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Research Article

A Note on Some Strongly Sequence Spaces

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We introduce and study new sequence spaces which arise from the notions of generalized de la Vallée-Poussin means, invariant means, and modulus functions.

1. Introduction

Let w be the set of all real or complex sequences and let l_∞, c, and c_0 be the Banach spaces of bounded, convergent, and null sequences $x = (x_k)$, respectively, with the usual norm $\|x\| = \sup_n |x_n|$.

A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if its Banach limit coincides. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [1] proved that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m t_{mn}(x) \text{ exist uniformly in } n \right\}, \tag{1.1}$$

where

$$t_{mn}(x) = \frac{x_n + x_{n+1} + \cdots + x_{n+m}}{m+1}. \tag{1.2}$$

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [2] as

$$[\hat{c}] = \left\{ x \in l_\infty : \lim_m t_{mn}(|x - \ell e|) \text{ exist uniformly in } n \text{ for some } \ell \in \mathbb{C} \right\}, \tag{1.3}$$

where $e = (1, 1, \ldots)$.

Let σ be a one-to-one mapping from the set of positive integers into itself such that $\sigma^m(n) = \sigma^{m-1}(\sigma(n))$, $m = 1, 2, 3, \ldots$, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ in n, see [3]. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ-mean, if and only if,

(i) $\varphi(x) \geq 0$, when the sequence $x = (x_n)$ is such that $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$, where $e = (1, 1, \ldots)$,

(iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$, for all $x \in l_\infty$.

For a certain kind of mapping σ, every invariant mean φ extends the functional limit on the space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences with equal σ-means. Schaefer [3] proved that

$$V_\sigma = \left\{ x \in l_\infty : \lim_k t_{km}(x) = L \text{ uniformly in } m \text{ for some } L = \sigma - \lim x \right\}, \tag{1.4}$$

where

$$t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \cdots + x_{\sigma^k(m)}}{k+1}, \qquad t_{-1,m} = 0. \tag{1.5}$$

Thus we say that a bounded sequence $x = (x_k)$ is σ-convergent, if and only if, $x \in V_\sigma$ such that $\sigma^k(n) \neq n$ for all $n \geq 0$, $k \geq 1$. Note that similarly as the concept of almost convergence leads naturally to the concept of strong almost convergence, the σ-convergence leads naturally to the concept of strong σ-convergence.

A sequence $x = (x_k)$ is said to be strongly σ-convergent (see, Mursaleen [4]), if there exists a number ℓ such that

$$\frac{1}{k}\sum_{i=1}^{k} \left| x_{\sigma^i(m)} - \ell \right| \longrightarrow 0, \tag{1.6}$$

as $k \to \infty$ uniformly in m. We write $[V_\sigma]$ to denote the set of all strong σ-convergent sequences and when (1.6) holds, we write $[V_\sigma] - \lim x = \ell$. Taking $\sigma(m) = m + 1$, we obtain $[V_\sigma] = [\hat{c}]$. Then the strong σ-convergence generalizes the concept of strong almost convergence. We also note that

$$[V_\sigma] \subset V_\sigma \subset l_\infty. \tag{1.7}$$

It is also well known that the concept of paranorm is closely related to linear metric spaces. In fact, it is a generalization of absolute value. Let X be a linear space. A function $p : X \to \mathbb{R}$ is called a paranorm, if

(P:1) $p(0) \geq 0$,

(P:2) $p(x) \geq 0$, for all $x \in X$,

(P:3) $p(-x) = p(x)$, for all $x \in X$, (P:4) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$ (triangle inequality),

(P:5) if (λ_n) is a sequence of scalars, with $\lambda_n \to \lambda$ $(n \to \infty)$, and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ (continuity of multiplication by scalars).

A complete linear metric space is said to be a Fréchet space. A Fréchet sequence space X is said to be an FK space, if its metric is stronger than the metric of w on X, that is, convergence in the sequence space X implies coordinatewise convergence (the letters F and K stand for Fréchet and Koordinate, the German word for coordinate).

Note that, by Ruckle in [5], a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$, if and only if, $x = 0$,

(ii) $f(x+y) \leq f(x) + f(y)$, for all $x, y \geq 0$,

(iii) f increasing,

(iv) f is continuous from the right at zero.

Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$. Furthermore, from condition (ii), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$, and thus

$$f(x) = f\left(nx\frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right), \tag{1.8}$$

hence

$$\frac{1}{n}f(x) \leq f\left(\frac{x}{n}\right), \quad \forall n \in \mathbb{N}. \tag{1.9}$$

In [5], Ruckle used the idea of a modulus function f in order to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}. \tag{1.10}$$

From the definition, we can easily see that the space $L(f)$ is closely related to the space l_1, if we consider $f(x) = x$ for all real numbers $x \geq 0$. Several authors study these types of spaces. For example, Maddox introduced and examined some properties of the sequence spaces $w_0(f)$, $w(f)$ and $w_\infty(f)$, defined by using a modulus f, which generalized the well-known spaces w_0, w and w_∞ of strongly summable sequences, see [6]. Similarly, Savaş in [7] generalized the concept of strong almost convergence by using a modulus f and examined some further properties of the corresponding new sequence spaces.

The generalized de la Vallé-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \tag{1.11}$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$. Then a sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L (see [8]), if $t_n(x) \to L$ as $n \to \infty$, and we write

$$[V,\lambda]_0 = \left\{ x : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\},$$

$$[V,\lambda] = \{ x : x - \ell e \in [V,\lambda]_0 \text{ for some } \ell \in \mathbb{C} \}, \tag{1.12}$$

$$[V,\lambda]_\infty = \left\{ x : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty \right\},$$

for the sets of sequences that are, respectively, strongly summable to zero, strongly summable, and strongly bounded by the de la Vallé-Poussin method. In the special case where $\lambda_n = n$, for $n = 1, 2, 3, \ldots$, the sets $[V,\lambda]_0$, $[V,\lambda]$, and $[V,\lambda]_\infty$ reduce to the sets w_0, w, and w_∞, which were introduced and studied by Maddox, see [6].

We also note that the sets of sequence spaces such as strongly σ-summable to zero, strongly σ-summable, and strongly σ-bounded with respect to the modulus function were defined by Nuray and Savaş in [9].

2. Main Results

Let $p = (p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k, and $\sup_k p_k < \infty$. This assumption is made throughout the rest of this paper. Then we now write

$$[V_\sigma,\lambda,f,p]_0 = \left\{ x : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \{ f(|x_{\sigma^k(m)}|) \}^{p_k} = 0, \text{ uniformly in } m \right\},$$

$$[V_\sigma,\lambda,f,p] = \{ x : x - \ell e \in [V_\sigma,\lambda,f,p]_0 \text{ for some } \ell \in \mathbb{C} \}, \tag{2.1}$$

$$[V_\sigma,\lambda,f,p]_\infty = \left\{ x : \sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} \{ f(|x_{\sigma^k(m)}|) \}^{p_k} < \infty \right\}.$$

In particular, if we take $p_k = 1$ for all k, we have

$$[V_\sigma,\lambda,f]_0 = \left\{ x : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{\sigma^k(m)}|) = 0, \text{ uniformly in } m \right\},$$

$$[V_\sigma,\lambda,f] = \{ x : x - \ell e \in [V_\sigma,\lambda,f]_0 \text{ for some } \ell \in \mathbb{C} \}, \tag{2.2}$$

$$[V_\sigma,\lambda,f]_\infty = \left\{ x : \sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{\sigma^k(m)}|) < \infty \right\}.$$

Similarly, when $\sigma(m) = m+1$, then $[V_\sigma,\lambda,f,p]_0$, $[V_\sigma,\lambda,f,p]$ and $[V_\sigma,\lambda,f,p]_\infty$ are reduced to

$$\begin{aligned}\left[\widehat{V},\lambda,f,p\right]_0 &= \left\{x : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\{f(|x_{k+m}|)\}^{p_k} = 0, \text{ uniformly in } m\right\},\\ \left[\widehat{V},\lambda,f,p\right] &= \left\{x : x-\ell e \in \left[\widehat{V},\lambda,f,p\right]_0 \text{ for some } \ell\in\mathbb{C}\right\},\\ \left[\widehat{V},\lambda,f,p\right]_\infty &= \left\{x : \sup_{n,m}\frac{1}{\lambda_n}\sum_{k\in I_n}\{f(|x_{k+m}|)\}^{p_k} < \infty\right\}, \text{ respectively.}\end{aligned} \tag{2.3}$$

In particular, when $p_k = p$ for all k, then we have the spaces

$$\left[\widehat{V},\lambda,f,p\right]_0 = \left[\widehat{V},\lambda,f\right]_0, \qquad \left[\widehat{V},\lambda,f,p\right] = \left[\widehat{V},\lambda,f\right], \qquad \left[\widehat{V},\lambda,f,p\right]_\infty = \left[\widehat{V},\lambda,f\right]_\infty, \tag{2.4}$$

which were introduced and studied by Malkowsky and Savaş in [10]. Further, when $\lambda_n = n$, for $n = 1,2,3,\ldots$, the sets $[\widehat{V},\lambda,f]_0$ and $[\widehat{V},\lambda,f]$ are reduced to $[\widehat{c}(f)]$ and $[\widehat{c}_0(f)]$ respectively, see [7]. Now, if we consider $f(x) = x$, then one can easily obtain

$$\begin{aligned}[V_\sigma,\lambda,p]_0 &= \left\{x : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_{\sigma^k(m)}\right|^{p_k} \text{ uniformly in } m\right\},\\ [V_\sigma,\lambda,p] &= \left\{x : x-\ell e\in\left[\widehat{V}_\sigma,\lambda,p\right]_0 \text{for some } \ell\in\mathbb{C}\right\},\\ [V_\sigma,\lambda,p]_\infty &= \left\{x : \sup_{n,m}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_{\sigma^k(m)}\right|^{p_k} < \infty\right\}.\end{aligned} \tag{2.5}$$

If $p_k = 1$ for all k, then we can obtain the spaces $[V_\sigma,\lambda]_0$, $[V_\sigma,\lambda]$, and $[V_\sigma,\lambda]_\infty$. Throughout this paper, we use the notation $f(|x_k|)^{p_k}$ instead of $\{f(|x_k|)\}^{p_k}$.

If $p\in l_\infty$, then it is clear that $[V_\sigma,\lambda,f,p]_0$, $[V_\sigma,\lambda,f,p]$, and $[V,\sigma,\lambda,f,p]_\infty$ are linear spaces over the complex field $\mathbb{C}$.

Lemma 2.1. *Let f be any modulus. Then*

$$[V_\sigma,\lambda,f]_\infty = \ell_\infty^\sigma(f) = \left\{x\in w : \left(f\left(\left|x_{\sigma^k(m)}\right|\right)\right)\in\ell_\infty\right\}. \tag{2.6}$$

Proof. Let $x\in[V_\sigma,\lambda,f]_\infty$. Then there is a constant $M>0$ such that

$$\frac{1}{\lambda_1}f\left(\left|x_{\sigma^k(m)}\right|\right) \le \sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}f\left(\left|x_{\sigma^k(m)}\right|\right)\le M, \tag{2.7}$$

for all m, and so $(f(|x_{\sigma^k(m)}|)) \in l_\infty$. Let $x \in \ell_\infty^\sigma(f)$. Then there is a constant $M > 0$ such that $(f(|x_{\sigma^k(m)}|)) \le M$ for all k and m, and so

$$\frac{1}{\lambda_n}\sum_{k\in I_n} f\left(\left|x_{\sigma^k(m)}\right|\right) \le M\frac{1}{\lambda_n}\sum_{k\in I_n} 1 \le M, \tag{2.8}$$

for all m and n. Thus $x \in [V_\sigma, \lambda, f]_\infty$. This completes the proof. □

If $x \in [V_\sigma, \lambda, f, p]$, with $(1/\lambda_n)\sum_{k\in I_n} f(|x_{\sigma^k(m)} - \ell e|)^{p_k} \to 0$ as $n \to \infty$ uniformly in m, then we write $x_k \to l[V_\sigma, \lambda, f, p]$.

The following well-known inequality ([11], page 190) will be used later.

If $0 \le p_k \le \sup p_k = H$ and $C = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \le C\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}, \tag{2.9}$$

for all k and $a_k, b_k \in \mathbb{C}$.

In the following theorem, we prove $x_k \to \ell$ implies $x_k \to \ell \in [V_\sigma, \lambda, f, p]$ and we also prove the uniqueness of the limit ℓ. To prove the theorem, we need the following lemma.

Lemma 2.2 (see [2]). *Let $p_k > 0$, $q_k > 0$. Then $c_o(q) \subset c_0(p)$, if and only if, $\lim_{k\to\infty}\inf p_k/q_k > 0$, where $c_0(p) = \{x : |x_k|^{p_k} \to 0$ as $k \to \infty\}$.*

Note that no other relation between (p_k) and (q_k) is needed in Lemma 2.2.

Theorem 2.3. *Let $\lim_{k\to\infty}\inf p_k > 0$. Then $x_k \to \ell$ implies $x_k \to \ell \in [V_\sigma, \lambda, f, p]$. Let $\lim_{k\to\infty} p_k = r > 0$. If $x_k \to \ell \in [V_\sigma, \lambda, f, p]$, then ℓ is unique.*

Proof. Let $x_k \to \ell$. By the definition of modulus, we have $f(|x_k - \ell|) \to 0$. Since $\lim_{k\to\infty}\inf p_k > 0$, it follows from the above lemma that $f(|x_k - \ell|)^{p_k} \to 0$ and consequently, $x_k \to \ell \in [V_\sigma, f, p]$.

Let $\lim_{k\to\infty} p_k = r > 0$. Suppose that $x_k \to \ell_1 \in [V_\sigma, \lambda, f, p]$, $x_k \to \ell_2 \in [V_\sigma, \lambda, f, p]$ and $|\ell_1 - \ell_2|^{p_k} = a > 0$. Now, from (2.9) and the definition of modulus, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} f(|\ell_1 - \ell_2|)^{p_k} &\le \frac{C}{\lambda_n}\sum_{k\in I_n} f\left(\left|x_{\sigma^k(m)} - \ell_1\right|\right)^{p_k} \\ &\quad + \frac{C}{\lambda_n}\sum_{k\in I_n} f\left(\left|x_{\sigma^k(m)} - \ell_2\right|\right)^{p_k}.\end{aligned} \tag{2.10}$$

Hence,

$$\frac{1}{\lambda_n}\sum_{k\in I_n} f(|\ell_1 - \ell_2|)^{p_k} = 0. \tag{2.11}$$

Further, $f(|\ell_1 - \ell_2|)^{p_k} \to f(a)^r$ as $k \to \infty$ and, therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} f(|\ell_1 - \ell_2|)^{p_k} = f(a)^r. \tag{2.12}$$

From (2.11) and (2.12), it follows that $f(a) = 0$ and by the definition of modulus, we have $a = 0$. Hence $\ell_1 = \ell_2$ and this completes the proof. □

Theorem 2.4. (i) *Let* $0 < \inf_k p_k \le p_k \le 1$. *Then,*

$$[V_\sigma, \lambda, f, p] \subset [V_\sigma, \lambda, f]. \tag{2.13}$$

(ii) *Let* $0 < p_k \le \sup_k p_k < \infty$. *Then,*

$$[V_\sigma, \lambda, f] \subset [V_\sigma, \lambda, f, p]. \tag{2.14}$$

Proof. (i) Let $x \in [V_\sigma, \lambda, f, p]$. Since $0 < \inf_k p_k \le 1$, we get

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ f\left(\left| x_{\sigma^k(m)} - \ell e \right| \right) \right\} \le \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ f\left(\left| x_{\sigma^k(m)} - \ell e \right| \right) \right\}^{p_k}, \tag{2.15}$$

and hence $x \in [V_\sigma, \lambda, f]$.

(ii) Let $p \ge 1$ for each k, and $\sup_k p_k < \infty$. Let $x \in [V_\sigma, \lambda, f]$. Then, for each k, $0 < \varepsilon < 1$, there exists a positive integer N such that

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ f\left(\left| x_{\sigma^k(m)} - \ell e \right| \right) \right\} \le \varepsilon < 1, \tag{2.16}$$

for all $m \ge N$. This implies that

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ f\left(\left| x_{\sigma^k(m)} - \ell e \right| \right) \right\}^{p_k} \le \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ f\left(\left| x_{\sigma^k(m)} - \ell e \right| \right) \right\}. \tag{2.17}$$

Therefore, $x \in [V_\sigma, \lambda, f, p]$. This completes the proof. □

Finally, we conclude this paper by stating the following theorem. We omit the proof, since it involves routine verification and can be obtained by using standard techniques.

Theorem 2.5. $[V_\sigma, \lambda, f, p]_0$ *and* $[V_\sigma, \lambda, f, p]$ *are complete linear topological spaces, with paranorm* g*, where* g *is defined by*

$$g(x) = \sup_{m,n} \left(\frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ f\left(\left| x_{\sigma^k(m)} \right| \right) \right\}^{p_k} \right)^M, \tag{2.18}$$

where $M = \max(1, \{\sup_k p_k\})$.

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