, Lemma 2.2 follows from (2.7)–(2.11) and the monotonicity of $g_1'''(t)$.

Lemma 2.3. *Let* $\alpha \in (1/4, 1)$, $p = (1-4\alpha)/3 \in (-1, 0)$, *and* $h(t) = 2\alpha(1-t^{p+1})t\log^2 t + (1-\alpha)(1+t^{p-1})(1+t)^2 t\log t + (1-\alpha)(1+t)^2(1-t)(t^p+1)$, *then* $h(t) > 0$ *for* $t \in (1, +\infty)$.

Proof. Let $h_1(t) = t^{-p}h''(t)$ and $h_2(t) = t^{p+2}h_1'(t)$, then simple computations lead to

$$h(1) = 0, \tag{2.15}$$

$h'(t) = 2\alpha[1-(p+2)t^{p+1}]\log^2 t + [(p+2-\alpha p - 6\alpha)t^{p+1} + 2(1-\alpha)(p+1)t^p + (1-\alpha)pt^{p-1} + 3(1-\alpha)t^2 + 4(1-\alpha)t + 3\alpha + 1]\log t - (1-\alpha)[(p+3)t^{p+2} + (p+1)t^{p+1} - (p+3)t^p - (p+1)t^{p-1} + 2t^2 - 2]$,

$$h'(1) = 0, \tag{2.16}$$

$h_1(t) = -2\alpha(p+1)(p+2)\log^2 t + [(p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2) + 2(1-\alpha)p(p+1)t^{-1} - (1-\alpha)p(1-p)t^{-2} + 6(1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + 4\alpha t^{-1-p}]\log t - (1-\alpha)(p+2)(p+3)t + (1-\alpha)(p^2+5p+2)t^{-1} + (1-\alpha)(p^2+p-1)t^{-2} - (1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + (1+3\alpha)t^{-1-p} - (1-\alpha)p^2 - (1-\alpha)p - 5\alpha + 1$,

$$h_1(1) = 0, \tag{2.17}$$

$h_2(t) = -[4\alpha(p+1)(p+2)t^{p+1} + 2(1-\alpha)p(p+1)t^p - 2(1-\alpha)p(1-p)t^{p-1} - 6(1-\alpha)(1-p)t^2 + 4(1-\alpha)pt + 4\alpha(1+p)]\log t - (1-\alpha)(p+2)(p+3)t^{p+2} + (p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2)t^{p+1} + (1-\alpha)(p^2 - 3p - 2)t^p - (1-\alpha)(p^2+3p-2)t^{p-1} + (1-\alpha)(p+5)t^2 + 4(1-\alpha)(1-p)t + \alpha - 3\alpha p - p - 1$,

$$h_2(1) = 0, \tag{2.18}$$

$h_2'(t) = -[4\alpha(p+1)^2(p+2)t^p + 2(1-\alpha)p^2(p+1)t^{p-1} + 2(1-\alpha)p(1-p)^2 t^{p-2} - 12(1-\alpha)(1-p)t + 4(1-\alpha)p]\log t - (1-\alpha)(p+2)^2(p+3)t^{p+1} + (p+1)(p^2 - \alpha p^2 - 15\alpha p + 3p - 22\alpha + 2)t^p + (1-\alpha)p(p^2 - 5p - 4)t^{p-1} + (1-\alpha)(1-p)(p^2+5p-2)t^{p-2} + 4(1-\alpha)(4-p)t - 4\alpha(p+1)t^{-1} + 4(1-\alpha)(1-2p)$,

$$\begin{aligned} h_2'(1) &= 0, \\ h_2''(t) &= f(t)\log t + g(t), \end{aligned} \tag{2.19}$$

where $f(t)$ and $g(t)$ are defined as in Lemmas 2.1 and 2.2, respectively.

From (2.19) and (2.10) together with Lemmas 2.1 and 2.2, we clearly see that $h_2(t)$ is strictly increasing in $[1, +\infty)$. □

Therefore, Lemma 2.3 follows from (2.15)–(2.18) and the monotonicity of $h_2(t)$.

3. Main Result

Theorem 3.1. *Inequality*

$$\alpha H(a,b) + (1-\alpha)L(a,b) > M_{(1-4\alpha)/3}(a,b) \tag{3.1}$$

holds for $\alpha \in [1/4, 1)$ *and all* $a, b > 0$ *with* $a \neq b$*, and* $M_{(1-4\alpha)/3}(a,b)$ *is the best possible lower power mean bound for the sum* $\alpha H(a,b) + (1-\alpha)L(a,b)$*.*

Proof. We divide the proof of inequality (3.1) into two cases.

Case 1 ($\alpha = 1/4$). Without loss of generality, we assume that $a > b$ and put $t = \sqrt{a/b} > 1$, then from (1.1) and (1.2), we have

$$\begin{aligned} &\alpha H(a,b) + (1-\alpha)L(a,b) - M_{(1-4\alpha)/3}(a,b) \\ &\quad = \frac{1}{4}[H(a,b) + 3L(a,b)] - \sqrt{ab} \\ &\quad = \frac{3t^4 - 4(2t^3 - t^2 + 2t)\log t - 3}{8(t^2+1)\log t} b. \end{aligned} \tag{3.2}$$

Let

$$F(t) = 3t^4 - 4\left(2t^3 - t^2 + 2t\right)\log t - 3, \tag{3.3}$$

then simple computations lead to

$$\begin{aligned} F(1) &= 0, \\ F'(t) &= 4\left(3t^3 - 2t^2 + t - 2\right) - 8\left(3t^2 - t + 1\right)\log t, \\ F'(1) &= 0, \\ F''(t) &= \frac{4}{t}F_1(t), \end{aligned} \tag{3.4}$$

where $F_1(t) = 9t^3 - 10t^2 + 3t - 2 - 2(6t-1)t\log t$,

$$\begin{aligned} F''(1) &= F_1(1) = 0, \\ F_1'(t) &= 27t^2 - 32t + 5 - 2(12t-1)\log t, \\ F_1'(1) &= 0, \\ F_1''(t) &= \frac{2}{t}F_2(t), \end{aligned} \tag{3.5}$$

where $F_2(t) = 27t^2 - 12t\log t - 28t + 1$,

$$F_1''(1) = F_2(1) = 0,$$
$$F_2'(t) = 54t - 12\log t - 40 > 0 \tag{3.6}$$

for $t > 1$.

Therefore, inequality (3.1) follows easily from (3.2)–(3.6).

Case 2 ($\alpha \in (1/4, 1)$). Without loss of generality, we assume that $a > b$. Let $p = (1-4\alpha)/3 \in (-1, 0)$ and $t = a/b > 1$, then from (1.1) and (1.2), one has

$$\begin{aligned}
&\alpha H(a,b) + (1-\alpha)L(a,b) - M_{(1-4\alpha)/3}(a,b) \\
&\quad = \alpha H(a,b) + (1-\alpha)L(a,b) - M_p(a,b) \\
&\quad = b\left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t} - \left(\frac{t^p+1}{2}\right)^{1/p}\right].
\end{aligned} \tag{3.7}$$

Let

$$G(t) = \log\left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t}\right] - \frac{1}{p}\log\frac{t^p+1}{2}. \tag{3.8}$$

Then simple computations lead to

$$\lim_{t\to 1} G(t) = 0, \tag{3.9}$$

$$G'(t) = \frac{h(t)}{t(t+1)(t^p+1)\log t\left[2\alpha t\ \log t + (1-\alpha)(t^2-1)\right]}, \tag{3.10}$$

where $h(t)$ is defined as in Lemma 2.3.

From Lemma 2.3 and (3.10), we clearly see that $G(t)$ is strictly increasing in $(1, +\infty)$. □

Therefore, inequality (3.1) follows from (3.7)–(3.9) and the monotonicity of $G(t)$.

Next, we prove that $M_{(1-4\alpha)/3}(a,b)$ is the best possible lower power mean bound for the sum $\alpha H(a,b) + (1-\alpha)L(a,b)$ if $\alpha \in [1/4, 1)$.

For any $\alpha \in [1/4, 1)$, $p > (1-4\alpha)/3$, and $x > 0$, one has

$$M_p(1+x,1) - \alpha H(1+x,1) - (1-\alpha)L(1+x,1) = \frac{J(x)}{2^{1/p}\left(1+\dfrac{x}{2}\right)\log(1+x)}, \tag{3.11}$$

where $J(x) = (1+x/2)[1+(1+x)^p]^{1/p}\log(1+x) - 2^{1/p}[\alpha(1+x)\log(1+x)+(1-\alpha)x(1+x/2)]$. Letting $x \to 0$ and making use of Taylor expansion, we have

$$J(x) = \frac{2^{1/p}}{8}\left(p - \frac{1-4\alpha}{3}\right)x^3 + o\left(x^3\right). \tag{3.12}$$

Equations (3.11) and (3.12) imply that for any $\alpha \in [1/4, 1)$ and $p > (1-4\alpha)/3$ there exists $\delta > 0$, such that $\alpha H(1+x,1) + (1-\alpha)L(1+x,1) < M_p(1+x,1)$ for $x \in (0,\delta)$.

Remark 3.2. If $0 < \alpha < 1/4$, then from (1.1) and (1.2), we have

$$\lim_{x\to+\infty}\frac{M_{(1-4\alpha)/3}(1,x)}{\alpha H(1,x)+(1-\alpha)L(1,x)} = 2^{3/(4\alpha-1)} \times \lim_{x\to+\infty}\frac{\left(1+x^{(4\alpha-1)/3}\right)^{3/(1-4\alpha)}}{2\alpha/(x+1)+\left((1-1/x)(1-\alpha)/\log x\right)} = +\infty. \tag{3.13}$$

Equation (3.13) implies that for any $0 < \alpha < 1/4$, there exists $X > 1$, such that $M_{(1-4\alpha)/3}(1,x) > \alpha H(1,x) + (1-\alpha)L(1,x)$ for $x \in (X, +\infty)$. Therefore, $\lambda = 1/4$ is the least value of λ in $(0,1)$ such that inequality (3.1) holds for all $a, b > 0$ with $a \neq b$.

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