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Research Article

On Convergents Infinite Products and Some Generalized Inverses of Matrix Sequences

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The definition of convergence of an infinite product of scalars is extended to the infinite usual and Kronecker products of matrices. The new definitions are less restricted invertibly convergence. Whereas the invertibly convergence is based on the invertible of matrices; in this study, we assume that matrices are not invertible. Some sufficient conditions for these kinds of convergence are studied. Further, some matrix sequences which are convergent to the Moore-Penrose inverses A^{+} and outer inverses $A^{(2)}_{T,S}$ as a general case are also studied. The results are derived here by considering the related well-known methods, namely, Euler-Knopp, Newton-Raphson, and Tikhonov methods. Finally, we provide some examples for computing both generalized inverses $A^{(2)}_{T,S}$ and A^{+} numerically for any arbitrary matrix $A_{m,n}$ of large dimension by using MATLAB and comparing the results between some of different methods.

1. Introduction and Preliminaries

A scalar infinite product $p = \prod_{m=1}^{\infty} b_m$ of complex numbers is said to converge if b_m is nonzero for m sufficiently large, say $m \geq N$, and $q = \lim_{m\to\infty} \prod_{m=1}^{\infty} b_m$ exists and is nonzero. If this is so then p is defined to be $p = q = \prod_{m=1}^{N-1} b_m$. With this definition, a convergent infinite product vanishes if and only if one of its factors vanishes.

Let $\{B_m\}$ be a sequence of $k \times k$ matrices, then

$$\prod_{m=r}^{s} B_m = \begin{cases} B_s B_{s-1} \cdots B_r & \text{if } r \leq s \\ I & \text{if } r > s. \end{cases} \tag{1.1}$$

In [1], Daubechies and Lagarias defined the converges of an infinite product of matrices without the adverb "invertibly" as follows.

(i) An infinite product $\prod_{m=1}^{\infty} B_m$ of $k \times k$ matrices is said to be *right converges* if $\lim_{m\to\infty} B_1 B_2 \cdots B_m$ exists, in which case

$$\prod_{m=1}^{\infty} B_m = \lim_{m\to\infty} B_1 B_2 \cdots B_m. \tag{1.2}$$

(ii) An infinite product $\prod_{m=1}^{\infty} B_m$ of $k \times k$ matrices is said to be *left converges* if $\lim_{m\to\infty} B_m \cdots B_2 B_1$ exists, in which case

$$\prod_{m=1}^{\infty} B_m = \lim_{m\to\infty} B_m \cdots B_2 B_1. \tag{1.3}$$

The idea of invertibly convergence of sequence of matrices was introduced by Trench [2, 3] as follows. An infinite product $\prod_{m=1}^{\infty} B_m$ of $k \times k$ matrices is said to be invertibly converged if there is an integer N such that B_m is invertible for $m \ge N$, and

$$Q = \lim_{n\to\infty} \prod_{m=N}^{n} B_m, \tag{1.4}$$

exists and is invertible. In this case,

$$\prod_{m=1}^{\infty} B_m = Q \prod_{m=1}^{N-1} B_m. \tag{1.5}$$

Let us recall some concepts that will be used below. Before starting, throughout we consider matrices over the field of complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$. The set of m-by-n complex matrices is denoted by $M_{m,n}(\mathbb{C}) = \mathbb{C}^{m\times n}$. For simplicity, we write $M_{m,n}$ instead of $M_{m,n}(\mathbb{C})$ and when $m = n$, we write M_n instead of $M_{n,n}$. The notations A^T, A^*, A^+, $A^{(2)}_{T,S}$, $\operatorname{rank}(A)$, $\operatorname{rang}(A)$, $\operatorname{null}(A)$, $\rho(A)$, $\|A\|_s$, $\|A\|_p$, and $\sigma(A)$ stand, respectively, for the transpose, conjugate transpose, Moore-Penrose inverse, outer inverse, rank, range, null space, spectral radius, spectrum norm, p-norm, and the set of all eigenvalues of matrix A.

The Moore-Penrose and outer inverses of an arbitrary matrix (including singular and rectangular) are very useful in various applications in control system analysis, statistics, singular differential and difference equations, Markov chains, iterative methods, least-square problem, perturbation theory, neural networks problem, and many other subjects were found in the literature (see, e.g., [4–14]).

It is well known that *Moore-Penrose inverse* (MPI) of a matrix $A \in M_{m,n}$ is defined to be the unique solution of the following four matrix equations (see, e.g. [4, 11, 14–20]):

$$AXA = A, \qquad XAX = X, \qquad (AX)^* = AX, \qquad (XA)^* = XA \tag{1.6}$$

and is often denoted by $X = A^+ \in M_{n,m}$. In particular, when A is a square and nonsingular matrix, then A^+ reduce to A^{-1}.

For $x = A^+ b$, $x' \in \mathbb{C}^n \setminus \{x\}$ arbitrary, it holds, (see, e.g., [14, 18]),

$$\|b - Ax\|_2^2 = (b - Ax)^*(b - Ax) \le \left\|b - Ax'\right\|_2^2, \tag{1.7}$$

and $\|b - Ax\|_2 = \|b - Ax'\|_2$, then

$$\|x\|_2^2 = x^* x < \|x'\|_2^2. \tag{1.8}$$

Thus, $x = A^+ b$ is the unique minimum least-squares solution of the following linear squares problem, (see, e.g., [14, 21, 22]),

$$\|b - Ax\|_2 = \min_{z \in C^n} \|b - Az\|_2. \tag{1.9}$$

It is well known also that the *singular value decomposition* of any rectangular matrix $A \in M_{m,n}$ with $\operatorname{rank}(A) = r \neq 0$ is given by

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^* : \qquad U^* U = I_m, \qquad V^* V = I_n, \tag{1.10}$$

where $D = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_r) \in M_r$ is a diagonal matrix with diagonal entries $\delta_i (i = 1, 2, \ldots, r)$, and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > 0$ are the *singular values of* A, that is, $\mu_i^2 (i = 1, 2, \ldots, r)$ are the nonzero eigenvalues of $A^* A$. This decomposition is extremely useful to represent the MPI of $A \in M_{m,n}$ by [20, 23]

$$A^+ = V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{1.11}$$

where $D^{-1} = \operatorname{diag}(\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_r^{-1}) \in M_r$ is a diagonal matrix with diagonal entries $\mu_i^{-1} (i = 1, \ldots, r)$.

Furthermore, the *spectral norm* of A is defined by

$$\|A\|_s = \max_{1 \leq i \leq r} \{\mu_i\} = \mu_1; \qquad \|A^+\|_s = \frac{1}{\mu_r}, \tag{1.12}$$

where $\mu_1$and μ_r are, respectively, the largest and smallest singular value of A.

Generally speaking, the outer inverse $A^{(2)}_{T,S}$ of a matrix $A \in M_{m,n}$, which is a unique matrix $X \in M_{n,m}$ satisfying the following equations (see, e.g., [20, 24–27]):

$$AXA = A, \qquad \operatorname{rang}(A) = T, \qquad \operatorname{null}(A) = S, \tag{1.13}$$

where T is a subspace of $\mathbb{C}^n$ of $s \leq r$, and S is a subspace of $\mathbb{C}^m$ of dimension $m - s$.

As we see in [13, 20, 24–29], it is well-known fact that several important generalized inverses, such as the Moore-Penrose inverse A^+, the weighted Moore-Penrose inverse $A^+_{M,N}$, the Drazin inverse A^D, and so forth, are all the generalized inverse $A^{(2)}_{T,S}$, which is having the prescribed range T and null space S of outer inverse of A. In this case, the Moore-Penrose inverse A^+ can be represented in outer inverse form as follows [27]:

$$A^+ = A^{(2)}_{\operatorname{rang}(A^*),\, \operatorname{null}(A^*)}. \tag{1.14}$$

Also, the representation and characterization for the outer generalized inverse $A^{(2)}_{T,S}$ have been considered by many authors (see, e.g., [15, 16, 20, 27, 30, 31])

Finally, given two matrices $A = [a_{ij}] \in M_{m,n}$, and $B = [b_{kl}] \in M_{p,q}$, then the Kronecker product of A and B is defined by (see, e.g., [5, 7, 32–35])

$$A \otimes B = \left[a_{ij}B\right] \in M_{mp,nq}. \tag{1.15}$$

Furthermore, the Kronecker product enjoys the following well-known and important properties:

(i) The Kronecker product is associative and distributive with respect to matrix addition.

(ii) If $A \in M_{m,n}$, $B \in M_{p,q}$, $C \in M_{n,r}$ and $D \in M_{q,s}$, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{1.16}$$

(iii) If $A \in M_m$, $B \in M_p$ are positive definite matrices, then for any real number r, we have:

$$(A \otimes B)^r = A^r \otimes B^r. \tag{1.17}$$

(iv) If $A \in M_{m,n}$, $B \in M_{p,q}$, then

$$(A \otimes B)^+ = A^+ \otimes B^+. \tag{1.18}$$

2. Convergent Moore-Penrose Inverse of Matrices

First, the need to compute A^+ by using sequences method. The key to such results below is the following two Lemmas, due to Wei [23] and Wei and Wu [17], respectively.

Lemma 2.1. *Let $A \in M_{m,n}$ ba a matrix. Then*

$$A^+ = \widehat{A}^{-1}A^*, \tag{2.1}$$

where $\widehat{A} = A^*A\rfloor_{\mathrm{rang}(A^*)}$ is the restriction of A^*A on $\mathrm{rang}(A^*)$.

Lemma 2.2. *Let $A \in M_{m,n}$ with* $\mathrm{rank}(A) = r$ *and $\widehat{A} = A^*A\rfloor_{\mathrm{rang}(A^*)}$. Suppose Ω is an open set such that $\sigma(\widehat{A}) \subset \Omega \subset (0,\infty)$. Let $\{S_n(x)\}$ be a family of continuous real valued function on Ω with* $\lim_{n\to\infty}S_n(x) = 1/x$ *uniformly on$\sigma(\widehat{A})$. Then*

$$A^+ = \lim_{n\to\infty} S_n\left(\widehat{A}\right)A^*. \tag{2.2}$$

Furthermore,

$$\left\|S_n\left(\widehat{A}\right)A^* - A^+\right\|_2 \le \sup_{x\in\sigma(\widehat{A})} |S_n(x)x - 1|\|A^+\|_2. \tag{2.3}$$

Moreover, for each $\lambda \in \sigma(\widehat{A})$, we have

$$\mu_r^2 \leq \lambda \leq \mu_1^2. \tag{2.4}$$

It is well known that the inverse of an invertible operator can be calculated by interpolating the function $1/x$, in a similar manner we will approximate the Moore-Penrose inverse by interpolating function $1/x$ and using Lemmas 2.1 and 2.2.

One way to produce a family of functions $\{S_n(x)\}$ which is suitable for use in the Lemma 2.2 is to employ the well known Euler-Knopp method. A series $\sum_{n=0}^{\infty} a_n$ is said to be Euler-Knopp summable with parameter $\alpha > 0$ to the value a if the sequence defined by

$$S_n = \alpha \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} (1-\alpha)^{k-j} \alpha^j \alpha_j \tag{2.5}$$

converges to a. If $a_k = (1-x)$ for $k = 0, 1, 2, \ldots$, then we obtain as the Euler-Knopp transform of the series $\sum_{k=0}^{\infty} (1-x)^k$, the sequence given by

$$S_n(x) = \alpha \sum_{k=0}^{n} (1-\alpha x)^k. \tag{2.6}$$

Clearly $\lim_{n\to\infty} S_n(x) = 1/x$ uniformly on any compact subset of the set

$$E_\alpha = \{x : |1-\alpha x| < 1\} = \left\{x : 0 < x < \frac{2}{\alpha}\right\}. \tag{2.7}$$

Another way to produce a family functions $\{S_n(x)\}$ which is suitable also for use in the Lemma 2.2 is to employ the well-known Newton-Raphson method. This can be done by generating a sequence y_n, where

$$y_{n+1} = y_n - \frac{s(y_n)}{s'(y_n)} = y_n(2 - x y_n), \tag{2.8}$$

for suitable y_0. Suppose that for $\alpha > 0$ we define a sequence of functions $\{S_n(x)\}$ by

$$S_0(x) = \alpha; \qquad S_{n+1}(x) = S_n(x)(2 - x S_n(x)). \tag{2.9}$$

In fact,

$$x S_{n+1}(x) - 1 = -(x S_n(x) - 1)^2. \tag{2.10}$$

Iterating on this equality, it follows that if x is confined to a compact subset of $E_\alpha = \{x : 0 < x < 2/\alpha\}$. Then there is a constant β (defining on this compact set) with $0 < \beta < 1$ and

$$|x S_n(x) - 1| = |\alpha x - 1|^{2^n} \leq \beta^{2^n} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.11}$$

According to the variational definition, A^+b is the vector $x \in \mathbb{C}^n$ which minimizes the functional $\|Ax - b\|_2$ and also has the smallest 2 norm among all such minimizing vectors. The idea of Tikhonov's regularization [36, 37] of order zero is to approximately minimize both the functional $\|Ax - b\|_2$ and the norm $\|x\|_2$ by minimizing the functional $g : \mathbb{C}^n \to \mathbb{R}$ defined by

$$g(x) = \|Ax - b\|_2^2 + t\|x\|_2^2, \tag{2.12}$$

where $t > 0$. The minimum of this functional will occur at the unique stationary point u of g, that is, the vector u which satisfies $\nabla g(u) = 0$. The gradient of g is given by

$$\nabla g(x) = 2(A^*Ax - A^*b) + 2tx, \tag{2.13}$$

and hence the unique minimizer u_t satisfies

$$u_t = (A^*A + tI)^{-1}A^*b. \tag{2.14}$$

On intuitive grounds, it seems reasonable to expect that

$$\lim_{t \to 0^+} u_t = (A^*A)^{-1}A^*b = A^+b. \tag{2.15}$$

Therefore, if we define a sequence of functions $\{S_n(x)\}$ by using Euler-Knopp method, Newton-Raphson method and the idea of Tikhonov's regularization that mentioned above, then we get the following nice Theorem.

Theorem 2.3. *Let* $A \in M_{m,n}$ *with* $\operatorname{rank}(A) = r$ *and* $0 < \alpha < 2\mu_1^{-2}$. *Then*

(i) *the sequence* $\{A_n\}$ *defined by*

$$A_0 = \alpha A^*; \qquad A_{n+1} = (1 - \alpha A^*A)A_n + \alpha A^* \tag{2.16}$$

converges to A^+. *Furthermore, the error estimate is given by*

$$\|A_n - A^+\|_2 \le \beta^{n+1}\|A^+\|_2, \tag{2.17}$$

where $0 < \beta < 1$.

(ii) *The sequence* $\{A_n\}$ *defined by*

$$A_0 = \alpha A^*; \qquad A_{n+1} = A_n(2I - AA_n) \tag{2.18}$$

converges to A^+.

Furthermore, the error estimate is given by

$$\|A_n - A^+\|_2 \le \beta^{2^n}\|A^+\|_2. \tag{2.19}$$

where $0 < \beta < 1$.

(iii) *for* $t > 0$,

$$A^{+} = \lim_{t \to 0^{+}} (tI + A^{*}A)^{-1}A^{*}. \tag{2.20}$$

Thus, the error estimate is given by

$$\left\| (tI + A^{*}A)^{-1}A^{*} - A^{+} \right\|_{2} \leq \frac{t}{\mu_r^2 + t} \|A^{+}\|_{2}. \tag{2.21}$$

Proof. (i) It follows from $\sigma(\widehat{A}) \subseteq [\mu_r^2, \mu_1^2]$ that $\sigma(\widehat{A}) \subset (0, \mu_1^2]$, and hence we apply Lemma 2.2 if we choose the parameter α is such a way that $(0, \mu_1^2] \subseteq E_\alpha$, where E_α is defined by (2.7). We may choose α such that $0 < \alpha < 2\mu_1^{-2}$. If we use the sequence defined by

$$S_0(x) = \alpha; \qquad S_{n+1}(x) = (1 - \alpha x)S_n(x) + \alpha, \tag{2.22}$$

it is easy to verify that

$$\lim_{n \to \infty} S_n(x) = \frac{1}{x}, \tag{2.23}$$

uniformly on any compact subset of E_α. Hence, if $0 < \alpha < 2\mu_1^{-2}$, then, applying Lemma 2.2, we get

$$\lim_{n \to \infty} S_n\left(\widehat{A}\right)A^{*} = A^{+}. \tag{2.24}$$

But it is easy to see from (2.22) that $S_n(\widehat{A})A^{*} = A_n$, where A_n is given by (2.16). This is surely the case if $0 < \alpha < 2\mu_1^{-2}$, then, for such α, we have the representation

$$A^{+} = \alpha \sum_{j=0}^{n} (1 - \alpha A^{*}A)^{j} A^{*}. \tag{2.25}$$

Note that if we set

$$A_n = \alpha \sum_{j=0}^{n} (1 - \alpha A^{*}A)^{j} A^{*}, \tag{2.26}$$

then we get (2.16).

To derive an error estimate for the Euler-Knopp method, suppose that $0 < \alpha < 2\mu_1^{-2}$. If the sequence $S_n(x)$ is defined as in (2.22), then

$$S_{n+1}(x)x - 1 = (1 - \alpha x)(S_n(x)x - 1). \tag{2.27}$$

Therefore, since $S_0 = \alpha$,

$$|S_n(x)x - 1| = |1 - \alpha x|^{n+1}. \tag{2.28}$$

By $\mu_r^2 \le x \le \mu_1^2$ for $x \in \sigma(\widehat{A})$ and $0 < \alpha < 2\mu_1^{-2}$, it follows that

$$|1 - \alpha x| < \beta, \tag{2.29}$$

where β is given by

$$\beta = \max\left\{\left|1 - \alpha\mu_1^2\right|, \left|1 - \alpha\mu_r^2\right|\right\}. \tag{2.30}$$

Clearly,

$$2 > \alpha\mu_1^2 \ge \alpha\mu_r^2 > 0, \tag{2.31}$$

and therefore $0 < \beta < 1$. From Lemma 2.2, we establish (2.17).

(ii) Using the Newton-Raphson iterations in (2.8)–(2.11) in conjunction with Lemma 2.2, we see that the sequence of $\{S_n(\widehat{A})\}$ defined by

$$S_0(x) = \alpha I; \qquad S_{n+1}\left(\widehat{A}\right) = S_n\left(\widehat{A}\right)\left(2I - A^*AS_n\left(\widehat{A}\right)\right) \tag{2.32}$$

has the property that $\lim_{n\to\infty} S_n(\widehat{A})A^* = A^+$ uniformly in $\mathbb{C}^{m\times n}$. If we set $A_n = S_n(\widehat{A})A^*$, then we get (2.18).

If $x \in \sigma(\widehat{A})$ and $0 < \alpha < 2\mu_1^{-2}$, then we see that $|1 - \alpha x| \le \beta$, where β is given by (2.30). It follows as in (2.11) and hence from Lemma 2.2, then we get the error bound as in (2.19).

(iii) If we set

$$S_t(x) = (t + x)^{-1} : \quad t > 0 \tag{2.33}$$

in Lemma 2.2 and using the idea of Tikhonov's regularization as in (2.12)–(2.15), then it is easy to get (2.20) and (2.21). □

Huang and Zhang [28] presented the following sequence which is convergent to A^+:

$$A_0 = \alpha_0 A^*; \qquad A_{n+1} = \alpha_{n+1}(2A_n - A_n A A_n), \quad n = 1, 2, \ldots, \tag{2.34}$$

where $\alpha_{n+1} \in [1, 2]$ is called an acceleration parameter and chosen so as to minimize that which is bound on the maximum distance of any nonzero singular value of $A_{n+1}A$ from 1. They chose A_0 according to the first term of sequence (2.34) with $\alpha_0 = 2/(\mu_1 + \mu_r)$, and let $p_0 = \alpha_0\mu_1$, then the acceleration parameters α_{n+1} and p_{n+1} have the following sequences:

$$\alpha_{n+1} = \frac{2}{1 + (2 - p_n)p_n}, \qquad p_{n+1} = \alpha_{n+1}(2 - p_n)p_n. \tag{2.35}$$

We point out that the iteration (2.18) is a special case of an acceleration iteration (2.34). Further, we note that the above methods and the first-order iterative methods used by Ben-Israel and Greville [4] for computing A^+ are a set of instructions for generating a sequence $\{A_n : n = 1, 2, 3, \ldots\}$ converging to A^+.

Similarly, Liu et al. [25] introduced some necessary and sufficient conditions for iterative convergence to the generalized inverse $A^{(2)}_{T,S}$ and its existence and estimated the error bounds of the iterative methods for approximating $A^{(2)}_{T,S}$ by defining the sequence $\{A_n\}$ in the following way:

$$A_n = A_{n-1} + \beta X(I - AA_{n-1}), \quad n = 1, 2, \ldots, \tag{2.36}$$

where $\beta \neq 0$ with $X \neq XAA_0$. Then the iteration (2.36) converges if and only $\rho(I - \beta AX) < 1$, equivalently, $\rho(I - \beta AX) < 1$. In which case, if $\mathrm{rang}(X) = T$, $\mathrm{null}(X) = S$, and $\mathrm{rang}(A_0) \subset T$. Then, $A^{(2)}_{T,S}$ exists and $\{A_n\}$ converges to $A^{(2)}_{T,S}$. Furthermore, the error estimate is given by

$$\left\| A^{(2)}_{T,S} - A_n \right\| \leq \frac{|\beta| q^n}{1 - q} \|X\| \|I - AA_0\| = R(\beta, n), \tag{2.37}$$

where $q = \min\{\|I - \beta XA\|, \|I - \beta AX\|\}$.

What is the best value β_{opt} such that $\rho(I - \beta AX)$ minimize in order to achieve good convergence? Unfortunately, it may be very difficult and still require further studies. If $\sigma(AX)$ is a subset of $\mathbb{R}$ and $\lambda_{\min} = \min\{\lambda : \lambda \in \sigma(AX)\} > 0$, analogous to [38, Example 4.1], we can have

$$\beta_{\mathrm{opt}} = \frac{2}{\lambda_{\min} + \rho(AX)}. \tag{2.38}$$

3. Convergent Infinite Products of Matrices

Trench [3, Definition 1] defined invertibly convergence of an infinite products matrices $\prod_{m=1}^{\infty} B_m$ as in the invertible of matrices B_m for all $m > N$ (where N is an integer number). Here, we define the less restricted definitions of convergence of an infinite products $\prod_{m=1}^{\infty} B_m$ and $\prod_{m=1}^{\infty} \otimes B_m$ for $k \times k$ complex matrices such that

$$\prod_{m=r}^{s} \otimes B_m = \begin{cases} B_s \otimes B_{s-1} \otimes \cdots \otimes B_r & \text{if } r \leq s \\ I & \text{if } r > s \end{cases} \tag{3.1}$$

as follows.

Definition 3.1. Let $\{B_m\}$ be a sequence of $k \times k$ matrices. Then An infinite product $\prod_{m=1}^{\infty} B_m$ is said to be convergent if there is an integer N such that $B_m \neq 0$ (may B_m is invertible or not) for $m \geq N$ and

$$Q = \lim_{n \to \infty} \prod_{m=N}^{n} B_m \tag{3.2}$$

exists and is nonzero. In this case, we define

$$\prod_{m=1}^{\infty} B_m = Q \prod_{m=1}^{N-1} B_m. \tag{3.3}$$

Similarly, an infinite product $\prod_{m=1}^{\infty} \otimes B_m$ converges if there is an integer N such that $B_m \neq 0$ for $m \geq N$, and

$$R = \lim_{n \to \infty} \prod_{m=N}^{n} \otimes B_m \tag{3.4}$$

exists and is nonzero. In this case, we define

$$\prod_{m=1}^{\infty} \otimes B_m = R \otimes \prod_{m=1}^{N-1} \otimes B_m. \tag{3.5}$$

In the above Definition 3.1, the matrix R may be singular even if B_m is nonsingular for all $m \geq 1$, but R may be singular if B_m is singular for some $m \geq 1$. However, this definition does not require that B_m is invertible for large m.

Definition 3.2. Let $\{B_m\}$ be a sequence of $k \times k$ matrices. Then an infinite product $\prod_{m=1}^{\infty} \otimes B_m$ is said to be invertibly convergent if there is an integer N such that B_m is invertible for $m \geq N$, and

$$R = \lim_{n \to \infty} \prod_{m=N}^{n} \otimes B_m \tag{3.6}$$

exists and invertible. In this case, we define

$$\prod_{m=1}^{\infty} B_m = R \otimes \prod_{m=1}^{N-1} \otimes B_m. \tag{3.7}$$

The Definitions 3.1 and 3.2 have the following obvious consequence.

Theorem 3.3. *Let $\{B_m\}$ be a sequence of $k \times k$ matrices such that the infinite products $\prod_{m=1}^{\infty} B_m$ and $\prod_{m=1}^{\infty} \otimes B_m$ are invertibly convergent. Then, both infinite products are convergent, but the converse, in general, is not true.*

Theorem 3.4. *Let $\{B_m\}$ be a sequence of $k \times k$ matrices such that*

(i) *if the infinite product $\prod_{m=1}^{\infty} B_m$ is convergent, then*

$$\lim_{m \to \infty} B_m = QQ^{+}, \tag{3.8}$$

where $Q = \lim_{n \to \infty} \prod_{m=N}^{n} B_m$.

(ii) *If the infinite product* $\prod_{m=1}^{\infty} \otimes B_m$ *is converge, then*

$$\lim_{m\to\infty} B_m = R \otimes R^{+}. \tag{3.9}$$

Proof. (i) Suppose that $\prod_{m=1}^{\infty} B_m$ is convergent such that $B_m \neq 0$ when $m \geq N$. Let $Q = \prod_{m=N}^{n} B_m$. Then $\lim_{n\to\infty} Q_n = Q$, where $Q \neq 0$. Therefore, $\lim_{n\to\infty} Q_n^{+} = Q^{+}$. Since $B_n = Q_n Q_{n-1}^{+}$, we then have

$$\lim_{n\to\infty} B_n = \left(\lim_{n\to\infty} Q_n\right)\left(\lim_{n\to\infty} Q_{n-1}^{+}\right) = QQ^{+}. \tag{3.10}$$

But if B_m is invertible when $m \geq N$, then Q is invertible and

$$\lim_{n\to\infty} B_n = \left(\lim_{n\to\infty} Q_n\right)\left(\lim_{n\to\infty} Q_{n-1}^{-1}\right) = QQ^{-1} = I. \tag{3.11}$$

Similarly, it is easy to prove (ii). □

If the infinite products $\prod_{m=1}^{\infty} B_m$ and $\prod_{m=1}^{\infty} \otimes B_m$ are invertibly convergent in Theorem 3.4, then we get the following corollary.

Corollary 3.5. *Let* $\{B_m\}$ *be a sequence of* $k \times k$ *matrices such that*

(i) *if the infinite product* $\prod_{m=1}^{\infty} B_m$ *is invertibly convergent, then*

$$\lim_{m\to\infty} B_m = I \tag{3.12}$$

(ii) *if the infinite product* $\prod_{m=1}^{\infty} \otimes B_m$ *is invertibly convergent, then*

$$\lim_{m\to\infty} B_m = R \otimes R^{-1}, \tag{3.13}$$

where $R = \lim_{n\to\infty} \prod_{m=N}^{n} \otimes B_m$.

The main reason for interest in these products above is to generate matrix sequences for solving such matrix problems such as singular linear systems and singular coupled matrix equations. For example, Cao [21] and Shi et al. [22] constructed the general stationary and nonstationary iterative process generated by $\{B_m\}_{m=1}^{\infty}$ for solving the singular linear system $Ax = b$, and Leizarowitz [39] established conditions for weak ergodicity of products, existence of optimal strategies for controlled Markov chains, and growth properties of certain linear nonautonomous differential equations based on a sequence (an infinite product) of stochastic matrices $\{B_m\}_{m=1}^{\infty}(\prod_{m=1}^{\infty} B_m)$. Also, as discussed in [2], the motivation of Definition 3.1 stems from a question about linear systems of difference equations and coupled matrix equations, under what conditions on $\{B_m\}_{m=1}^{\infty}$ of, for instance, the system $x_m = B_m x_{m-1}$, $m = 1, 2, \ldots$, approach a finite nonzero limit whenever $x_0 \neq 0$? A system with property linear asymptotic equilibrium if and only if B_m is invertible for every $m \geq 1$

and $\prod_{m=1}^{\infty} B_m$ invertibly convergent, but a system with the so-called least-squares linear asymptotic equilibrium if $B_m \neq 0$ for every $m \geq 1$ and $\prod_{m=1}^{\infty} B_m$ converges.

Because of Theorem 3.4, we consider only infinite product of the form $\prod_{m=1}^{\infty}(I + A_m)$, where $\lim_{m\to\infty} A_m = 0$. We will write

$$P_n = \prod_{m=1}^{n}(I + A_m); \qquad P = \prod_{m=1}^{\infty}(I + A_m). \tag{3.14}$$

The following Theorem provides the convergence and invertibly convergence of the infinite product $\prod_{m=1}^{\infty}(I + A_m)$, and the proof here is omitted.

Theorem 3.6. *The infinite product $\prod_{m=1}^{\infty}(I+A_m)$ converges (invertibly converges) if $\sum_{m=1}^{\infty} \|A_m\|_p < \infty$, for some p-norm $\|\cdot\|$.*

The following theorem relates convergence of an infinite product to the asymptotic behavior of least-square solutions of a related system of difference equations.

Theorem 3.7. *The infinite product $\prod_{m=1}^{\infty}(I + A_m)$ converges if and only if for some integer $N \geq 1$ the matrix difference equation*

$$X_{n+1} = (I + A_n)X_n : \quad n \geq N \tag{3.15}$$

has a least-square solution $\{X_n\}_{n=N}^{\infty}$ such that $\lim_{n\to\infty} X_n = I$. In this case,

$$P_n = X_{n+1}X_N^{+}\prod_{m=1}^{N-1}(I + A_m); \qquad P = X_N^{+}\prod_{m=1}^{N-1}(I + A_m). \tag{3.16}$$

Proof. Suppose that $\prod_{m=1}^{\infty}(I + A_m)$ converges. Choose N so that $I + A_m \neq 0$ for $m \geq N$, and let $Q = \prod_{m=N}^{n}(I + A_m)$. Define

$$X_n = \left\{\prod_{m=N}^{n-1}(I + A_m)\right\}Q^{+} : \quad n \geq N. \tag{3.17}$$

Then $\{X_n\}_{n=N}^{\infty}$ is a solution of (3.15) such that $\lim_{n\to\infty} X_n = I$.

Conversely, suppose that (3.15) has a least-square solution $\{X_n\}_{n=N}^{\infty}$ such that $\lim_{n\to\infty} X_n = I$. Then

$$X_n = \left\{\prod_{m=N}^{n-1}(I + A_m)\right\}X_N : \quad n \geq N, \tag{3.18}$$

where $X_N \neq 0$. Therefore,

$$\prod_{m=N}^{n-1}(I + A_m) = X_nX_N^{+} : \quad n \geq N. \tag{3.19}$$

Letting $n \to \infty$ shows that

$$\prod_{m=N}^{\infty}(I + A_m) = X_N^{+} \tag{3.20}$$

which implies that $\prod_{m=1}^{\infty}(I + A_m)$ converges.

From (3.14) and (3.18), we get

$$P_n = \prod_{m=1}^{n}(I + A_m) = X_{n+1}X_N^{+}\left\{\prod_{m=1}^{N-1}(I + A_m)\right\}, \tag{3.21}$$

which proves the first expression in (3.16). From (3.14) and (3.20), we get

$$P = \prod_{m=1}^{\infty}(I + A_m) = X_N^{+}\prod_{m=1}^{N-1}(I + A_m). \tag{3.22}$$

which proves the second expression in (3.16). □

Remark 3.8. If the infinite product $\prod_{m=1}^{\infty}(I + A_m)$ is invertibly convergent in Theorem 3.7, then

$$P_n = X_{n+1}X_N^{-1}\prod_{m=1}^{N-1}(I + A_m); \quad P = X_N^{-1}\prod_{m=1}^{N-1}(I + A_m). \tag{3.23}$$

Theorem 3.7 indicates the connection between convergence of an infinite product of $k \times k$ matrices $\prod_{m=1}^{\infty}(I + A_m)$ and the asymptotic properties of least-squares solutions of matrix difference equation as defined in (3.15). We can say that (3.15) has a least-squares linear asymptotic equilibrium if every least-squares solution of (3.15) for which $X_N \neq 0$ that approaches I as $n \to \infty$.

For example, Ding and Chen [5] and generalized later by Kılıçman and Al-Zhour [8] studied the convergence of least-square solutions to the coupled Sylvester matrix equations

$$AX + YB = C; \quad DX + YE = F, \tag{3.24}$$

where $A, D \in M_m$, $B, E \in M_n$, $C, F \in M_{m,n}$ are given constant matrices, and $X, Y \in M_{m,n}$ are the unknown matrices to be solved. If the coupled Sylvester matrix equation determined by (3.24) has a unique solution X and Y, then the following iterative solution X_{n+1} and Y_{n+1} given by [5, 8]:

$$\begin{gathered} X_{n+1} = X_n + \mu G^{+}\begin{bmatrix} C - AX_n - Y_nB \\ F - DX_n - Y_nE \end{bmatrix}; \\ Y_{n+1} = Y_n + \mu\left[C - AX_n - Y_nB \quad F - DX_n - Y_nE\right]H^{+}, \end{gathered} \tag{3.25}$$

where $G = \left[\begin{smallmatrix} A \\ D \end{smallmatrix}\right]$ and $H = [\,B\ E\,]$ are full column and full row-rank matrices, respectively;

$$\mu = \frac{1}{m+n} \quad \text{or } \mu = \frac{1}{\lambda_{\max}\left[G(G^*G)^{-1}G^*\right] + \lambda_{\max}\left[H^*(HH^*)^{-1}H^*\right]}. \tag{3.26}$$

converges to X and Y for any finite initial values X_0 and Y_0.

The convergence factor μ in (3.26) may not be the best and may be conservative. In fact, there exists a best μ such that the fast convergence rate of X_k to X and Y_k to Y can be obtained as in numerical examples given by Cao [21] and Kılıçman and Al-Zhour [8]. How to find the connections between convergence of an infinite products of $k \times k$ matrices and least-square solutions of coupled Sylvester matrix equation in (3.24) requires further research.

4. Numerical Examples

Here, we give some numerical example for computing outer inverse $A^{(2)}_{T,S}$ and Moore-Penrose inverse $A^{(2)}_{T,S}$ by applying sequences methods which are studied and derived in Section 2. Our results are obtained in this Section by choosing Frobenius norm ($\|\cdot\|_2$) and using MATLAB software.

Example 4.1. Consider the matrix

$$A = \begin{bmatrix} 2 & 0.4 & 0.4 & 0.4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{5\times 4}. \tag{4.1}$$

Let $T = \mathbb{C}^4,\ e = [0,0,0,0,1]^T \in \mathbb{C}^5,\ S = \operatorname{span}\{e\}$.

Take

$$X = A_0 = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 5}. \tag{4.2}$$

Here $R(\beta,n) = |\beta| q^n (1-q)^{-1} \|X\|_2 \|I - AA_0\|_2$. Clearly $\operatorname{rang}(X) = T$, $\operatorname{null}(X) = S$, and $\operatorname{rang}(A_0) \subset T$. By computing, we have

$$A^{(2)}_{T,S} = \begin{bmatrix} 0.5 & -0.1 & -0.1 & -0.1 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 5}. \tag{4.3}$$

Table 1: Results for Example 4.1 using the iterations (2.36) and (2.37) for $\beta = 1.25$.

n	$E(n) = \|A_{T,S}^{(2)} - A_n\|_2$	$R(\beta, n)$	$\|A_n - A_{n-1}\|_2$
1	$3.464101615138 \times 10^{-2}$	0.58943384582443	0.24331050121193
2	$2.020636405220133 \times 10^{-16}$	0.20418587373373	$3.464101615138 \times 10^{-2}$
3	$2.020636405220133 \times 10^{-16}$	$7.073206149893 \times 10^{-2}$	0
4	$2.020636405220133 \times 10^{-16}$	$2.450230484805 \times 10^{-2}$	0
5	$2.020636405220133 \times 10^{-16}$	$8.48784737987 \times 10^{-3}$	0
8	$2.020636405220133 \times 10^{-16}$	$3.528331898118804 \times 10^{-4}$	0
10	$2.020636405220133 \times 10^{-16}$	$4.233998277742565 \times 10^{-5}$	0
12	$2.020636405220133 \times 10^{-16}$	$5.080797933291078 \times 10^{-6}$	0
14	$2.020636405220133 \times 10^{-16}$	$6.096957519949294 \times 10^{-7}$	0

Table 2: Results for Example 4.1 using the iterations (4.4) and (4.5).

n	$E(n) = \|A_{T,S}^{(2)} - A_n\|_2$	$R(n)$	$\|A_n - A_{n-1}\|_2$
5	$2.844222213540086 \times 10^{-4}$	$2.06917386097900 \times 10^{-2}$	$9.230211265187883 \times 10^{-4}$
7	$1.485461921416498 \times 10^{-5}$	$4.89980370280000 \times 10^{-3}$	$5.070609777923677 \times 10^{-5}$
10	$1.609342324440632 \times 10^{-7}$	$5.64613461842000 \times 10^{-4}$	$5.734399999925342 \times 10^{-7}$
13	$1.625817851681746 \times 10^{-9}$	$6.50614556479466 \times 10^{-5}$	$5.938877957045854 \times 10^{-9}$
16	$1.571929418012274 \times 10^{-11}$	$7.497152117521204 \times 10^{-6}$	$5.835397257624137 \times 10^{-11}$
19	$1.474087541668810 \times 10^{-13}$	$8.639107335285478 \times 10^{-7}$	$5.537232598086118 \times 10^{-13}$
22	$1.295028878277394 \times 10^{-15}$	$9.955003497416000 \times 10^{-8}$	$5.130710307439707 \times 10^{-15}$
24	$1.942890293094024 \times 10^{-16}$	$2.357344828188108 \times 10^{-8}$	$2.403703357979455 \times 10^{-16}$
25	$2.020636405220133 \times 10^{-16}$	$1.147133503351594 \times 10^{-8}$	$4.807406715958910 \times 10^{-17}$

In order to satisfy $q = \min\{\|I - \beta XA\|_2, \|I - \beta AX\|_2\} < 1$, we get that β should satisfy the following $0.63474563020816 < \beta < 1.79243883581125$.

From the iteration (2.5) in [24, Theorem 2.2], Let $A \in \mathbb{C}^{m\times r}$, and T and S be given subspaces of $\mathbb{C}^{m\times r}$ such that there exists $A_{T,S}^{(2)}$. Then the sequence $(A_n)_n$ in $\mathbb{C}^{m\times r}$ defined in the following way:

$$\begin{aligned} R_n &= P_{A(T),S} - P_{A(T),S} A A_n, \\ A_{n+1} &= A_0 R_n + A_n \quad n = 0, 1, 2, \dots \end{aligned} \tag{4.4}$$

converges to $A_{T,S}^{(2)}$ if and only if $\operatorname{rang}(A_0) \subset T$ and $\rho(R_0) < 1$ (where $R(n) = (\|R_0\|^{n+1}/(1 - \|R_0\|))\|A_0\|$ and $R_0 = P_{A(T),S} - P_{A(T),S} A A_0$).

In this case, if $\|R_0\| = q < 1$, then

$$\left\| A_{T,S}^{(2)} - A_n \right\| \le \frac{q^{n+1}}{1-q} \|A_0\|. \tag{4.5}$$

Thus we have Tables 1 and 2 respectively, where

$$R(\beta, n) = \frac{|\beta| q^n}{1-q} \|X\| \|I - AA_0\|, \qquad R(n) = \frac{\|R_0\|^{n+1}}{1 - \|R_0\|} \|A_0\|. \tag{4.6}$$

Table 1 illustrates that $\beta = 1.25$ is best value such that $\|A_{T,S}^{(2)} - A_n\|$ reaches $2.020636405220133 \times 10^{-16}$ iterating the least number of steps, the reason for which is that such a β is calculating by using (2.38). Thus, for an appropriate β, the iteration is better than the iteration (4.4) (cf. Tables 1 and 2). And with respect to the error bound, the iterations for almost all are also better. Let us take the error bound smaller than 10^{-16}; for instance, the number of steps of iterations in Table 1 is smaller than that of the iterations in Table 2. But, in practice, we consider also the quantity $\|A_n - A_{n-1}\|$ in order to cease iteration since there exist such cases as $\beta = 1.25$. For example, for $\|A_n - A_{n-1}\| < \mu\|A_n\|$, where μ is the machine precision, the iteration for $\beta = 1.25$ only needs 3 steps. Therefore, in general, the iteration of (2.36) is better than the iteration (4.4) for an appropriate β. Note that the iterations in both Tables 1 and 2 indicate a fast convergence for the quantity $\|A_{T,S}^{(2)} - A_n\|$ more than the quantity $R(\beta, n)$ in Table 1 and the quantity $R(n)$ in Table 2 since each of $R(\beta, n)$ and $R(n)$ is an upper bound of the quantity $\|A_{T,S}^{(2)} - A_n\|$, and, to find the best or least upper bound for the quantity, $\|A_{T,S}^{(2)} - A_n\|$ requires further research.

Example 4.2. Consider the matrix

$$A = \begin{bmatrix} 22 & 10 & 2 & 3 & 7 \\ 14 & 7 & 10 & 0 & 8 \\ -1 & 13 & -1 & -11 & 3 \\ -3 & -2 & 13 & -2 & 4 \\ 9 & 8 & 1 & -2 & 4 \\ 9 & 1 & -7 & 5 & -1 \\ 2 & -6 & 6 & 5 & 1 \\ 4 & 5 & 0 & -2 & 2 \end{bmatrix} \in \mathbb{C}^{8\times 5}. \tag{4.7}$$

Then by computing we have

$$\begin{aligned} A^{+} &= \begin{bmatrix} 2.1129808\times10^{-2} & 4.6153846\times10^{-3} & -2.1073718\times10^{-3} & 7.6041667\times10^{-3} & 3.8060897\times10^{-3} \\ 9.3108974\times10^{-3} & 2.2115385\times10^{-3} & 2.0528846\times10^{-3} & -2.0833333\times10^{-3} & 1.0016026\times10^{-3} \\ -1.1097756\times10^{-2} & 2.7403846\times10^{-2} & -3.8862179\times10^{-3} & -2.7604167\times10^{-2} & 4.2067308\times10^{-3} \\ -7.9166667\times10^{-3} & -5.0000000\times10^{-3} & 3.3750000\times10^{-2} & -5.4166667\times10^{-3} & 1.0416667\times10^{-3} \\ 5.5128205\times10^{-3} & 9.8076923\times10^{-3} & -8.9743590\times10^{-4} & -5.0000000\times10^{-3} & 3.2051282\times10^{-3} \\ 1.4318910\times10^{-2} & -2.5961538\times10^{-3} & -2.0136218\times10^{-2} & 1.2812500\times10^{-3} & -6.2099359\times10^{-3} \\ 4.8958333\times10^{-3} & -1.5000000\times10^{-2} & 1.5312500\times10^{-2} & 1.2395833\times10^{-2} & 2.4604166\times10^{-3} \\ 1.5064103\times10^{-3} & 7.4038462\times10^{-3} & -1.6987179\times10^{-3} & -5.0000000\times10^{-3} & 1.6025641\times10^{-3} \end{bmatrix}^{T} \\ &\in \mathbb{C}^{5\times 8}. \end{aligned} \tag{4.8}$$

Thus, (see Tables 3 and 4).

Table 3: Results for Example 4.2 using an acceleration iteration (2.34).

n	$\|XAX-X\|$	$\|AXA-A\|$	$\|(AX)^*-AX\|$	$\|(XA)^*-XA\|$
1	0.010134	8.8184	1.7808×10^{-15}	5.5517×10^{-16}
2	0.0017505	1.1564	2.25×10^{-15}	4.2698×10^{-16}
3	2.7532×10^{-5}	0.018018	2.2253×10^{-15}	6.3027×10^{-16}
4	7.2707×10^{-9}	4.6827×10^{-6}	2.6984×10^{-15}	4.1249×10^{-16}
5	$4.4545e\times10^{-16}$	2.9558×10^{-13}	2.7911×10^{-15}	5.4318×10^{-16}
6	7.4467×10^{-17}	$6.4138e\times10^{-15}$	3.2971×10^{-15}	1.0954×10^{-16}
7	1.4818×10^{-17}	5.2442×10^{-15}	2.9214×10^{-15}	4.6700×10^{-16}
8	2.9927×10^{-18}	4.3997×10^{-15}	2.7295×10^{-15}	7.1699×10^{-16}

Table 4: Results for Example 4.2 using the iteration (2.18).

n	$\|XAX-X\|$	$\|AXA-A\|$	$\|(AX)^*-AX\|$	$\|(XA)^*-XA\|$
1	0.013054	14.088	6.8532×10^{-16}	3.7436×10^{-16}
2	0.0072362	5.6172	9.7451×10^{-16}	3.3446×10^{-16}
3	0.0013284	0.89207	9.6044×10^{-16}	3.7163×10^{-16}
4	3.4199×10^{-5}	0.022434	1.2231×10^{-16}	5.2064×10^{-16}
5	2.1567×10^{-8}	1.4143×10^{-5}	1.9419×10^{-15}	9.9378×10^{-16}
6	8.5747×10^{-15}	5.6179×10^{-12}	2.1874×10^{-15}	7.6483×10^{-16}
7	1.2675×10^{-16}	5.8744×10^{-15}	2.7634×10^{-15}	8.3034×10^{-16}
8	2.5102×10^{-16}	3.0005×10^{-15}	2.5232×10^{-15}	8.1248×10^{-16}

Table 5: Results for a random matrix $A\in\mathbb{C}^{100\times80}$ using an acceleration iteration (2.34).

n	$\|XAX-X\|$	$\|AXA-A\|$	$\|(AX)^*-AX\|$	$\|(XA)^*-XA\|$
1	0.094713	45.137	4.1569×10^{-12}	3.2203×10^{-12}
2	0.12329	45.032	$1.6962e\times10^{-12}$	1.42×10^{-12}
3	0.74538	45.195	6.3723×10^{-13}	5.9666×10^{-13}
4	0.85654	44.985	2.1027×10^{-13}	2.3662×10^{-13}
5	0.95103	44.599	6.4066×10^{-14}	8.656×10^{-14}
6	1.2811	42.945	1.7448×10^{-14}	2.9065×10^{-14}
7	1.6118	36.615	4.8196×10^{-15}	6.8183×10^{-15}
8	1.4848	21.984	1.1672×10^{-15}	8.3827×10^{-16}
9	0.50821	6.0397	1.8502×10^{-15}	5.3961×10^{-16}
10	0.038445	0.40952	$2.0718e\times10^{-16}$	1.5552×10^{-16}
11	0.00017627	0.0018439	2.0888×10^{-16}	1.222×10^{-16}
12	3.5682×10^{-9}	3.746×10^{-8}	2.0885×10^{-16}	1.2319×10^{-17}
13	4.661×10^{-15}	6.7753×10^{-14}	2.0898×10^{-16}	1.588×10^{-17}
14	5.3525×10^{-15}	6.805×10^{-14}	2.0897×10^{-16}	1.4264×10^{-17}
15	4.0198×10^{-15}	6.8871×10^{-14}	2.0891×10^{-16}	1.4191×10^{-17}

Example 4.3. We generate a random matrix, $A\in\mathbb{C}^{100\times80}$ by using MATLAB, and then we obtain the results as in Tables 5 and 6.

Note that from Tables 3, 4, 5, and 6, it is clear that the quantities $\|XAX-X\|$, and $\|AXA-A\|$, $\|(AX)^*-AX\|$, $\|(XA)^*-XA\|$ are becoming smaller and smaller and goes to zero as n increases in both iterations (2.34) and (2.18). We can also conclude that both iterations

Table 6: Results for a random matrix $A \in \mathbb{C}^{100\times 80}$ using the iteration (2.18).

n	$\|XAX-X\|$	$\|AXA-A\|$	$\|(AX)^*-AX\|$	$\|(XA)^*-XA\|$
1	0.060276	45.097	2.0509	1.5239×10^{-4}
2	0.089542	45.082	3.9427×10^{-1}	4.0096×10^{-5}
3	0.10119	45.081	8.3223×10^{-2}	8.8403×10^{-6}
4	0.096287	45.121	1.627×10^{-3}	1.8814×10^{-6}
5	0.1096	45.185	$3.2587\times \times 10^{-4}$	3.8609×10^{-7}
6	0.12111	45.207	6.1535×10^{-5}	7.7762×10^{-8}
7	0.13823	45.146	1.1418×10^{-5}	1.4853×10^{-9}
8	0.17182	44.971	2.0472×10^{-6}	2.9817×10^{-10}
9	0.21594	44.623	3.5587×10^{-7}	$5.422e\times 10^{-11}$
10	0.27022	43.962	6.0384×10^{-8}	9.337×10^{-12}
11	0.34782	42.688	9.8614×10^{-9}	1.5714×10^{-12}
12	0.46134	40.249	1.5947×10^{-9}	2.5064×10^{-13}
13	0.62169	35.766	2.5794×10^{-10}	3.6162×10^{-14}
14	0.74641	28.232	4.0931×10^{-11}	4.6385×10^{-15}
15	0.73592	17.588	6.0751×10^{-12}	5.0276×10^{-16}
16	0.39382	6.8259	8.0089×10^{-13}	3.3054×10^{-16}
17	0.068122	1.0281	9.0865×10^{-14}	2.3201×10^{-17}
18	0.0015796	0.023321	9.2857×10^{-15}	2.1781×10^{-17}
19	$8.1324e-007$	$1.2001e-005$	9.2907×10^{-16}	1.7884×10^{-17}

almost have same fast of convergence when the dimension of any arbitrary matrix A is not so large, but the acceleration iteration (2.34) is better more than the iteration (2.18) when the dimension of any arbitrary matrix A is so large with an appropriate acceleration parameter $\alpha_{n+1} \in [1,2]$.

5. Concluding Remarks

In this paper, we have studied some matrix sequences convergence to the Moore-Penrose inverse A^+ and outer inverse $A^{(2)}_{T,S}$ of an arbitrary matrix $A \in M_{m,n}$. The key to derive matrix sequences which are convergent to weighted Drazin and weighted Moore-Penrose inverses is the Lemma 2.2. Some sufficient conditions for infinite products $\prod_{m=1}^{\infty} B_m$ and $\prod_{m=1}^{\infty} \otimes B_m$ of $k \times k$ matrices are also derived. In our opinion, it is worth establishing some connections between convergence of an infinite products of $k \times k$ matrices and least-square solutions of such linear singular systems as well as the singular coupled matrix equations.

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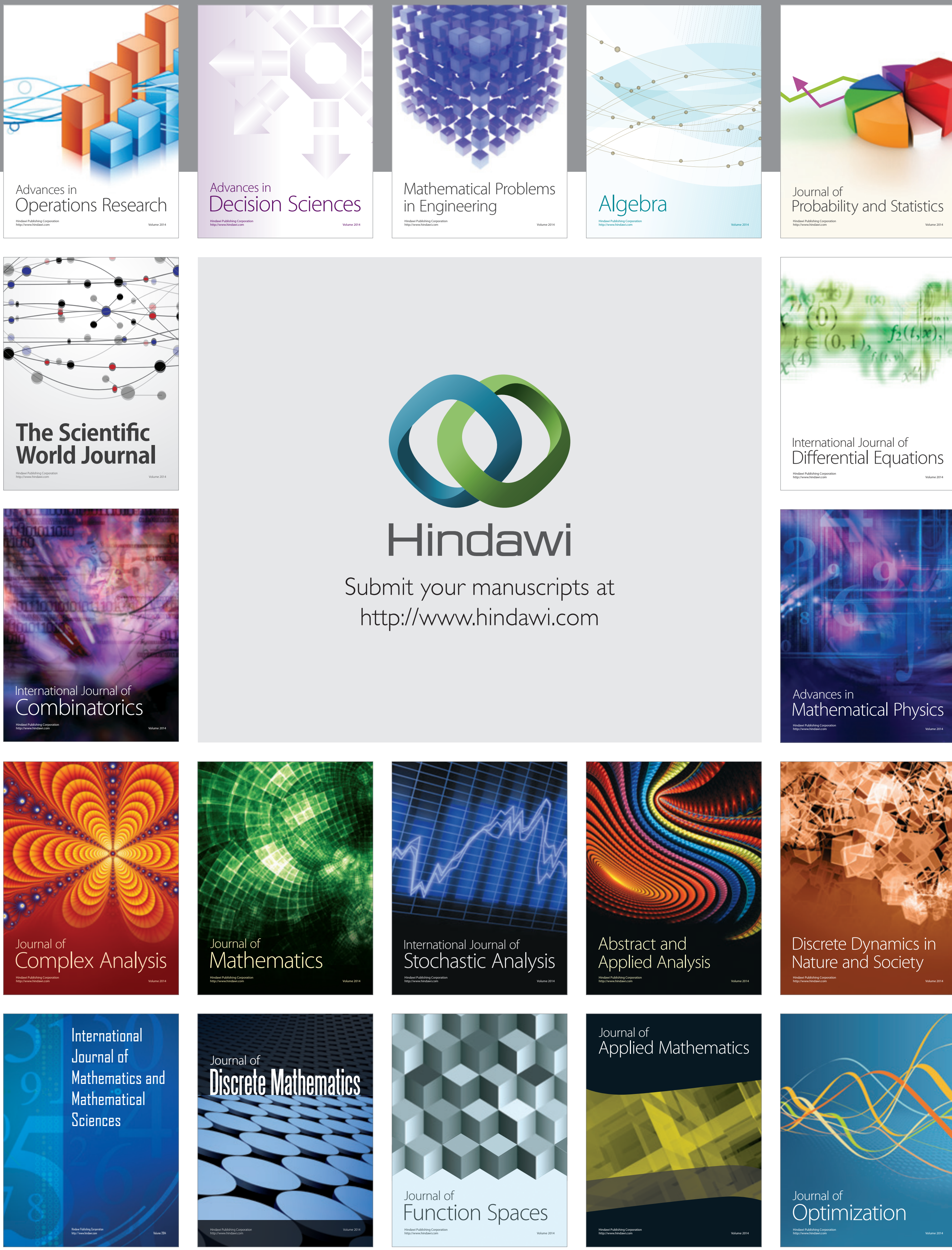
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