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Research Article

A Halpern-Mann Type Iteration for Fixed Point Problems of a Relatively Nonexpansive Mapping and a System of Equilibrium Problems

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A new modified Halpern-Mann type iterative method is constructed. Strong convergence of the scheme to a common element of the set of fixed points of a relatively nonexpansive mapping and the set of common solutions to a system of equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth is proved. The results presented in this work improve on the corresponding ones announced by many others.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a Banach space, E^* the dual space of E, and C a nonempty closed convex subset of E. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. The equilibrium problems include fixed point problems, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases. Some methods have been proposed to solve the equilibrium problems (see, e.g., [1, 2]). In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty, and they also proved a strong convergence theorem.

Let E be a smooth Banach space and J the normalized duality mapping from E to E^*. Alber [4] considered the following functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E). \tag{1.2}$$

Using this functional, Matsushita and Takahashi [5, 6] studied and investigated the following mappings in Banach spaces. A mapping $S : C \to E$ is *relatively nonexpansive* if the following properties are satisfied:

(R1) $F(S) \neq \emptyset$,

(R2) $\varphi(p, Sx) \leq \varphi(p, x)$ for all $p \in F(S)$ and $x \in C$,

(R3) $F(S) = \widehat{F}(S)$,

where $F(S)$ and $\widehat{F}(S)$ denote the set of fixed points of S and the set of asymptotic fixed points of S, respectively. It is known that S satisfies condition (R3) if and only if $I - S$ is demiclosed at zero, where I is the identity mapping; that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Sx_n\}$ converges strongly to 0, it follows that $p \in F(S)$. In a Hilbert space H, the duality mapping J is an identity mapping and $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $S : C \to H$ is nonexpansive (i.e., $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$), then it is relatively nonexpansive. Several articles have appeared providing methods for approximating fixed points of relatively nonexpansive mappings (see, e.g., [5–19] and the references therein). Matsushita and Takahashi [5] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n) \quad n = 1, 2, \ldots, \tag{1.3}$$

where $x_1 \in C$ is arbitrary, $\{\alpha_n\}$ is an appropriate sequence in $[0, 1]$, S is a relatively nonexpansive mapping, and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges *weakly* to a fixed point of T. Moreover, Matsushita and Takahashi [6] proposed the following modification of iteration (1.3):

$$\begin{aligned}
&x_1 \in C \quad \text{is arbitrary}, \\
&y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\
&C_n = \{z \in C : \varphi(z, y_n) \leq \varphi(z, x_n)\}, \\
&Q_n = \{z \in C : \langle x_n - z, Jx_1 - Jx_n\rangle \geq 0\}, \\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \quad n = 1, 2, \ldots,
\end{aligned} \tag{1.4}$$

and proved that the sequence $\{x_n\}$ converges *strongly* to $\Pi_{F(S)} x_1$. The iteration (1.4) is called *the hybrid method.* To generate the iterative sequence, we use the generalized metric projection onto $C_n \cap Q_n$ for $n \in \mathbb{N}$. It always exists, because each $C_n \cap Q_n$ is nonempty, closed, and convex. However, in a practical case, it is not easy to be calculated. In particular, as n becomes larger, the shape of $C_n \cap Q_n$ becomes more complicate, and the projection will take much more time to be calculated.

In order to overcome this difficulty, Nilsrakoo and Saejung [15] modified Halpern and Mann's iterations for finding a fixed point of a relatively nonexpansive mapping in a Banach space as follows: $x \in E$, $x_1 \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx + \beta_n Jx_n + \gamma_n JSx_n), \quad n = 1, 2, \ldots, \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are appropriate sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n \equiv 1$, and they proved that $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$.

Many authors studied the problems of finding a common element of the set of fixed points for a mapping and the set of common solutions to a system of equilibrium problems in the setting of Hilbert space and uniformly smooth and uniformly convex Banach space, respectively (see, e.g., [20–33] and the references therein). In a Hilbert space H, S. Takahashi and W. Takahashi [34] introduced the iteration as follows: sequence $\{x_n\}$ generated by $x, x_1 \in C$,

$$\begin{gathered} u_n \in C \quad \text{such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)Su_n, \quad n = 1, 2, \ldots, \end{gathered} \tag{1.6}$$

where $\{\alpha_n\}$ is an appropriate sequence in $[0,1]$, S is nonexpansive, and $\{r_n\}$ is an appropriate positive real sequence. They proved that $\{x_n\}$ converges strongly to an element in $F(S) \cap \mathrm{EP}(F)$. In 2009, Takahashi and Zembayashi [30] proposed the iteration in a uniformly smooth and uniformly convex Banach space as follows: a sequence $\{x_n\}$ generated by $u_1 \in E$,

$$\begin{gathered} x_n \in C \quad \text{such that } F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \quad n = 1, 2, \ldots, \end{gathered} \tag{1.7}$$

where S is relatively nonexpansive, $\{\alpha_n\}$ is an appropriate sequence in $[0,1]$, and $\{r_n\}$ is an appropriate positive real sequence. They proved that if J is weakly sequentially continuous, then $\{x_n\}$ converges *weakly* to an element in $F(S) \cap \mathrm{EP}(F)$. Consequently, there are many results presented strong convergence theorems for finding a common element of the set of fixed points for a mapping and the set of common solutions to a system of equilibrium problems by using the hybrid method. However, Nilsrakoo [35] introduced the Halpern-Mann iteration guaranteeing the strong convergence as follows: $x \in C$, $u_1 \in E$ and

$$\begin{gathered} x_n \in C \quad \text{such that } F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jx_n), \\ u_{n+1} = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSy_n), \quad n = 1, 2, \ldots, \end{gathered} \tag{1.8}$$

and proved that $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{F(S)\cap \mathrm{EP}(F)}x$.

Motivated by Nilsrakoo and Saejung [15] and Nilsrakoo [35], we present a strong convergence theorem of a new modified Halpern-Mann iterative scheme to find a common element of the set of fixed points of a relatively nonexpansive mapping and the set of common solutions to a system of equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth. The results in this work improve on the corresponding ones announced by many others.

2. Preliminaries

We collect together some definitions and preliminaries which are needed in this paper. We say that a Banach space E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \quad \text{imply} \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x-y\| \geq \varepsilon \quad \text{imply} \left\| \frac{x+y}{2} \right\| \leq 1-\delta. \tag{2.2}$$

It is known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex. We say that E is *uniformly smooth* if the dual space E^* of E is uniformly convex. A Banach space E is *smooth* if the limit $\lim_{t\to 0}((\|x+ty\|-\|x\|)/t)$ exists for all norm one elements x and y in E. It is not hard to show that if E is reflexive, then E is smooth if and only if E^* is strictly convex.

Let E be a smooth Banach space. The function $\varphi : E \times E \to \mathbb{R}$ (see [4]) is defined by

$$\varphi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E), \tag{2.3}$$

where the *duality mapping* $J : E \to E^*$ is given by

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2 \quad (x \in E). \tag{2.4}$$

It is obvious from the definition of the function φ that

$$(\|x\| - \|y\|)^2 \leq \varphi(x,y) \leq (\|x\| + \|y\|)^2, \tag{2.5}$$

$$\varphi(x,y) = \varphi(x,z) + \varphi(z,y) + 2\langle x-z, Jz-Jy\rangle, \tag{2.6}$$

for all $x, y, z \in E$. Moreover,

$$\varphi\left(x, J^{-1}\left(\sum_{i=1}^{n}\lambda_i J y_i\right)\right) \leq \sum_{i=1}^{n}\lambda_i \varphi(x, y_i), \tag{2.7}$$

for all $\lambda_i \in [0,1]$ with $\sum_{i=1}^{n}\lambda_i = 1$ and $x, y_i \in E$.

The following lemma is an analogue of Xu's inequality [36, Theorem 2] with respect to φ.

Lemma 2.1 (see [15, Lemma 2.2]). *Let E be a uniformly smooth Banach space and $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\varphi\Big(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\Big) \leq \lambda\varphi(x, y) + (1-\lambda)\varphi(x, z) - \lambda(1-\lambda)g(\|Jy - Jz\|), \tag{2.8}$$

for all $\lambda \in [0, 1]$, $x \in E$ and $y, z \in B_r := \{z \in E : \|z\| \leq r\}$.

It is also easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $x_n - y_n \to 0$ implies that $\varphi(x_n, y_n) \to 0$.

Lemma 2.2 (see [37, Proposition 2]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\varphi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

Remark 2.3. For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E, we have

$$\varphi(x_n, y_n) \longrightarrow 0 \iff x_n - y_n \longrightarrow 0 \iff Jx_n - Jy_n \longrightarrow 0. \tag{2.9}$$

Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. It is known that [4, 37] for any $x \in E$, there exists a unique point $\widehat{x} \in C$ such that

$$\varphi(\widehat{x}, x) = \min_{y \in C} \varphi(y, x). \tag{2.10}$$

Following Alber [4], we denote such an element $\widehat{x}$ by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. It is easy to see that in a Hilbert space, the mapping Π_C coincides with the metric projection P_C. Concerning the generalized projection, the followings are well known.

Lemma 2.4 (see [37, Propositions 4 and 5]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E, $x \in E$ and $\widehat{x} \in C$. Then,*

(a) $\widehat{x} = \Pi_C x$ *if and only if* $\langle y - \widehat{x}, Jx - J\widehat{x}\rangle \leq 0$ *for all* $y \in C$,

(b) $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x)$ *for all* $y \in C$.

Remark 2.5. The generalized projection mapping Π_C above is relatively nonexpansive and $F(\Pi_C) = C$.

Let E be a reflexive, strictly convex, and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*; that is, $J^* = J^{-1}$. We make use of the following mapping $V : E \times E^* \to \mathbb{R}$ studied in Alber [4]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{2.11}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma (see [4] and [38, Lemma 3.2]).

Lemma 2.6. *Let E be a reflexive, strictly convex, and smooth Banach space, and let V be as in* (2.11). *Then*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \le V(x, x^* + y^*), \tag{2.12}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.7 (see [39, Lemma 2.1]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \gamma_n \delta_n \tag{2.13}$$

for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in $(0,1)$ and $\{\delta_n\}$ in $\mathbb{R}$ satisfy conditions: $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, *and* $\limsup_{n\to\infty}\delta_n \le 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 (see [40, Lemma 3.1]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{2.14}$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

For solving the equilibrium problem, we usually assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions (see, e.g., [1, 3, 30]):

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$, for all $x, y \in C$,

(A3) for all $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \le F(x, y)$,

(A4) for all $x \in C$, $F(x, \cdot)$ is convex and lower semicontinuous.

The following lemma is a result which appeared in Blum and Oettli [1, Corollary 1].

Lemma 2.9 (see [1, Corollary 1]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0 \quad \forall y \in C. \tag{2.15}$$

The following lemma gives a characterization of a solution of an equilibrium problem.

Lemma 2.10 (see [30, Lemma 2.8]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and uniformly smooth Banach space E. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* *conditions (A1)–(A4). For $r > 0$, define a mapping $T_r^F : E \to C$ so-called the resolvent of F as follows:*

$$T_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \; \forall y \in C \right\}, \tag{2.16}$$

for all $x \in E$. Then, the followings hold:

(i) *T_r is single-valued,*

(ii) *T_r is a firmly nonexpansive-type mapping [11], that is, for all $x, y \in E$*

$$\left\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y \right\rangle \leq \left\langle T_r^F x - T_r^F y, Jx - Jy \right\rangle, \tag{2.17}$$

(iii) *for all $x \in E$ and $p \in \mathrm{EP}(F)$,*

$$\varphi\left(p, T_r^F x\right) \leq \varphi\left(z, T_r^F x\right) + \varphi\left(T_r^F x, x\right) \leq \varphi(p, x), \tag{2.18}$$

(iv) $F(T_r^F) = \mathrm{EP}(F)$,

(v) $\mathrm{EP}(F)$ *is closed and convex.*

Remark 2.11. Some well-known examples of resolvents of bifunctions satisfying conditions (A1)–(A4) are presented in [3, Lemma 2.15].

Lemma 2.12 (see [8, Lemma 2.3]). *Let C be a nonempty closed convex subset of a Banach space E, F a bifunction from $C \times C \to \mathbb{R}$ satisfying conditions (A1)–(A4), and $z \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $F(y, z) \leq 0$ for all $y \in C$.*

Lemma 2.13 (see [6], Proposition 2.4). *Let C be a nonempty closed convex subset of a strictly convex and smooth Banach space E and $S : C \to E$ a relatively nonexpansive mapping. Then $F(S)$ is closed and convex.*

3. Main Results

In this section, we introduce a modified Halpern-Mann type iteration without *using the generalized metric projection* and prove a strong convergence theorem for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions to a system of equilibrium problems in a uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let E a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, $\{F_i\}_{i=1}^m$ be a finite family of a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a relatively nonexpansive mapping such that $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) \neq \emptyset$. Let $\{T_{r_{i,n}}^{F_i}\}_{i=1}^m$ be a finite family of the resolvents of F_i with positive real sequences $\{r_{i,n}\}$ such that*

$\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + \beta_n Jx_n + \gamma_n JST_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \cdots T_{r_{1,n}}^{F_1} x_n\right) \quad (n \geq 1), \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, $\{x_n\}$ *converges strongly to* $\Pi_\Omega x$.

Proof. For each $n \geq 1$, setting

$$\begin{aligned} z_n^k &= T_{r_{k,n}}^{F_k} T_{r_{k1,n}}^{F_{k-1}} \cdots T_{r_{1,n}}^{F_1} x_n, \quad (k = 1, 2, \ldots, m), \\ y_n &= J^{-1}\left(\frac{\beta_n}{1-\alpha_n} Jx_n + \frac{\gamma_n}{1-\alpha_n} JSz_n^m\right). \end{aligned} \tag{3.2}$$

We can see that $z_n^k = T_{k,n}^{F_k} z_n^{k-1}$. Since Ω is nonempty, closed, and convex, we put $\hat{x} = \Pi_\Omega x$. By Lemma 2.10(iii), we get

$$\begin{aligned} \varphi(\hat{x}, z_n^m) &\leq \varphi\left(\hat{x}, z_n^{m-1}\right) - \varphi\left(z_n^m, z_n^{m-1}\right) \\ &\leq \varphi\left(\hat{x}, z_n^{m-2}\right) - \varphi\left(z_n^{m-1}, z_n^{m-2}\right) - \varphi\left(z_n^m, z_n^{m-1}\right) \\ &\;\;\vdots \\ &\leq \varphi(\hat{x}, x_n) - \sum_{k=1}^{m} \varphi\left(z_n^k, z_n^{k-1}\right), \end{aligned} \tag{3.3}$$

where $z_n^0 = x_n$. This together with (2.7) gives

$$\begin{aligned} \varphi(\hat{x}, y_n) &\leq \frac{\beta_n}{1-\alpha_n} \varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n} \varphi(\hat{x}, Sz_n^m) \\ &\leq \frac{\beta_n}{1-\alpha_n} \varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n} \varphi(\hat{x}, z_n^m) \\ &\leq \varphi(\hat{x}, x_n). \end{aligned} \tag{3.4}$$

By Lemma 2.6, we obtain

$$\begin{aligned}
\varphi(\hat{x}, x_{n+1}) &= V(\hat{x}, Jx_{n+1}) \\
&\leq V(\hat{x}, Jx_{n+1} - \alpha_n(Jx - J\hat{x})) - 2\langle x_{n+1} - \hat{x}, -\alpha_n(Jx - J\hat{x})\rangle \\
&= \varphi\Big(\hat{x}, J^{-1}(\alpha_n J\hat{x} + (1-\alpha_n)Jy_n)\Big) + 2\alpha_n\langle x_{n+1} - \hat{x}, Jx - J\hat{x}\rangle \\
&\leq \alpha_n\varphi(\hat{x}, \hat{x}) + (1-\alpha_n)\varphi(\hat{x}, y_n) + 2\alpha_n\langle x_{n+1} - \hat{x}, Jx - J\hat{x}\rangle \\
&\leq (1-\alpha_n)\varphi(\hat{x}, x_n) + 2\alpha_n\langle x_{n+1} - \hat{x}, Jx - J\hat{x}\rangle.
\end{aligned} \tag{3.5}$$

Next, we show that $\{x_n\}$ is bounded. We consider

$$\begin{aligned}
\varphi(\hat{x}, x_{n+1}) &\leq \varphi\Big(\hat{x}, J^{-1}(\alpha_n Jx + \beta_n Jx_n + \gamma_n JSz_n^m)\Big) \\
&= \varphi\Big(\hat{x}, J^{-1}(\alpha_n Jx + (1-\alpha_n)Jy_n\Big) \\
&\leq \alpha_n\varphi(\hat{x}, x) + (1-\alpha_n)\varphi(\hat{x}, y_n) \\
&\leq \alpha_n\varphi(\hat{x}, x) + (1-\alpha_n)\varphi(\hat{x}, x_n) \\
&\leq \max\{\varphi(\hat{x}, x), \varphi(\hat{x}, x_n)\}.
\end{aligned} \tag{3.6}$$

By induction, we have

$$\varphi(\hat{x}, x_{n+1}) \leq \max\{\varphi(\hat{x}, x), \varphi(\hat{x}, x_1)\}, \tag{3.7}$$

for all $n \geq 1$. This implies that $\{x_n\}$ is bounded, and so are $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n^m\}$, and $\{Sz_n^m\}$. Let $g : [0, 2r] \to [0, \infty)$ be a function satisfying the properties of Lemma 2.1, where $r = \sup\{\|x_n\|, \|Sz_n^m\| : n \geq 1\}$. It follows from (3.3) that

$$\begin{aligned}
\varphi(\hat{x}, y_n) &\leq \frac{\beta_n}{1-\alpha_n}\varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(\hat{x}, Sz_n^m) - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}g(\|Jx_n - JSz_n^m\|) \\
&\leq \frac{\beta_n}{1-\alpha_n}\varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(\hat{x}, z_n^m) - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}g(\|Jx_n - JSz_n^m\|) \\
&\leq \varphi(\hat{x}, x_n) - \frac{\gamma_n}{1-\alpha_n}\sum_{k=1}^{m}\varphi\Big(z_n^k, z_n^{k-1}\Big) - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}g(\|Jx_n - JSz_n^m\|).
\end{aligned} \tag{3.8}$$

The rest of the proof will be divided into two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\hat{x}, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(\hat{x}, x_n)\}$ is then convergent. Then,

$$\varphi(\hat{x}, x_n) - \varphi(\hat{x}, x_{n+1}) \longrightarrow 0. \tag{3.9}$$

Notice that

$$\varphi(\hat{x}, x_{n+1}) \leq \alpha_n \varphi(\hat{x}, x) + (1-\alpha_n)\varphi(\hat{x}, y_n). \tag{3.10}$$

From condition (ii),

$$\begin{aligned}\varphi(\hat{x}, x_n) - \varphi(\hat{x}, y_n) &= \varphi(\hat{x}, x_n) - \varphi(\hat{x}, x_{n+1}) + \varphi(\hat{x}, x_{n+1}) - \varphi(\hat{x}, y_n) \\ &\leq \varphi(\hat{x}, x_n) - \varphi(\hat{x}, x_{n+1}) + \alpha_n (\varphi(\hat{x}, x) - \varphi(\hat{x}, y_n)) \longrightarrow 0.\end{aligned} \tag{3.11}$$

It follows from (3.8) that

$$\frac{\gamma_n}{1-\alpha_n}\sum_{k=1}^{m}\varphi\left(z_n^k, z_n^{k-1}\right) + \frac{\beta_n\gamma_n}{(1-\alpha_n)^2} g(\|Jx_n - JSz_n^m\|) \longrightarrow 0. \tag{3.12}$$

By the assumptions (i), (ii), and (iv),

$$\varphi\left(z_n^k, z_n^{k-1}\right) \longrightarrow 0 \quad (k = 1, 2, \dots, m), \qquad g(\|Jx_n - JSz_n^m\|) \longrightarrow 0. \tag{3.13}$$

By Remark 2.3, we get

$$z_n^k - z_n^{k-1} \longrightarrow 0 \quad (k = 1, 2, \dots, m). \tag{3.14}$$

From g is continuous strictly increasing with $g(0) = 0$, we have

$$z_n^m - Sz_n^m \longrightarrow 0, \qquad \varphi(x_n, Sz_n^m) \longrightarrow 0. \tag{3.15}$$

Consequently,

$$\begin{gathered}\varphi(x_n, y_n) \leq \frac{\beta_n}{1-\alpha_n}\varphi(x_n, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(x_n, Sz_n^m) = \frac{\gamma_n}{1-\alpha_n}\varphi(x_n, Sz_n^m) \longrightarrow 0, \\ \varphi(y_n, x_{n+1}) \leq \alpha_n\varphi(y_n, x) + (1-\alpha_n)\varphi(y_n, y_n) = \alpha_n\varphi(y_n, x) \longrightarrow 0.\end{gathered} \tag{3.16}$$

This implies that

$$x_{n+1} - x_n \longrightarrow 0. \tag{3.17}$$

Since $\{x_n\}$ is bounded and E is reflexive, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle x_n - \hat{x}, Jx - J\hat{x}\rangle = \lim_{j\to\infty}\left\langle x_{n_j} - \hat{x}, Jx - J\hat{x}\right\rangle = \langle w - \hat{x}, Jx - J\hat{x}\rangle. \tag{3.18}$$

Let $k = 1, 2, \dots, m$ be fixed. Then, $z_{n_j}^k \rightharpoonup w$ as $j \to \infty$. From $\liminf_{n\to\infty} r_{k,n} > 0$ and (3.14), we have

$$\lim_{n\to\infty} \frac{1}{r_{k,n}} \left\| J z_n^k - J z_n^{k-1} \right\| = 0. \tag{3.19}$$

Then,

$$F_k\left(z_n^k, y\right) + \frac{1}{r_{k,n}} \left\langle y - z_n^k, J z_n^k - J z_n^{k-1} \right\rangle \geq 0, \quad \forall y \in C. \tag{3.20}$$

Replacing n by n_j, we have from (A2) that

$$\frac{1}{r_{k,n_j}} \left\langle y - z_{n_j}^k, J z_{n_j}^k - J z_{n_j}^{k-1} \right\rangle \geq -F_k\left(z_{n_j}^k, y\right) \geq F_k\left(y, z_{n_j}^k\right), \quad \forall y \in C. \tag{3.21}$$

Letting $j \to \infty$, we have from (3.19) and (A4) that

$$F_k(y, w) \leq 0, \quad \forall y \in C. \tag{3.22}$$

From Lemma 2.12, we have $w \in \mathrm{EP}(F_k)$. Since S satisfies condition (R3) and $z_n^m - S z_n^m \to 0$, we have $w \in F(S)$. It follows that $w \in \Omega$. By Lemma 2.4(a), we immediately obtain that

$$\limsup_{n\to\infty} \langle x_{n+1} - \hat{x}, Jx - J\hat{x} \rangle = \limsup_{n\to\infty} \langle x_n - \hat{x}, Jx - J\hat{x} \rangle = \langle w - \hat{x}, Jx - J\hat{x} \rangle \leq 0. \tag{3.23}$$

It follows from Lemma 2.7 and (3.5) that $\varphi(\hat{x}, x_n) \to 0$. Then, $x_n \to \hat{x}$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\hat{x}, x_{n_i}) < \varphi(\hat{x}, x_{n_i+1}), \tag{3.24}$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.8, there exists a nondecreasing sequence of positive integer numbers $\{\ell_j\}$ such that $\ell_j \to \infty$,

$$\varphi\left(\hat{x}, x_{\ell_j}\right) \leq \varphi\left(\hat{x}, x_{\ell_j+1}\right), \qquad \varphi(\hat{x}, x_j) \leq \varphi\left(\hat{x}, x_{\ell_j+1}\right), \tag{3.25}$$

for all sufficiently large numbers j. We may assume without loss of generality that $\alpha_{\ell_j} > 0$ for all sufficiently large numbers j. Since

$$\varphi\left(\hat{x}, x_{\ell_j+1}\right) \leq \alpha_{\ell_j} \varphi(\hat{x}, x) + \left(1 - \alpha_{\ell_j}\right) \varphi\left(\hat{x}, y_{\ell_j}\right), \tag{3.26}$$

we obtain

$$\begin{aligned}\varphi\left(\hat{x}, x_{\ell_j}\right) - \varphi\left(\hat{x}, y_{\ell_j}\right) &= \varphi\left(\hat{x}, x_{\ell_j}\right) - \varphi\left(\hat{x}, x_{\ell_j+1}\right) + \varphi\left(\hat{x}, x_{\ell_j+1}\right) - \varphi\left(\hat{x}, y_{\ell_j}\right) \\ &\leq \alpha_{\ell_j}\left(\varphi(\hat{x}, x) - \varphi\left(\hat{x}, y_{\ell_j}\right)\right) \longrightarrow 0.\end{aligned} \tag{3.27}$$

It follows from (3.8) that

$$\frac{\gamma_{\ell_j}}{1-\alpha_{\ell_j}} \sum_{k=1}^{m} \varphi\left(z_{\ell_j}^{k}, z_{\ell_j}^{k-1}\right) + \frac{\beta_{\ell_j}\gamma_{\ell_j}}{\left(1-\alpha_{\ell_j}\right)^2} g\left(\left\| Jx_{\ell_j} - JSz_{\ell_j}^{m}\right\|\right) \longrightarrow 0. \tag{3.28}$$

Using the same proof of *Case 1*, we also obtain

$$\limsup_{j\to\infty}\left\langle x_{\ell_j+1} - \hat{x}, Jx - J\hat{x}\right\rangle \leq 0. \tag{3.29}$$

From (3.5), we have

$$\varphi\left(\hat{x}, x_{\ell_j+1}\right) \leq \left(1-\alpha_{\ell_j}\right)\varphi\left(\hat{x}, x_{\ell_j}\right) + 2\alpha_{\ell_j}\left\langle x_{\ell_j+1} - \hat{x}, Jx - J\hat{x}\right\rangle. \tag{3.30}$$

Since $\varphi(\hat{x}, x_{\ell_j}) \leq \varphi(\hat{x}, x_{\ell_j+1})$, we have

$$\begin{aligned}\alpha_{\ell_j}\varphi(\hat{x}, x_{\alpha}) &\leq \varphi\left(\hat{x}, x_{\ell_j}\right) - \varphi\left(\hat{x}, x_{\ell_j+1}\right) + 2\alpha_{\ell_j}\left\langle x_{\ell_j+1} - \hat{x}, Jx - J\hat{x}\right\rangle \\ &\leq 2\alpha_{\ell_j}\left\langle x_{\ell_j+1} - \hat{x}, Jx - J\hat{x}\right\rangle.\end{aligned} \tag{3.31}$$

In particular, since $\alpha_{\ell_j} > 0$, we get

$$\varphi(\hat{x}, x_{m_k}) \leq 2\left\langle x_{\ell_j+1} - \hat{x}, Jx - J\hat{x}\right\rangle. \tag{3.32}$$

It follows from (3.29) that $\varphi(\hat{x}, x_{\ell_j}) \to 0$. This together with (3.30) gives

$$\varphi\left(\hat{x}, x_{\ell_j+1}\right) \longrightarrow 0. \tag{3.33}$$

But $\varphi(\hat{x}, x_j) \leq \varphi(\hat{x}, x_{\ell_j+1})$ for all sufficiently large numbers j, we conclude that $x_j \to \hat{x}$.

From the two cases, we can conclude that $\{x_n\}$ converges strongly to $\hat{x}$ and the proof is finished. □

Setting $m = 1$, $F_1 = F \equiv 0$, and $r_{1,n} \equiv r_n$ in Theorem 3.1, we have the following.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$*

be a relatively nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \neq \emptyset$. *Let* $T_{r_n}^F$ *be the resolvent of* F *with a positive real sequence* $\{r_n\}$ *such that* $\liminf_{n\to\infty} r_n > 0$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + \beta_n Jx_n + \gamma_n JST_{r_n}^F x_n\right) \quad (n \geq 1), \tag{3.34}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap \mathrm{EP}(F)} x$.

Setting $F_1 \equiv 0$ and $r_{1,n} \equiv 1$ in Corollary 3.2, we have the following result.

Corollary 3.3. *Let* E *be a uniformly convex and uniformly smooth Banach space,* C *a nonempty closed convex subset of* E, *and* $S : C \to E$ *a relatively nonexpansive mapping such that* $F(S) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}(\alpha_n Jx + \beta_n Jx_n + \gamma_n JS\Pi_C x_n) \quad (n \geq 1), \tag{3.35}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_{F(S)} x$.

Next, we prove a strong convergence theorem for finding an element of the set of solutions to a system of equilibrium problems in a uniformly convex and uniformly smooth Banach space.

Theorem 3.4. *Let* E *be a uniformly convex and uniformly smooth Banach space,* C *a nonempty closed convex subset of* E, $\{F_i\}_{i=1}^m$ *a finite family of a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying conditions (A1)–(A4), and* $\cap_{i=1}^m \mathrm{EP}(F_i) \neq \emptyset$. *Let* $\{T_{r_{i,n}}^{F_i}\}_{i=1}^m$ *be a finite family of the resolvents of* F_i *with positive real sequences* $\{r_{i,n}\}$ *such that* $\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + \beta_n Jx_n + \gamma_n JT_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} x_n\right) \quad (n \geq 1), \tag{3.36}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *or* $\liminf_{n\to\infty} \beta_n = 0$.

Then, $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i=1}^{m} \mathrm{EP}(F_i)} x$.

Proof. For each $n \geq 1$, setting

$$
\begin{aligned}
z_n^k &= T_{r_{k,n}}^{F_k} T_{r_{k1,n}}^{F_{k-1}} \cdots T_{r_{1,n}}^{F_1} x_n, \quad (k = 1, 2, \ldots, m), \\
y_n &= J^{-1}\left(\frac{\beta_n}{1-\alpha_n} J x_n + \frac{\gamma_n}{1-\alpha_n} J z_n^m \right).
\end{aligned}
\tag{3.37}
$$

Since $\cap_{i=1}^{m} \mathrm{EP}(F_i)$ is nonempty, closed, and convex, we put $\widehat{x} = \Pi_{\cap_{i=1}^{m} \mathrm{EP}(F_i)} x$. Using the same proof of Theorem 3.1 when S is the identity operator, we can see that

$$
\varphi(\widehat{x}, y_n) \leq \varphi(\widehat{x}, x_n) - \frac{\gamma_n}{1-\alpha_n} \sum_{k=1}^{m} \varphi\left(z_n^k, z_n^{k-1} \right), \tag{3.38}
$$

$$
\varphi(\widehat{x}, x_{n+1}) \leq (1-\alpha_n)\varphi(\widehat{x}, x_n) + 2\alpha_n \langle x_{n+1} - \widehat{x}, Jx - J\widehat{x} \rangle. \tag{3.39}
$$

The rest of the proof will be divided into two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\widehat{x}, x_n)\}_{n=n_0}^{\infty}$ is non-increasing. In this situation, $\{\varphi(\widehat{x}, x_n)\}$ is then convergent. Then,

$$
\varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) \longrightarrow 0. \tag{3.40}
$$

Notice that

$$
\varphi(\widehat{x}, x_{n+1}) \leq \alpha_n \varphi(\widehat{x}, x) + (1-\alpha_n)\varphi(\widehat{x}, y_n). \tag{3.41}
$$

From condition (ii),

$$
\begin{aligned}
\varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, y_n) &= \varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) + \varphi(\widehat{x}, x_{n+1}) - \varphi(\widehat{x}, y_n) \\
&\leq \varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) + \alpha_n(\varphi(\widehat{x}, x) - \varphi(\widehat{x}, y_n)) \longrightarrow 0.
\end{aligned}
\tag{3.42}
$$

It follows from (3.38) that

$$
\frac{\gamma_n}{1-\alpha_n} \sum_{k=1}^{m} \varphi\left(z_n^k, z_n^{k-1} \right) \longrightarrow 0. \tag{3.43}
$$

By the assumptions (i), (ii), and (iv),

$$
\varphi\left(z_n^k, z_n^{k-1} \right) \longrightarrow 0 \quad (k = 1, 2, \ldots, m). \tag{3.44}
$$

By Remark 2.3, we get

$$z_n^k - z_n^{k-1} \longrightarrow 0 \quad (k = 1, 2, \ldots, m). \tag{3.45}$$

Consequently,

$$\begin{gathered}\varphi(x_n, y_n) \leq \frac{\beta_n}{1-\alpha_n}\varphi(x_n, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(x_n, z_n^m) = \frac{\gamma_n}{1-\alpha_n}\varphi\left(z_n^0, z_n^m\right) \longrightarrow 0, \\ \varphi(y_n, x_{n+1}) \leq \alpha_n\varphi(y_n, x) + (1-\alpha_n)\varphi(y_n, y_n) = \alpha_n\varphi(y_n, x) \longrightarrow 0.\end{gathered} \tag{3.46}$$

This implies that

$$x_{n+1} - x_n \longrightarrow 0. \tag{3.47}$$

Since $\{x_n\}$ is bounded and E is reflexive, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle x_n - \widehat{x}, Jx - J\widehat{x}\rangle = \lim_{j\to\infty}\left\langle x_{n_j} - \widehat{x}, Jx - J\widehat{x}\right\rangle = \langle w - \widehat{x}, Jx - J\widehat{x}\rangle. \tag{3.48}$$

Let $k = 1, 2, \ldots, m$ be fixed. Then, $z_{n_j}^k \rightharpoonup w$ as $j \to \infty$. From $\liminf_{n\to\infty} r_{k,n} > 0$ and (3.14), we have

$$\lim_{n\to\infty}\frac{1}{r_{k,n}}\left\|Jz_n^k - Jz_n^{k-1}\right\| = 0. \tag{3.49}$$

Then,

$$F_k\left(z_n^k, y\right) + \frac{1}{r_{k,n}}\left\langle y - z_n^k, Jz_n^k - Jz_n^{k-1}\right\rangle \geq 0, \qquad \forall y \in C. \tag{3.50}$$

Replacing n by n_j, we have from (A2) that

$$\frac{1}{r_{k,n_j}}\left\langle y - z_{n_j}^k, Jz_{n_j}^k - Jz_{n_j}^{k-1}\right\rangle \geq -F_k\left(z_{n_j}^k, y\right) \geq F_k\left(y, z_{n_j}^k\right), \qquad \forall y \in C. \tag{3.51}$$

Letting $j \to \infty$, we have from (3.49) and (A4) that

$$F_k(y, w) \leq 0, \qquad \forall y \in C. \tag{3.52}$$

From Lemma 2.12, we have $w \in \mathrm{EP}(F_k)$. By Lemma 2.4(a), we immediately obtain that

$$\limsup_{n\to\infty}\langle x_{n+1} - \widehat{x}, Jx - J\widehat{x}\rangle = \limsup_{n\to\infty}\langle x_n - \widehat{x}, Jx - J\widehat{x}\rangle = \langle w - \widehat{x}, Jx - J\widehat{x}\rangle \leq 0. \tag{3.53}$$

It follows from Lemma 2.7 and (3.39) that $\varphi(\widehat{x}, x_n) \to 0$. Then, $x_n \to \widehat{x}$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\hat{x}, x_{n_i}) < \varphi(\hat{x}, x_{n_i+1}), \tag{3.54}$$

for all $i \in \mathbb{N}$. Using the same proof of *Case 2* in Theorem 3.1, we also conclude that $x_j \to \hat{x}$.

From the two cases, we can conclude that $\{x_n\}$ converges strongly to $\hat{x}$. □

Finally, we give two explicit examples validating the assumptions in Theorem 3.1 as follows.

Example 3.5 (Optimization). *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty bounded closed convex subset of E, and $f : C \to \mathbb{R}$ a lower semicontinuous and convex functional. For instance, let $E = \mathbb{R}$, $C = [0,1]$ and $f : [0,1] \to \mathbb{R}$ be defined dy*

$$f(x) = \begin{cases} 0, & \text{if } \; x = 0, 1; \\ x \log x + (1-x)\log(1-x), & \text{if } \; x \in (0,1). \end{cases} \tag{3.55}$$

Then f is lower semicontinuous and convex. For each $i = 1, 2, \ldots, m$, let $F_i : C \times C \to \mathbb{R}$ be defined by $F_i(x,y) := f(y) - f(x)$ for all $x, y \in C$. It is known [1, 11] that F_i satisfies conditions (A1)–(A4), and $\mathrm{EP}(F_i) \neq \emptyset$. Let $S = \Pi_C$. Then, S is relatively nonexpansive of E into C (see [5, 6]) and $F(S) = C$. Then, $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) = \mathrm{EP}(F_i) \neq \emptyset$. Applying Theorem 3.1, we conclude that the sequence defined by (3.1) converges strongly to $\Pi_\Omega x$.

Example 3.6 (The convex feasibility problem). *Let E be a real Hilbert space, let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of E satisfying $C := \cap_{i=1}^m C_i \neq \emptyset$ (e.g., $C_1 = C_2 = \cdots = C_m = C \neq \emptyset$). Let $\{F_i\}_{i=1}^m$ be a finite family of bifunctions of $E \times E$ into $\mathbb{R}$ defined by*

$$F_i(x,y) = \frac{1}{2}\langle y - x, x - P_{C_i}x\rangle \quad \forall x, y \in E, \tag{3.56}$$

where P_{C_i} is a metric projection from E onto C_i. It is known [3, Lemma 2.15(iv)] that F_i satisfies conditions (A1)–(A4) and $\mathrm{EP}(F_i) = C_i$. Let $S = P_C$. Then, S is relatively nonexpansive of E into C (see [5, 6]) and then $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) = C \neq \emptyset$. Applying Theorem 3.1, we conclude that the sequence defined by (3.1) converges strongly to $\Pi_\Omega x$.

4. Deduced Theorems in Hilbert Spaces

In Hilbert spaces, if S is quasi-nonexpansive such that $I - S$ is demiclosed at zero, then S is relatively nonexpansive. We obtain the following result.

Theorem 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{F_i\}_{i=1}^m$ a finite family of a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) \neq \emptyset$. Let*

$\{T_{r_{i,n}}^{F_i}\}_{i=1}^{m}$ *be a finite family of the resolvents of* F_i *with real sequences* $\{r_{i,n}\}$ *such that* $\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in H$ *and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} x_n \quad (n \geq 1), \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n (1 - \beta_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $P_\Omega x$.

Applying Theorem 4.1 and using the technique in [41], we have the following result.

Theorem 4.2. *Let* H *be a Hilbert space,* C *a nonempty closed convex subset of* H, f *a contraction of* H *into itself (i.e., there is* $a \in (0,1)$ *such that* $\|f(x) - f(y)\| \leq a\|x - y\|$ *for all* $x, y \in H$*),* $\{F_i\}_{i=1}^{m}$ *a finite family of a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying conditions (A1)–(A4), and* $S : C \to E$ *be a nonexpansive mapping such that* $\Omega := F(S) \cap (\cap_{i=1}^{m} \mathrm{EP}(F_i)) \neq \emptyset$. *Let* $\{T_{r_{i,n}}^{F_i}\}_{i=1}^{m}$ *be a finite family of the resolvents of* F_i *with real sequences* $\{r_{i,n}\}$ *such that* $\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in H$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} x_n, \quad (n \geq 1), \tag{4.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n (1 - \beta_n) > 0$.

Then, $\{x_n\}$ *converges strongly to* z *such that* $z = P_\Omega f(z)$.

Proof. We note that $P_\Omega f$ is contraction. By Banach contraction principle, let z be the fixed point of $P_\Omega f$ and $\{y_n\}$ a sequence generated by $y_1 = x_1 \in H$ and

$$y_{n+1} = \alpha_n f(z) + \beta_n y_n + \gamma_n S T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} y_n, \quad (n \geq 1). \tag{4.3}$$

Using Theorem 4.1, we have $y_n \to z = P_\Omega f(z)$. Since S and $T^{F_k}_{r_{k,n}}$ $(k = 1,2,\ldots,m)$ are nonexpansive,

$$\begin{aligned}
\|y_{n+1} - x_{n+1}\| &\le \alpha_n \|f(x_n) - f(z)\| + \beta_n \|y_n - x_n\| \\
&\quad + \gamma_n \left\| S T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \ldots, T^{F_1}_{r_{1,n}} y_n - S T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \ldots, T^{F_1}_{r_{1,n}} x_n \right\| \\
&\le \alpha_n a \|x_n - z\| + (\beta_n + \gamma_n) \|y_n - n_n\| \\
&\le \alpha_n a (\|x_n - y_n\| + \|y_n - z\|) + (\beta_n + \gamma_n) \|x_n - y_n\| \\
&= (1 - \alpha_n(1-a)) \|y_n - x_n\| + \alpha_n (1-a) \left(\frac{a}{1-a} \|y_n - z\| \right).
\end{aligned} \tag{4.4}$$

Applying Lemma 2.7, $y_n - x_n \to 0$ and so $x_n \to z = P_\Omega f(z)$. □

Setting $m = 1$, $F_1 = F \equiv 0$, and $r_{1,n} \equiv r_n$ in Theorem 4.1, we have the following.

Corollary 4.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $F(S) \cap \mathrm{EP}(F) \ne \emptyset$. Let $T^F_{r_n}$ be the resolvent of F with a positive real sequence $\{r_n\}$ such that $\liminf_{n\to\infty} r_n > 0$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S T^F_{r_n} x_n \quad (n \ge 1), \tag{4.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n (1 - \beta_n) > 0$.

Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{EP}(F)} x$.

Corollary 4.4. *Let H be a Hilbert space, C a nonempty closed convex subset of H, f a contraction of H into itself, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \ne \emptyset$. Let $T^F_{r_n}$ be the resolvent of F with a positive real sequence $\{r_n\}$ such that $\liminf_{n\to\infty} r_n > 0$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S T^F_{r_n} x_n \quad (n \ge 1), \tag{4.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n (1 - \beta_n) > 0$.

Then, $\{x_n\}$ converges strongly to z such that $z = P_{F(S)\cap \mathrm{EP}(F)} f(z)$.

Remark 4.5. Corollary 4.4 improves and extends [42, Theorem 5]. More precisely, the conditions $\lim_{n\to\infty}(r_{n+1} - r_n) = \infty$ are removed.

Setting $F \equiv 0$ and $r_n \equiv 1$ in Corollary 4.3, we have the following.

Corollary 4.6. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $S : C \to E$ a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S P_C x_n \quad (n \geq 1), \tag{4.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty}\beta_n(1 - \beta_n) > 0$.

Then, $\{x_n\}$ converges strongly to $P_{F(S)}x$.

Applying Theorem 3.4, we have the following result.

Theorem 4.7. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{F_i\}_{i=1}^{m}$ a finite family of a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $\cap_{i=1}^{m}\mathrm{EP}(F_i) \neq \emptyset$. Let $\{T_{r_{i,n}}^{F_i}\}_{i=1}^{m}$ be a finite family of the resolvents of F_i with positive real sequences $\{r_{i,n}\}$ such that $\liminf_{n\to\infty} r_{i,n} > 0$ for all $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} x_n \quad (n \geq 1), \tag{4.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,*s*

(iv) $\liminf_{n\to\infty}\beta_n(1 - \beta_n) > 0$ *or* $\liminf_{n\to\infty}\beta_n = 0$.

Then $\{x_n\}$ converges strongly to $P_{\cap_{i=1}^{m}\mathrm{EP}(F_i)}x$.

Setting $m = 1$, $F_1 = F \equiv 0$, $r_{1,n} \equiv r_n$, and $\beta_n \equiv 0$ in Theorem 4.4, we have the following result.

Corollary 4.8 (see [35, Corollary 4.4]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $\mathrm{EP}(F) \neq \emptyset$. Let $T_{r_n}^F$ the resolvent of F with a positive real sequence $\{r_n\}$ such that $\liminf_{n\to\infty} r_n > 0$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_{r_n}^F x_n \quad (n \geq 1), \tag{4.9}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,

Then, $\{x_n\}$ *converges strongly to* $P_{\mathrm{EP}(F)}x$.

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