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Research Article

Solutions of Smooth Nonlinear Partial Differential Equations

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The method of order completion provides a general and type-independent theory for the existence and basic regularity of the solutions of large classes of systems of nonlinear partial differential equations (PDEs). Recently, the application of convergence spaces to this theory resulted in a significant improvement upon the regularity of the solutions and provided new insight into the structure of solutions. In this paper, we show how this method may be adapted so as to allow for the infinite differentiability of generalized functions. Moreover, it is shown that a large class of smooth nonlinear PDEs admit generalized solutions in the space constructed here. As an indication of how the general theory can be applied to particular nonlinear equations, we construct generalized solutions of the parametrically driven, damped nonlinear Schrödinger equation in one spatial dimension.

1. Introduction

In the 1994 monograph [1] Oberguggenberger and Rosinger presented a general and type-independent theory for the existence and basic regularity of the solutions of a large class of systems of nonlinear PDEs, based on the Dedekind order completion of spaces of piecewise smooth functions. In the mentioned monograph, it is shown that the solutions satisfy a blanket regularity property. Namely, the solutions may be assimilated with usual real measurable functions, or even nearly finite Hausdorff continuous functions [2], defined on the Euclidean domain of definition the respective system of equations. The latter result is based on the highly nontrivial characterization of the Dedekind order completion of sets of continuous functions in terms of spaces of Hausdorff continuous interval valued functions [3]. Recently, the regularity of the solutions constructed through the order completion method has been significantly improved upon by introducing suitable uniform convergence structures on appropriate spaces of piecewise smooth functions; see [4–7]. This new approach also gives new insight into the structure of the solutions obtained through the original order completion method [1].

The generality and type independence of the solution method introduced in [1, 4–6] has to date not been obtained in any of the usual theories of generalized solutions of linear and nonlinear PDEs. Indeed, and perhaps as a result of the insufficiency of the spaces of generalized functions that are typical in the study of generalized solutions of PDEs, at least from the point of view of the existence of solutions of PDEs, it is often believed that such a general theory is not possible, see for instance [8, 9]. Within the setting of the linear topological spaces of generalized functions that form the basis for most studies of PDEs, this may perhaps turn out to be the case. As a clarification and motivation of the above remarks, the following general comments may be of interest.

For over 135 years by now, there has been a general and type-independent existence and regularity result for the solutions of systems of analytic nonlinear PDEs. Indeed, in 1875 Kovalevskaia [10], upon the suggestion of Weierstrass, gave a rigorous proof of an earlier result of Cauchy, published in 1821 in his Course d'Analyse. This result, although restricted to the realm of analytic PDEs, is completely general as far as the type of nonlinearities involved are concerned. The analytic solutions of such a systems of PDEs can, however, be guaranteed to exists only on a neighborhood of the noncharacteristic analytic hypersurface on which the analytic initial data is specified. The nonexistence of solutions of a system of analytic PDEs on the whole domain of definition of the respective system of equations is not due to the particular techniques used in the proof of the result, but may rather be attributed to the very nature of nonlinear PDEs. Indeed, rather simple examples, such as the nonlinear conservation law

$$U_t + U_x U = 0, \quad t > 0, \; x \in \mathbb{R} \tag{1.1}$$

with the initial condition

$$U(0, x) = u(x), \quad x \in \mathbb{R} \tag{1.2}$$

show that, irrespective of the smoothness of the initial data (1.2), the solution of the initial value problem may fail to exist on the whole domain of definition $[0, \infty) \times \mathbb{R}$ of the equation, see for instance [11, 12]. Furthermore, in the case of the nonlinear conservation law (1.1), it is exactly the points where the solution fails to exists that are of interest, since these may represent the formation and propagation of shock waves, as well as other chaotic phenomena such as turbulence.

In view of the above remarks, it is clear that any general and type-independent theory for the existence and regularity of the solutions of nonlinear PDEs must alow for sufficiently singular objects to act as generalized solutions of such equations. In particular, the solutions may fail to be continuous, let alone sufficiently smooth, on the whole domain of definition of respective system of equations. In many cases, it happens that the spaces of generalized functions that are used in the study of PDEs do not admit such sufficiently singular objects. Indeed, we may recall that, due to the well-known Sobolev Embedding theorem, see for instance [13], the Sobolev Space $H^k(\mathbb{R}^n)$ will, for sufficiently large values of k, contain only continuous functions.

Moreover, even in case a given system of PDEs admits a solution which is classical, indeed even analytic, everywhere except at a single point of its Euclidean domain of definition, it may happen that such a solution does not belong to any of the customary spaces of generalized functions. For example, given a function

$$u : \mathbb{C} \setminus \{z_0\} \longrightarrow \mathbb{C} \tag{1.3}$$

which is analytic everywhere except at the single point $z_0 \in \mathbb{C}$, and with an essential singularity at z_0, Picard's Theorem states that u attains every complex value, with possibly one exception, in every neighborhood of z_0. Clearly such a function does not satisfy any of the usual *growth conditions* that are, rather as a rule, imposed on generalized functions. Indeed, we may recall that the elements of a Sobolev space are locally integrable, while the elements of the Colombeau algebras [14], which contain the $\mathfrak{D}'$ distributions, must satisfy certain polynomial type growth conditions near singularities. Therefore these concepts of generalized functions cannot accommodate the mentioned singularity of the function in (1.3).

In this paper, we present further developments of the general and type-independent solution method presented in [1], and in particular the uniform convergence spaces of generalized functions introduced in [4–6]. Furthermore, and in contradistinction with the spaces of generalized functions introduced in [6], we construct here a space of generalized functions that admit generalized partial derivatives of arbitrary order. While, following the methods introduced in [6], one may easily construct such a space of generalized functions, the existence of generalized solutions of systems of nonlinear PDEs in this space is nontrivial. Here we present the mentioned construction of the space of generalized functions, and show how generalized solutions of a large class of $\mathcal{C}^\infty$-smooth nonlinear PDEs may be obtained in this space.

As an application of the general theory, we discuss also the existence and regularity of generalized solutions of the parametrically driven, damped nonlinear Schrödinger equation in one spatial dimension. In this regard, we show that for a large class of $\mathcal{C}^\infty$-smooth initial values, the mentioned Schrödinger equation admits a generalized solution that satisfies the initial condition in a suitable generalized sense. We also introduce the concept of a strongly generic weak solution of this equation, and show that the solution we construct is such a weak solution.

The paper is organized as follows. In Section 2 we recall some basic facts concerning normal lower semicontinuous functions from the literature. The construction of spaces of generalized functions is given in Section 3, while Section 4 is concerned with the existence of generalized solutions of $\mathcal{C}^\infty$-smooth nonlinear PDEs. Lastly, in Section 5, we apply the general method to the parametrically driven, damped nonlinear Schrödinger equation in one spatial dimension. For all details on convergence spaces we refer the reader to the excellent book [15] and the paper [16].

2. Normal Lower Semicontinuous Functions

The concept of a normal lower semicontinuous function was first introduced by Dilworth [17] in connection with his attempts at characterizing the Dedekind order completion of spaces of continuous functions, a problem that was solved only recently by Anguelov [3]. Here we recall some facts concerning normal semicontinuous functions. For more details, and the proofs of some of the results, we refer the reader to the more recent presentations in [4, 18].

Denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ the extended real line, ordered as usual. The set of all extended real-valued functions on a topological space X is denoted $\mathcal{A}(X)$. A function $u \in \mathcal{A}(X)$ is said to be *nearly finite* whenever

$$\{x \in X : u(x) \in \mathbb{R}\} \text{ is open and dense.} \tag{2.1}$$

Two fundamental operations on the space $\mathcal{A}(X)$ are the Lower and Upper Baire Operators

$$\begin{gathered} I : \mathcal{A}(X) \longrightarrow \mathcal{A}(X), \\ S : \mathcal{A}(X) \longrightarrow \mathcal{A}(X), \end{gathered} \tag{2.2}$$

introduced by Baire [19], see also [3], which are defined by

$$I(u) : X \ni x \longmapsto \sup\{\inf\{u(y) : y \in V\} : V \in \mathcal{V}_x\}, \tag{2.3}$$

$$S(u) : X \ni x \longmapsto \inf\{\sup\{u(y) : y \in V\} : V \in \mathcal{V}_x\}, \tag{2.4}$$

respectively, with $\mathcal{V}_x$ denoting the neighborhood filter at $x \in X$. Clearly, the Baire operators I and S satisfy

$$I(u) \leq u \leq S(u), \quad u \in \mathcal{A}(X), \tag{2.5}$$

when $\mathcal{A}(X)$ is equipped with the usual pointwise order

$$u \leq v \Longleftrightarrow (\forall x \in X : u(x) \leq v(x)). \tag{2.6}$$

Furthermore, the Baire operators, as well as their compositions, are idempotent and monotone with respect to the pointwise order. That is,

$$\begin{aligned} &\forall u \in \mathcal{A}(\Omega) : \\ &\quad (1)\ I(I(u)) = I(u), \\ &\quad (2)\ S(S(u)) = S(u), \\ &\quad (3)\ (I \circ S)((I \circ S)(u)) = (I \circ S)(u), \\ &\forall u, v \in \mathcal{A}(\Omega) : \\ &\quad u \leq v \Longrightarrow \begin{pmatrix} (1)\ I(u) \leq I(v) \\ (2)\ S(u) \leq S(v) \\ (3)\ (I \circ S)(u) \leq (I \circ S)(v) \end{pmatrix}. \end{aligned} \tag{2.7}$$

The operators I and S, as well as their compositions $I \circ S$ and $S \circ I$, are useful tools for the study of (extended) real-valued functions. In this regard, we may mention that these mappings characterize certain continuity properties of functions in $\mathcal{A}(X)$. In particular, we have

$$\begin{aligned} &u \in \mathcal{A}(X) \text{ is lower semicontinuous} \Longleftrightarrow I(u) = u, \\ &u \in \mathcal{A}(X) \text{ is upper semicontinuous} \Longleftrightarrow S(u) = u. \end{aligned} \tag{2.8}$$

Furthermore, a function $u \in \mathcal{A}(X)$ is *normal lower semicontinuous* on X whenever

$$(I \circ S)(u) = u. \tag{2.9}$$

We denote the set of nearly finite normal lower semicontinuous functions on X by $\mathcal{NL}(X)$. The concept of normal lower semicontinuity of extended real-valued functions extends that of continuity of usual real-valued functions. In particular, each continuous function is nearly finite and normal lower semicontinuous so that we have the inclusion

$$\mathcal{C}(X) \subseteq \mathcal{NL}(X). \tag{2.10}$$

Conversely, a normal lower semicontinuous function is *generically continuous* in the sense that

$$\begin{aligned} &\forall u \in \mathcal{NL}(X): \\ &\exists B \subseteq X \text{ of first Baire category}: \\ &\quad x \in X \setminus B \Longrightarrow u \text{ continuous at } x. \end{aligned} \tag{2.11}$$

In particular, in case X is a Baire space, it follows that each $u \in \mathcal{NL}(X)$ is continuous on some residual set, which is dense in X. Furthermore, the following well-known property of continuous functions holds also for normal lower semicontinuous functions. Namely, we have

$$\begin{aligned} &\forall u, v \in \mathcal{NL}(X), \quad D \subseteq X \text{ dense}: \\ &\quad (\forall x \in D : u(x) \leq v(x)) \Longrightarrow u \leq v. \end{aligned} \tag{2.12}$$

With respect to the usual pointwise order, the space $\mathcal{NL}(X)$ is a fully distributive lattice. That is, suprema and infima of finite sets always exists and

$$\begin{aligned} &\forall A \subset \mathcal{NL}(X), \quad v \in \mathcal{NL}(X): \\ &u_0 = \sup A \Longrightarrow \inf\{v, u_0\} = \sup\{\inf\{u, v\} : u \in A\}. \end{aligned} \tag{2.13}$$

Furthermore, $\mathcal{NL}(X)$ is Dedekind order complete. That is, every set which is order bounded from above, respectively, below, has a least upper bound, respectively, greatest lower bound. In particular, the supremum and infimum of a set $A \subset \mathcal{NL}(X)$, is given by

$$\begin{aligned} \sup A &= (I \circ S)(\varphi), \\ \inf A &= (I \circ S)(\psi), \end{aligned} \tag{2.14}$$

respectively, with φ and ψ given by

$$\varphi : X \ni x \longmapsto \sup\{u(x) : u \in A\} \in \overline{\mathbb{R}}, \tag{2.15}$$

$$\psi : X \ni x \longmapsto \inf\{u(x) : u \in A\} \in \overline{\mathbb{R}}. \tag{2.16}$$

A useful characterization of order bounded sets in $\mathcal{NL}(X)$ is the following: If X is a Baire space, then for any set $A \subset \mathcal{NL}(X)$ we have

$$\begin{aligned} &\exists u_0 \in \mathcal{NL}(X) : \\ &u \leq u_0, \quad u \in A \end{aligned} \tag{2.17}$$

if and only if

$$\begin{aligned} &\exists R \subseteq X \text{ a residual set} : \\ &\forall x \in R : \\ &\sup\{u(x) : u \in A\} < \infty. \end{aligned} \tag{2.18}$$

Indeed, suppose that a set $A \subset \mathcal{NL}(X)$ satisfies (2.18). Then the function φ associated with A through (2.15) satisfies

$$\varphi(x) < \infty, \quad x \in R. \tag{2.19}$$

The function $u_0 = (I \circ S)(\varphi)$ is normal lower semicontinuous and satisfies

$$u \leq u_0, \quad u \in A. \tag{2.20}$$

It is sufficient to show that u_0 is finite on a dense subset of X. To see that u_0 satisfies this condition, assume the opposite. That is, we assume

$$\begin{aligned} &\exists V \subseteq X \text{ nonempty and open} : \\ &\forall x \in V : \\ &u_0(x) = \infty. \end{aligned} \tag{2.21}$$

It follows by (2.5) that

$$S(\varphi)(x) = \infty, \quad x \in V. \tag{2.22}$$

Thus (2.4) implies that

$$\begin{aligned} &\forall x \in V, \quad W \in \mathcal{V}_x, \quad M > 0 : \\ &\exists x_M \in V \cap W : \\ &\quad \varphi(x_M) > M. \end{aligned} \tag{2.23}$$

Note that, since each $u \in A$ is lower semicontinuous, the function φ is also lower semicontinuous. Therefore (2.23) implies

$$\begin{aligned} &\forall M > 0 : \\ &\exists D_M \subseteq V \text{ open and dense in } V : \\ &\quad \varphi(x) > M, \quad x \in D_M. \end{aligned} \tag{2.24}$$

There is therefore a residual set $R' \subseteq V$ such that

$$\varphi(x) = \infty, \quad x \in R'. \tag{2.25}$$

Since X is a Baire space, so is the open set V in the subspace topology. Furthermore, $R \cap V$ is a residual set in V. But $R \cap V \subseteq V \setminus R'$ so that $R \cap V$ must be of first Baire category in V, which is a contradiction. Therefore u_0 is finite on a dense subset of X. The dual statement for sets bounded from below also holds.

A subspace of $\mathcal{NL}(X)$ which is of particular interest to us here is the space $\mathcal{ML}(X)$ which consists of all functions in $\mathcal{NL}(X)$ which are real-valued and continuous on an open and dense subset of X. That is,

$$\mathcal{ML}(X) = \{u \in \mathcal{NL}(X) \mid \exists \Gamma \subset X \text{ closed nowhere dense} : u \in \mathcal{C}(X \setminus \Gamma)\}. \tag{2.26}$$

The space $\mathcal{ML}(X)$ is a sublattice of $\mathcal{NL}(X)$. As such, it is also fully distributive. Furthermore, whenever X is a metric space, the following order denseness property is satisfied:

$$\forall u \in \mathcal{NL}(X) : \sup\{v \in \mathcal{ML}(X) : v \leq u\} = u = \inf\{w \in \mathcal{ML}(X) : u \leq w\}. \tag{2.27}$$

Moreover, the following sequential version of (2.27) holds:

$$\begin{aligned}
&\forall u \in \mathcal{NL}(X):\\
&\exists (\lambda_n), (\mu_n) \subset \mathcal{ML}(X):\\
&\quad (1)\ \lambda_n \leq \lambda_{n+1} \leq u \leq \mu_{n+1} \leq \mu_n, \quad n \in \mathbb{N},\\
&\quad (2)\ \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\}.
\end{aligned} \tag{2.28}$$

We may note that the space $\mathcal{ML}(X)$, and suitable subspaces of it, also play an important role in the theory of rings of continuous functions [20], and the theory of differential algebras of generalized functions [21] where such spaces arise in connection with the so-called closed nowhere dense ideals. In the next section we construct a space of generalized functions as the completion of a suitable uniform convergence space, the elements of which are functions in $\mathcal{ML}(X)$, when X is an open subset of Euclidean n-space $\mathbb{R}^n$.

3. Spaces of Generalized Functions

We now consider the construction of spaces of generalized functions based on the spaces of normal lower semicontinuous functions discussed in Section 2. This follows closely the method used in [6], with the exception that we consider here the case of infinitely differentiable functions, this being the main topic of the current investigation.

In this regard, let Ω be an open, nonempty and possibly unbounded subset of $\mathbb{R}^n$. For $m \in \mathbb{N} \cup \{\infty\}$ we denote by $\mathcal{ML}^m(\Omega)$ the set of those functions in $\mathcal{NL}(\Omega)$ that are $\mathcal{C}^m$-smooth everywhere except on some closed nowhere dense set $\Gamma \subset \Omega$. That is,

$$\mathcal{ML}^m(\Omega) = \{u \in \mathcal{NL}(\Omega) \mid \exists \Gamma \subset \Omega \text{ closed nowhere dense} : u \in \mathcal{C}^m(\Omega \setminus \Gamma)\}. \tag{3.1}$$

One should note that, while the singularity set Γ associated with a function $u \in \mathcal{ML}^m(\Omega)$ through (3.1) is a topologically small set, it may be large in the sense of measure [22]. That is, the set Γ may have arbitrarily large positive Lebesgue measure. Furthermore, a function $u \in \mathcal{ML}^m(\Omega)$ typically does not satisfy any of the usual growth conditions that are imposed on generalized functions. In particular, u is, in general, not locally integrable on any neighborhood of any point $x \in \Gamma$. Moreover, u will typically not satisfy any of the polynomial type growth conditions that are imposed on elements of the Colombeau algebras of generalized functions [14].

For $m = 0$, the space (3.1) reduces to $\mathcal{ML}^0(\Omega) = \mathcal{ML}(\Omega)$, as defined in (2.26). For $m = \infty$, the usual partial differential operators

$$D^\alpha : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega) \subset \mathcal{C}^0(\Omega), \quad \alpha \in \mathbb{N}^n \tag{3.2}$$

extend in a straight forward way to mappings

$$\mathfrak{D}^\alpha : \mathcal{ML}^\infty(\Omega) \longrightarrow \mathcal{ML}^\infty(\Omega) \subseteq \mathcal{ML}^0(\Omega), \quad \alpha \in \mathbb{N}^n \tag{3.3}$$

which are defined through

$$\mathfrak{D}^{\alpha} : \mathcal{ML}^{\infty}(\Omega) \ni u \longmapsto (I \circ S)(D^{\alpha}u) \in \mathcal{ML}^{\infty}(\Omega), \quad \alpha \in \mathbb{N}^{n}. \tag{3.4}$$

The space of generalized functions we consider here are constructed as the completion of the space $\mathcal{ML}^{\infty}(\Omega)$ equipped with a suitable uniform convergence structure. In this regard, see [4], we consider on $\mathcal{ML}^{0}(\Omega)$ the uniform order convergence structure.

Definition 3.1. Let Σ consist of all nonempty order intervals in $\mathcal{ML}^{0}(\Omega)$. The family $\mathcal{J}_{o}$ of filters on $\mathcal{ML}^{0}(\Omega) \times \mathcal{ML}^{0}(\Omega)$ consists of all filters that satisfy the following: there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}
&\forall i = 1, \ldots, k : \\
&\exists \Sigma_{i} = \left(I_{n}^{i}\right) \subseteq \Sigma, \quad u_{i} \in \mathcal{NL}(\Omega) : \\
&\quad (1)\ I_{n+1}^{i} \subseteq I_{n}^{i}, \quad n \in \mathbb{N}, \\
&\quad (2)\ \sup\left\{\inf I_{n}^{i} : n \in \mathbb{N}\right\} = u_{i} = \inf\left\{\sup I_{n}^{i} : n \in \mathbb{N}\right\}, \\
&\quad (3)\ ([\Sigma_{1}] \times [\Sigma_{1}]) \cap \cdots \cap ([\Sigma_{k}] \times [\Sigma_{k}]) \subseteq \mathcal{U}.
\end{aligned} \tag{3.5}$$

The family of filters $\mathcal{J}_{o}$ is a uniformly Hausdorff and first countable uniform convergence structure. Furthermore, the induced convergence structure is the order convergence structure [23]. That is, a filter $\mathcal{F}$ on $\mathcal{ML}^{0}(\Omega)$ converges to $u \in \mathcal{ML}^{0}(\Omega)$ if and only if

$$\begin{aligned}
&\exists (\lambda_{n}), (\mu_{n}) \subset \mathcal{ML}^{0}(\Omega) : \\
&\quad (1)\ \lambda_{n} \leq \lambda_{n+1} \leq u \leq \mu_{n+1} \leq \mu_{n}, \quad n \in \mathbb{N}, \\
&\quad (2)\ \sup\{\lambda_{n} : n \in \mathbb{N}\} = u = \inf\{\mu_{n} : n \in \mathbb{N}\}, \\
&\quad (3)\ [\lambda_{n}, \mu_{n}] \in \mathcal{F}, \quad n \in \mathbb{N}.
\end{aligned} \tag{3.6}$$

The completion of the uniform convergence space $\mathcal{ML}^{0}(\Omega)$ may be characterized in terms of the space $\mathcal{NL}(\Omega)$ equipped with a suitable uniform convergence structure $\mathcal{J}_{o}^{\sharp}$, see [4]. In particular, this means that $\mathcal{NL}(\Omega)$ is complete and contains $\mathcal{ML}^{0}(\Omega)$ as a dense subspace. Furthermore, given any complete Hausdorff uniform convergence space Y, and any uniformly continuous mapping

$$\Psi : \mathcal{ML}^{0}(\Omega) \longrightarrow Y, \tag{3.7}$$

there exists a unique uniformly continuous mapping

$$\Psi^{\sharp} : \mathcal{NL}(\Omega) \longrightarrow Y \tag{3.8}$$

which extends Ψ.

The space $\mathcal{ML}^{\infty}(\Omega)$ is equipped with the initial uniform convergence structure with respect to the family of mappings (3.3). That is,

$$\mathcal{U} \in \mathcal{J}_D \Longleftrightarrow (\forall \alpha \in \mathbb{N}^n : (\mathfrak{D}^{\alpha} \times \mathfrak{D}^{\alpha})(\mathcal{U}) \in \mathcal{J}_o). \tag{3.9}$$

Each of the mappings (3.3) is uniformly continuous with respect to the uniform convergence structures $\mathcal{J}_D$ and $\mathcal{J}_o$ on $\mathcal{ML}^{\infty}(\Omega)$ and $\mathcal{ML}^{0}(\Omega)$, respectively. In fact, (3.9) is the finest uniform convergence structure on $\mathcal{ML}^{\infty}(\Omega)$ making the mappings (3.3) uniformly continuous.

Since the family of mappings (3.3) is countable and separates the points of $\mathcal{ML}^{\infty}(\Omega)$, that is,

$$\begin{aligned} &\forall u, v \in \mathcal{ML}^{\infty}(\Omega) : \\ &\exists \alpha \in \mathbb{N}^n : \\ &\quad \mathfrak{D}^{\alpha} u \neq \mathfrak{D}^{\alpha} v, \end{aligned} \tag{3.10}$$

it follows from the corresponding properties of $\mathcal{ML}^{0}(\Omega)$ that the uniform convergence structure $\mathcal{J}_D$ is uniformly Hausdorff and first countable. Furthermore, a filter $\mathcal{F}$ on $\mathcal{ML}^{\infty}(\Omega)$ converges to $u \in \mathcal{ML}^{\infty}(\Omega)$ with respect to the convergence structure λ_D induced by $\mathcal{J}_D$ if and only if

$$\begin{aligned} &\forall \alpha \in \mathbb{N}^n : \\ &\quad \mathfrak{D}^{\alpha}(\mathcal{F}) \text{ converges to } \mathfrak{D}^{\alpha} u \text{ in } \mathcal{ML}^{0}(\Omega). \end{aligned} \tag{3.11}$$

The completion of $\mathcal{ML}^{\infty}(\Omega)$, which we denote by $\mathcal{NL}^{\infty}(\Omega)$, is related to the completion $\mathcal{NL}(\Omega)$ of $\mathcal{ML}^{0}(\Omega)$ in the following way. Since $\mathcal{ML}^{\infty}(\Omega)$ carries the initial uniform convergence structure with respect to the family of mappings (3.3), it follows [16] that the mapping

$$\mathbf{D} : \mathcal{ML}^{\infty}(\Omega) \ni u \longmapsto (\mathfrak{D}^{\alpha} u)_{\alpha \in \mathbb{N}^n} \in \mathcal{ML}^{0}(\Omega)^{\mathbb{N}^n} \tag{3.12}$$

is a uniformly continuous embedding, with $\mathcal{ML}^0(\Omega)^{\mathbb{N}^n}$ equipped with the product uniform convergence structure with respect to $\mathcal{J}_o$. In particular, the diagram

$$\begin{tikzcd} \mathcal{ML}^\infty(\Omega) \arrow[rr, "\mathbf{D}"] \arrow[dr, "\mathcal{D}^\alpha"'] & & \mathcal{ML}^0(\Omega)^{\mathbb{N}^n} \arrow[dl, "\pi_\alpha"] \\ & \mathcal{ML}^0(\Omega) & \end{tikzcd} \tag{3.13}$$

commutes for each $\alpha \in \mathbb{N}^n$, with π_α the projection. In view of the uniform continuity of the mappings (3.3) and (3.12), there are unique extensions of these mappings to the completion $\mathcal{NL}^\infty(\Omega)$ of $\mathcal{ML}^\infty(\Omega)$. That is, we have uniformly continuous mappings

$$\mathcal{D}^{\alpha\sharp} : \mathcal{NL}^\infty(\Omega) \longrightarrow \mathcal{NL}(\Omega), \quad \alpha \in \mathbb{N}^n, \tag{3.14}$$

$$\mathbf{D}^\sharp : \mathcal{NL}^\infty(\Omega) \longrightarrow \left(\mathcal{ML}^0(\Omega)^{\mathbb{N}^n}\right)^\sharp \tag{3.15}$$

which extend the mappings (3.3) and (3.12), respectively. Here $(\mathcal{ML}^0(\Omega)^{\mathbb{N}^n})^\sharp$ denotes the completion of $\mathcal{ML}^0(\Omega)^{\mathbb{N}^n}$. In particular, the mapping (3.15) is *injective*. Note that there exists a canonical, bijective uniformly continuous mapping

$$\iota : \left(\mathcal{ML}^0(\Omega)^{\mathbb{N}^n}\right)^\sharp \longrightarrow \mathcal{NL}(\Omega)^{\mathbb{N}^n}, \tag{3.16}$$

see [16]. We may therefore consider (3.15) as an injective uniformly continuous mapping

$$\mathbf{D}^\sharp : \mathcal{NL}^\infty(\Omega) \longrightarrow \mathcal{NL}(\Omega)^{\mathbb{N}^n}, \tag{3.17}$$

where $\mathcal{NL}(\Omega)^{\mathbb{N}^n}$ carries the product uniform convergence structure with respect to $\mathcal{J}_o^\sharp$. Furthermore, the commutative diagram (3.13) extends to the diagram

$$\begin{tikzcd} \mathcal{NL}^\infty(\Omega) \arrow[rr, "\mathbf{D}^\sharp"] \arrow[dr, "\mathcal{D}^{\alpha\sharp}"'] & & \mathcal{NL}(\Omega)^{\mathbb{N}^n} \arrow[dl, "\pi_\alpha^\sharp"] \\ & \mathcal{NL}(\Omega) & \end{tikzcd} \tag{3.18}$$

The interpretation of the existence of the injective, uniformly continuous mapping (3.17) and the commutative diagram (3.18) is as follows. Each generalized function $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ may be represented in a canonical way through its generalized partial derivatives

$\mathfrak{D}^{\alpha\sharp}u^\sharp$, which are usual nearly finite normal lower semicontinuous functions. This gives a first clarification of the structure of generalized functions. Furthermore, this also provides a basic blanket regularity for the generalized functions in $\mathcal{NL}^\infty(\Omega)$. Namely, each such generalized function is identified with the vector of normal lower semicontinuous functions

$$\mathbf{D}^\sharp u^\sharp = \left(\mathfrak{D}^{\alpha\sharp}u^\sharp\right). \tag{3.19}$$

Now, in view of (2.11), we have

$$\begin{aligned}
&\forall u^\sharp \in \mathcal{NL}^\infty(\Omega):\\
&\exists R \subseteq \Omega \text{ a residual set}:\\
&\forall \alpha \in \mathbb{N}^n:\\
&\quad x \in R \Longrightarrow \mathfrak{D}^{\alpha\sharp}u^\sharp \text{ continuous at } x.
\end{aligned} \tag{3.20}$$

Thus the singularity set associated with each generalized function $u^\sharp \in \mathcal{NL}^\infty(\Omega)$, that is, the set where $u^\sharp$ or any of its generalized partial derivatives are discontinuous, is of first Baire category. This set, while small in a topological sense, may be dense in Ω. Furthermore, it may have arbitrarily large positive Lebesgue measure [22]. We note that such highly singular objects may be of interest in connection with turbulence in fluids and other types chaotical phenomena.

4. Existence of Generalized Solutions

In the previous section we discussed the structure of spaces of generalized functions which are obtained as the completion of suitable uniform convergence spaces, the elements of which are nearly finite normal lower semicontinuous functions. This construction is an extension of that given in [6] for spaces $\mathcal{NL}^m(\Omega)$ of generalized functions which admit only generalized partial derivatives of an arbitrary but fixed finite order m, to the case of infinitely differentiable functions.

It is shown in [6] that a large class of systems of nonlinear PDEs admit solutions, in a suitable generalized sense, in the spaces $\mathcal{NL}^m(\Omega)$. In this section we discuss the existence of such generalized solutions in the space $\mathcal{NL}^\infty(\Omega)$. In this regard, consider nonlinear PDE of order m of the form

$$T(x,D)u(x) = f(x), \quad x \in \Omega. \tag{4.1}$$

Here the right hand term f is supposed to be $\mathcal{C}^\infty$-smooth on Ω, while the nonlinear partial differential operator $T(x,D)$ is defined by a $\mathcal{C}^\infty$-smooth mapping

$$F : \Omega \times \mathbb{R}^K \longrightarrow \mathbb{R} \tag{4.2}$$

through

$$T(x, D)u(x) = F(x, \ldots, D^{\alpha}u(x), \ldots), \quad |\alpha| \leq m \tag{4.3}$$

for any sufficiently smooth function u defined on Ω. For each $\beta \in \mathbb{N}^n$, we denote by F^{β} the mapping

$$F^{\beta} : \Omega \times \mathbb{R}^{M_{\beta}} \longrightarrow \mathbb{R} \tag{4.4}$$

such that

$$D^{\beta}(T(x, D)u(x)) = F^{\beta}(x, \ldots, D^{\alpha}u(x), \ldots), \quad |\alpha| \leq m + |\beta| \tag{4.5}$$

for all functions $u \in \mathcal{C}^{\infty}(\Omega)$. Consider the mapping

$$F^{\infty} : \Omega \times \mathbb{R}^{\mathbb{N}^n} \ni (x, (\xi_{\alpha})_{\alpha \in \mathbb{N}^n}) \longmapsto \left(F^{\beta}(x, \ldots, \xi_{\alpha}, \ldots)\right)_{\beta \in \mathbb{N}^n} \in \mathbb{R}^{\mathbb{N}^n}. \tag{4.6}$$

We will assume that the nonlinear PDE (4.1) satisfies the condition

$$\begin{aligned}
&\forall x \in \Omega : \\
&\exists \xi(x) \in \mathbb{R}^{\mathbb{N}^n}, \quad F^{\infty}(x, \xi(x)) = \left(D^{\beta} f(x)\right)_{\beta \in \mathbb{N}^n} : \\
&\exists V \in \mathcal{V}_x, \quad W \in \mathcal{V}_{\xi(x)} : \\
&\quad F^{\infty} : V \times W \longrightarrow \mathbb{R}^{\mathbb{N}^n} \text{open},
\end{aligned} \tag{4.7}$$

where $\mathbb{R}^{\mathbb{N}^n}$ is equipped with the product topology.

With the nonlinear operator (4.3) we may associate a mapping

$$T : \mathcal{C}^{\infty}(\Omega) \longrightarrow \mathcal{C}^{\infty}(\Omega). \tag{4.8}$$

This mapping may be extended so as to act on $\mathcal{ML}^{\infty}(\Omega)$. In this regard, we set

$$T : \mathcal{ML}^{\infty}(\Omega) \ni u \longmapsto (I \circ S)(F(\cdot, u, \ldots, \mathfrak{D}^{\alpha}u, \ldots)) \in \mathcal{ML}^{\infty}(\Omega). \tag{4.9}$$

Furthermore, the partial derivatives $\mathfrak{D}^{\beta}(Tu)$ of Tu, for $u \in \mathcal{ML}^{\infty}(\Omega)$, may be represented through the mappings (4.4). In particular,

$$\begin{aligned}
&\forall \beta \in \mathbb{N}^n : \\
&\forall u \in \mathcal{ML}^{\infty}(\Omega) : \\
&\quad \mathfrak{D}^{\beta}(Tu) = T^{\beta}u,
\end{aligned} \tag{4.10}$$

where the T^{β}, with $\beta \in \mathbb{N}^n$, are the mappings defined in terms of (4.4) as

$$T^{\beta} : \mathcal{ML}^{\infty}(\Omega) \ni u \longmapsto (I \circ S)\Big(F^{\beta}(\cdot, \ldots, \mathfrak{D}^{\alpha}u, \ldots)\Big) \in \mathcal{ML}^{\infty}(\Omega) \subset \mathcal{ML}^{0}(\Omega), \tag{4.11}$$

where $\alpha \leq m + |\beta|$. We denote by T^{∞} the mapping

$$T^{\infty} : \mathcal{ML}^{\infty}(\Omega) \ni u \longmapsto \left(T^{\beta}u\right)_{\beta \in \mathbb{N}^n} \in \mathcal{ML}^{0}(\Omega)^{\mathbb{N}^n}. \tag{4.12}$$

From (4.10) to (4.12) if follows that the diagram

$$\begin{tikzcd} \mathcal{ML}^{\infty}(\Omega) \arrow[rr, "T"] \arrow[dr, "T^{\infty}"'] & & \mathcal{ML}^{\infty}(\Omega) \arrow[dl, "\mathbf{D}"] \\ & \mathcal{ML}^{0}(\Omega)^{\mathbb{N}^n} & \end{tikzcd} \tag{4.13}$$

commutes, with $\mathbf{D}$ the mapping (3.12).

Through the mapping (4.9) we obtain a *first extension* of the nonlinear PDE (4.1). Namely, the equation

$$Tu = f, \tag{4.14}$$

where T is the mapping (4.9) and the unknown u is supposed to belong to $\mathcal{ML}^{\infty}(\Omega)$. Equation (4.14) generalizes (4.1) in the sense that any solution $u \in \mathcal{C}^{\infty}(\Omega)$ of (4.1) is also a solution of (4.14). Conversely, any solution $u \in \mathcal{ML}^{\infty}(\Omega)$ of (4.14) satisfies (4.1) everywhere except on the closed nowhere dense set Γ associated with u through (3.1). That is,

$$T(x, D)u(x) = f(x), \quad x \in \Omega \setminus \Gamma. \tag{4.15}$$

A further generalization of (4.1) is obtained by extending the mapping (4.9) to the completion $\mathcal{NL}^{\infty}(\Omega)$ of $\mathcal{ML}^{\infty}(\Omega)$. In order for the concept of generalized solution of (4.1) obtained through such an extension to be a sensible one, the extension of (4.9) to $\mathcal{NL}^{\infty}(\Omega)$ must be constructed in a *canonical* way. In this regard, the following is the fundamental result.

Theorem 4.1. *The mapping* $T : \mathcal{ML}^{\infty}(\Omega) \rightarrow \mathcal{ML}^{\infty}(\Omega)$ *associated with the nonlinear partial differential operator* (4.3) *through* (4.9) *is uniformly continuous.*

Proof. In view of the commutative diagram (4.13) it is sufficient to show that the mapping (4.12) is uniformly continuous. In this regard, we claim that each of the mappings

$$T^{\beta} : \mathcal{ML}^{\infty}(\Omega) \longrightarrow \mathcal{ML}^{0}(\Omega) \tag{4.16}$$

is uniformly continuous. To see that this is so, we represent each mapping T^{β} through the diagram

$$\begin{tikzcd} \mathcal{ML}^{\infty}(\Omega) \arrow[rr, "\mathbf{D}^{|\beta|}"] \arrow[dr, "T^{\beta}"'] & & \mathcal{ML}^{0}(\Omega)^{M_{|\beta|}} \arrow[dl, "\overline{F}^{\beta}"] \\ & \mathcal{ML}^{0}(\Omega) & \end{tikzcd} \tag{4.17}$$

where

$$\mathbf{D}^{|\beta|} : \mathcal{ML}^{\infty}(\Omega) \ni u \longmapsto (\mathfrak{D}^{\alpha}u)_{|\alpha|\leq m+|\beta|} \in \mathcal{ML}^{0}(\Omega)^{M_{|\beta|}}, \tag{4.18}$$

$$\overline{F}^{\beta} : \mathcal{ML}^{0}(\Omega)^{M_{\beta}} \ni \mathbf{u} = (u_{\alpha})_{|\alpha|\leq m+|\beta|} \longmapsto (I \circ S)(F(\cdot,\ldots,u_{\alpha},\ldots)) \in \mathcal{ML}^{0}(\Omega). \tag{4.19}$$

Clearly the mapping (4.18) is uniformly continuous. As such, it suffices to show that (4.19) is uniformly continuous. In this regard, for each $|\alpha| \leq m + |\beta|$ consider a sequence (I_n^{α}) of order intervals that satisfies (1) and (2) of (3.5). For each $n \in \mathbb{N}$ there is an order interval I_n in $\mathcal{ML}^{0}(\Omega)$ such that

$$\overline{F}^{\beta}\left(\prod_{|\alpha|\leq m+|\beta|} I_n\right) \subseteq I_n. \tag{4.20}$$

Indeed, there exists a closed nowhere dense set $\Gamma \subset \Omega$ such that

$$\begin{aligned} &\forall K \subset \Omega \setminus \Gamma \text{ compact}: \\ &\exists M_K > 0: \\ &\forall n \in \mathbb{N}, \quad x \in K, \quad \mathbf{u}_n = (u_{n,\alpha}) \in \prod_{|\alpha|\leq m+|\beta|} I_n^{\alpha}: \\ &\quad -M_K < u_{\alpha}(x) < M_K, \quad |\alpha| \leq m + |\beta|. \end{aligned} \tag{4.21}$$

The inclusion (4.20) now follows by the continuity of the mapping (4.4) and the definitions of the operators I and S, respectively. For each $n \in \mathbb{N}$, set

$$\begin{aligned} \lambda_n &= \inf \overline{F}^{\beta}\left(\prod_{|\alpha|\leq m+|\beta|} I_n^{\alpha}\right) \in \mathcal{NL}(\Omega), \\ \mu_n &= \sup \overline{F}^{\beta}\left(\prod_{|\alpha|\leq m+|\beta|} I_n^{\alpha}\right) \in \mathcal{NL}(\Omega). \end{aligned} \tag{4.22}$$

Clearly the sequence (λ_n) is increasing, while (μ_n) is decreasing and

$$\lambda_n \leq \mu_n, \quad n \in \mathbb{N}. \tag{4.23}$$

We show that

$$\overline{u} = \sup\{\lambda_n : n \in \mathbb{N}\} = \inf\{\mu_n : n \in \mathbb{N}\} = \underline{u}. \tag{4.24}$$

From condition (2) of (3.5) it follows that

$$\begin{aligned}
&\forall \epsilon > 0,\ V \subseteq \Omega \text{ nonempty and open :}\\
&\exists N_{\epsilon,V} \in \mathbb{N}, \quad x \in V :\\
&\forall n \geq N_{\epsilon,V} :\\
&\quad u^{\alpha}(x) - \epsilon < u^{n,\alpha}(x) < u^{\alpha}(x) + \epsilon, \quad \mathbf{u}_n = (u_{n,\alpha}) \in \prod_{|\alpha| \leq m+|\beta|} I_n^{\alpha},
\end{aligned} \tag{4.25}$$

where $u^{\alpha} \in \mathcal{NL}(\Omega)$ is the function associated with (I_n^{α}) through (2) of (3.5). In fact, due to (2.12), the inequality in (4.25) holds on a nonempty, open subset of V. That is,

$$\begin{aligned}
&\exists V' \subseteq V \text{ nonempty and open :}\\
&\forall x \in V' :\\
&\quad u^{\alpha}(x) - \epsilon < u^{n,\alpha}(x) < u^{\alpha}(x) + \epsilon, \quad \mathbf{u}_n = (u_{n,\alpha}) \in \prod_{|\alpha| \leq m+|\beta|} I_n^{\alpha}.
\end{aligned} \tag{4.26}$$

Therefore the continuity of the mapping (4.4) implies

$$\begin{aligned}
&\forall \epsilon > 0,\ V \subseteq \Omega \text{ nonempty and open :}\\
&\exists N_{\epsilon,V} \in \mathbb{N},\ V' \subseteq V \text{ nonempty and open :}\\
&\forall x \in V', \quad n \geq N_{\epsilon,V} :\\
&\quad \left|\overline{F}^{\beta}(\mathbf{u}_n)(x) - \overline{F}^{\beta}(\mathbf{v}_n)(x)\right| < \epsilon, \quad \mathbf{u}_n, \mathbf{v}_n \in \prod_{|\alpha| \leq m+|\beta|} I_n^{\alpha}
\end{aligned} \tag{4.27}$$

so that

$$\begin{aligned}
&\forall \epsilon > 0,\ V \subseteq \Omega \text{ nonempty and open :}\\
&\exists N_{\epsilon,V} \in \mathbb{N},\ V' \subseteq V \text{ nonempty and open :}\\
&\forall x \in V', \quad n \geq N_{\epsilon,V} :\\
&\quad 0 < \lambda_n(x) - \mu_n(x) < \epsilon
\end{aligned} \tag{4.28}$$

which verifies (4.24). From the sequential order denseness (2.28) of $\mathcal{ML}^0(\Omega)$ in $\mathcal{NL}(\Omega)$ we obtain

$$\begin{aligned}
&\forall n \in \mathbb{N} : \\
&\exists (\lambda_{n,m}), (\mu_{n,m}) \subset \mathcal{ML}^0(\Omega) : \\
&\quad (1)\ \lambda_{n,m} \leq \lambda_{n,m+1} \leq \mu_{n,m+1} \leq \mu_{n,m}, \quad m \in \mathbb{N}, \\
&\quad (2)\ \sup\{\lambda_{n,m} : m \in \mathbb{N}\} = \lambda_n, \quad \inf\{\mu_{n,m} : m \in \mathbb{N}\} = \mu_n.
\end{aligned} \tag{4.29}$$

By [23, Lemma 36] it follows that

$$\begin{aligned}
&\exists (\lambda'_n), (\mu'_n) \subset \mathcal{ML}^0(\Omega) : \\
&\quad (1)\ \lambda'_n \leq \lambda'_{n+1} \leq \mu'_{n+1} \leq \mu'_n, \quad n \in \mathbb{N}, \\
&\quad (2)\ \lambda'_n \leq \lambda_n \leq \mu_n \leq \mu'_n, \quad n \in \mathbb{N}, \\
&\quad (3)\ \sup\{\lambda'_n : n \in \mathbb{N}\} = \underline{u} = \overline{u} = \inf\{\mu'_n : n \in \mathbb{N}\}.
\end{aligned} \tag{4.30}$$

Therefore the sequence of order intervals $(I_n) = ([\lambda'_n, \mu'_n])$ satisfies (1) and (2) of (3.5) and

$$\forall n \in \mathbb{N} : \overline{F}^{\beta}\left(\prod_{|\alpha| \leq m + |\beta|} I_n^{\beta} \right) \subseteq I_n \tag{4.31}$$

which shows that $\overline{F}^{\beta}$ is uniformly continuous. Therefore, according to the diagram (4.17) each of the mappings T^{β} is also uniformly continuous.

The uniform continuity of the mapping (4.12) now follows by the commutative diagram

$$\begin{tikzcd}
\mathcal{ML}^{\infty}(\Omega) \arrow[rr, "T^{\infty}"] \arrow[dr, "T^{\beta}"'] & & \mathcal{ML}^0(\Omega)^{\mathbb{N}^n} \arrow[dl, "\pi_{\beta}"] \\
& \mathcal{ML}^0(\Omega) &
\end{tikzcd} \tag{4.32}$$

This completes the proof. □

As a consequence of Theorem 4.1 we obtain a *canonical* extension of the mapping (4.9) to a mapping

$$T^{\sharp} : \mathcal{ML}^{\infty}(\Omega) \longrightarrow \mathcal{ML}^{\infty}(\Omega). \tag{4.33}$$

Indeed, since T is uniformly continuous, there exists a *unique* uniformly continuous extension of (4.9). This extension of the nonlinear partial differential operator to the space of generalized functions gives rise to a concept of generalized solution of (4.1). Namely, any solution $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ of the extended equation

$$T^\sharp u^\sharp = f \tag{4.34}$$

corresponding to the nonlinear PDE (4.1) is interpreted as a generalized solution of (4.1).

In proving the uniform continuity of the mapping (4.9) in Theorem 4.1, we also showed that the mappings (4.11) and (4.12) are uniformly continuous. Since the mapping (3.12) is a uniformly continuous embedding, the diagram (4.13) may be extended to

$$\begin{tikzcd}
\mathcal{NL}^\infty(\Omega) \arrow[rr, "T^\sharp"] \arrow[dr, "T^{\infty\sharp}"'] & & \mathcal{NL}^\infty(\Omega) \arrow[dl, "\mathbf{D}^\sharp"] \\
& \mathcal{NL}(\Omega) &
\end{tikzcd} \tag{4.35}$$

where $T^{\infty\sharp}$ is the uniformly continuous extension of (4.12). Furthermore, each of the mappings (4.11) extend uniquely to uniformly continuous mappings

$$T^{\beta\sharp} : \mathcal{NL}^\infty(\Omega) \longrightarrow \mathcal{NL}(\Omega), \quad \beta \in \mathbb{N}^n. \tag{4.36}$$

Since (4.10) coincides with $\mathfrak{D}^{\beta\sharp} \circ T^\sharp$ on the dense subspace $\mathcal{ML}^\infty(\Omega)$ of $\mathcal{NL}^\infty(\Omega)$, it follows [24] that

$$\begin{gathered}
\forall u^\sharp \in \mathcal{NL}^\infty(\Omega) : \\
T^{\beta\sharp} u^\sharp = \mathfrak{D}^{\beta\sharp}\left(T^\sharp u^\sharp\right).
\end{gathered} \tag{4.37}$$

From the commutative diagram (4.35), and the identity (3.15) it follows that the mapping $T^{\infty\sharp}$ may be represented as

$$T^{\infty\sharp} : \mathcal{NL}^\infty(\Omega) \ni u \longmapsto \left(T^{\beta\sharp} u^\sharp\right)_{\beta\in\mathbb{N}^n} \in \mathcal{NL}(\Omega)^{\mathbb{N}^n}. \tag{4.38}$$

The meaning of this is that the usual situation encountered when dealing with *classical*, $\mathcal{C}^\infty$-smooth solution of (4.1), namely,

$$\begin{gathered}
\forall u \in \mathcal{C}^\infty(\Omega) \text{ a solution of (4.1)}, \quad \beta \in \mathbb{N}^n : \\
D^\beta(T(x, D)u)(x) = D^\beta f(x), \quad x \in \Omega,
\end{gathered} \tag{4.39}$$

remains valid, in a generalized sense, for any generalized solution $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ of (4.34). That is,

$$\begin{aligned} &\forall \beta \in \mathbb{N}^n : \\ &T^{\beta\sharp}u^\sharp = \mathfrak{D}^{\beta\sharp}\left(T^\sharp u^\sharp\right) = D^\beta f. \end{aligned} \tag{4.40}$$

The main result of this paper, concerning the existence of solutions of (4.34), is the following.

Theorem 4.2. *Consider a nonlinear PDE of the form* (4.1). *If the nonlinear operator* (4.9) *satisfies* (4.7)*, then there exists some* $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ *that satisfies* (4.34).

Proof. Let us express Ω as

$$\Omega = \bigcup_{\nu\in\mathbb{N}} C_\nu, \tag{4.41}$$

where, for $\nu \in \mathbb{N}$, the compact sets C_ν are n-dimensional intervals

$$C_\nu = [a_\nu, b_\nu] \tag{4.42}$$

with $a_\nu = (a_{\nu,1}, \dots, a_{\nu,n})$, $b_\nu = (b_{\nu,1}, \dots, b_{\nu,n}) \in \mathbb{R}^n$ and $a_{\nu,j} \leq b_{\nu,j}$ for every $j = 1, \dots, n$. We assume that $\{C_\nu : \nu \in \mathbb{N}\}$ is locally finite, that is,

$$\begin{aligned} &\forall x \in \Omega : \\ &\exists V \subseteq \Omega \text{ a neighborhood of } x : \\ &\quad \{\nu \in \mathbb{N} : C_\nu \cap V \neq \emptyset\} \text{ is finite.} \end{aligned} \tag{4.43}$$

Such a partition of Ω exists, see for instance [25].

Fix $\nu \in \mathbb{N}$. To each $x_0 \in C_\nu$ we apply (4.7) so that we obtain

$$\begin{aligned} &\forall x_0 \in C_\nu : \\ &\exists \xi(x_0) = (\xi^\alpha(x_0)) \in \mathbb{R}^{\mathbb{N}^n}, \quad F^\infty(x_0, \xi(x_0)) = \left(D^\beta f(x_0)\right)_{\beta\in\mathbb{N}^n} : \\ &\exists \delta_{x_0}, \epsilon_{x_0} > 0 : \\ &\quad (1)\ F^\infty : B_{\delta_{x_0}}(x_0) \times W^1_{x_0} \longrightarrow \mathbb{R}^{\mathbb{N}} \text{ open} \\ &\quad (2)\ \left(D^\beta f(x)\right)_{\beta\in\mathbb{N}^n} \in F^\infty\left(\{x\} \times W^1_{x_0}\right), \quad x \in B_{\delta_{x_0}}(x_0), \end{aligned} \tag{4.44}$$

where $W^1_{\epsilon_{x_0}}$ is the neighborhood of $\xi(x_0)$ defined as

$$W^1_{x_0} = \prod_{|\alpha|\leq M_{x_0,1}} B_{\epsilon_{x_0}}(\xi^\alpha(x_0)) \times \mathbb{R}^{\mathbb{N}^n} \tag{4.45}$$

for $M_1 > m$ a sufficiently large integer. Note that our choice of integer M_1 does not depend on the set C_ν or the point $x_0 \in C_\nu$. Consider now a function $U_{x_0} \in \mathcal{C}^\infty(\Omega)$ such that

$$\begin{aligned} &\forall \alpha \in \mathbb{N}^n : \\ &D^\alpha U_{x_0}(x_0) = \xi^\alpha(x_0). \end{aligned} \tag{4.46}$$

From the continuity of U_{x_0} and (4.44) it follows that

$$\begin{aligned} &\forall x_0 \in C_\nu : \\ &\exists \delta_{x_0} > 0 : \\ &\forall x \in B_{\delta_{x_0}}(x_0), \quad |\beta| \leq M_1 - m : \\ &\quad (1)\ F^\infty : B_{\delta_{x_0}}(x_0) \times W^1_{x_0} \longrightarrow \mathbb{R}^{\mathbb{N}} \text{ open}, \\ &\quad (2)\ D^\beta f(x) - \epsilon_{x_0} < T^\beta U_{x_0}(x) < D^\beta f(x) + \epsilon_{x_0}, \\ &\quad (3)\ (D^\alpha U_{x_0}(x))_{\alpha \in \mathbb{N}^n} \in W^1_{x_0}, \\ &\quad (4)\ \left(D^\beta f(x)\right)_{\beta \in \mathbb{N}^n} \in F^\infty\left(\{x\} \times W^1_{x_0}\right). \end{aligned} \tag{4.47}$$

Since C_ν is compact (4.47) may be strengthened to

$$\begin{aligned} &\exists \delta_\nu > 0 : \\ &\forall x_0 \in C_\nu, \quad x \in B_{\delta_\nu}(x_0), \quad |\beta| \leq M_1 - m : \\ &\quad (1)\ F^\infty : B_{\delta_\nu}(x_0) \times W^1_{x_0} \longrightarrow \mathbb{R}^{\mathbb{N}} \text{ open}, \\ &\quad (2)\ D^\beta f(x) - \epsilon_{x_0} < T^\beta U_{x_0}(x) < D^\beta f(x) + \epsilon_{x_0}, \\ &\quad (3)\ (D^\alpha U_{x_0}(x))_{\alpha \in \mathbb{N}^n} \in W^1_{x_0}, \\ &\quad (4)\ \left(D^\beta f(x)\right)_{\beta \in \mathbb{N}^n} \in F^\infty\left(\{x\} \times W^1_{x_0}\right). \end{aligned} \tag{4.48}$$

Now subdivide C_ν into locally finite, compact n-dimensional intervals $C_{\nu,1}, \ldots, C_{\nu,K_\nu}$ with pairwise disjoint interiors such that each $C_{\nu,i}$ has diameter not exceeding δ_ν. Let $a_{\nu,i}$ denote the midpoint of $C_{\nu,i}$. Then, from (4.48), it follows that

$$\begin{aligned}
&\exists U_{\nu,i} \in \mathcal{C}^{\infty}(\Omega), \quad \epsilon_{\nu,i} > 0 : \\
&\forall x \in C_{\nu,i}, \quad \left|\beta\right| \leq M_1 - m : \\
&\quad (1)\ F^{\infty} : \operatorname{int} C_{\nu,i} \times W^1_{\nu,i} \longrightarrow \mathbb{R}^{\mathbb{N}} \text{ open}, \\
&\quad (2)\ D^{\beta} f(x) - \epsilon_{\nu,i} < T^{\beta} U_{\nu}(x) < D^{\beta} f(x) + \epsilon_{\nu,i}, \\
&\quad (3)\ \left(D^{\alpha} U_{\nu}(x)\right)_{\alpha \in \mathbb{N}^n} \in W^1_{\nu,i}, \\
&\quad (4)\ \left(D^{\beta} f(x)\right)_{\beta \in \mathbb{N}^n} \in F^{\infty}\left(\{x\} \times W^1_{\nu,i}\right),
\end{aligned} \tag{4.49}$$

where

$$W^1_{\nu,i} = \prod_{|\alpha| \leq M_{\nu,1}} B_{\epsilon_{\nu}}\left(\xi^{\alpha}(a_{\nu,i})\right) \times \mathbb{R}^{\mathbb{N}^n}. \tag{4.50}$$

Now consider the function

$$V_1 = \sum_{\nu \in \mathbb{N}} \left(\sum_{i=1}^{K_{\nu}} \chi_{\nu,i} U_{\nu,i} \right), \tag{4.51}$$

with $\chi_{\nu,i}$ the characteristic function of $\operatorname{int} C_{\nu,i}$, the interior of $C_{\nu,i}$. Clearly we have $V_1 \in \mathcal{C}^{\infty}(\Omega \setminus \Gamma_1)$ where $\Gamma_1 \subset \Omega$ is the closed nowhere dense set

$$\Gamma_1 = \Omega \setminus \left(\bigcup_{\nu \in \mathbb{N}} \left(\bigcup_{i=1}^{K_{\nu}} \operatorname{int} C_{\nu,i} \right) \right). \tag{4.52}$$

Upon application of (4.49) we find

$$\begin{aligned}
&\forall \nu \in \mathbb{N}, \quad i = 1, \ldots, K_{\nu} : \\
&\forall x \in \operatorname{int} C_{\nu,i}, \quad \left|\beta\right| \leq M_1 - m : \\
&\quad D^{\beta} f(x) - \epsilon_{\nu,i} < D^{\beta}\left(T(x,D)V_1\right)(x) < D^{\beta} f(x) + \epsilon_{\nu,i}.
\end{aligned} \tag{4.53}$$

Furthermore,

$$\begin{aligned}
&\forall \nu \in \mathbb{N}, \quad i = 1, \ldots, K_{\nu} : \\
&\forall x \in \operatorname{int} C_{\nu,i}, \quad |\alpha| \leq M_1 : \\
&\quad \lambda_1^{\alpha}(x) < D^{\alpha} V_1(x) < \mu_1^{\alpha}(x),
\end{aligned} \tag{4.54}$$

where $\lambda_1^\alpha, \mu_1^\alpha \in \mathcal{C}^0(\Omega)$ are defined as

$$\lambda_1^\alpha(x) = \xi^\alpha(a_{\nu,i}) - \epsilon_{\nu,i} \quad \text{if } x \in C_{\nu,i},$$
$$\mu_1^\alpha(x) = \xi^\alpha(a_{\nu,i}) + \epsilon_{\nu,i} \quad \text{if } x \in C_{\nu,i}. \tag{4.55}$$

Clearly the functions λ_1^α and μ_1^α satisfy

$$\mu_1^\alpha(x) - \lambda_1^\alpha(x) = 2\epsilon_{\nu,i} \quad \text{if } x \in C_{\nu,i}. \tag{4.56}$$

Continuing in this way, we may construct a sequence (Γ_n) of closed nowhere dense subsets of Ω such that $\Gamma_n \subseteq \Gamma_{n+1}$ for each $n \in \mathbb{N}$, a strictly increasing sequence of integers (M_n) and functions $V_n \in \mathcal{C}^\infty(\Omega \setminus \Gamma_n)$ such that

$$\begin{aligned}
&\forall \nu \in \mathbb{N}, \quad i = 1, \ldots, K_\nu : \\
&\forall x \in \operatorname{int} C_{\nu,i} \setminus \Gamma_n, \quad |\beta| \le M_n - m : \\
&D^\beta f(x) - \frac{\epsilon_{\nu,i}}{n} < T^\beta V_n(x) < D^\beta f(x) + \frac{\epsilon_{\nu,i}}{n}.
\end{aligned} \tag{4.57}$$

Furthermore, for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\forall |\alpha| \le M_n : \\
&\exists \lambda_n^\alpha, \mu_n^\alpha \in \mathcal{C}^0(\Omega \setminus \Gamma_n) : \\
&\forall x \in \Omega \setminus \Gamma_n, \nu \in \mathbb{N}, \quad i = 1, \ldots, K_\nu : \\
&\quad (1)\ \lambda_n^\alpha(x) < D^\alpha V_n(x) < \mu_n^\alpha(x), \\
&\quad (2)\ \mu_n^\alpha(x) - \lambda_n^\alpha(x) < \frac{\epsilon_{\nu,i}}{n}, \quad x \in \operatorname{int} C_{\nu,i} \setminus \Gamma_n, \\
&\quad (3)\ \mu_n^\alpha \le \mu_{n+1}^\alpha \le \lambda_{n+1}^\alpha \le \lambda_n^\alpha.
\end{aligned} \tag{4.58}$$

Consider now the functions

$$u_n = (I \circ S)(V_n) \in \mathcal{ML}^\infty(\Omega). \tag{4.59}$$

From (4.57) as well as the monotonicity and idempotency of the operator $I \circ S$, it follows that the sequence (Tu_n) converges to f in $\mathcal{ML}^\infty(\Omega)$. Furthermore, (4.58) implies that the sequence (u_n) is a Cauchy sequence in $\mathcal{ML}^\infty(\Omega)$. As such, it follows by Theorem 4.1 that there is some $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ that satisfies (4.34). □

It should be noted that the concept of generalized solution of nonlinear PDEs introduced here is similar to many of those that are typical in the literature, at least as far as the way in which the concept of a generalized solution is arrived at. In this regard, we may recall a construction of generalized solutions of PDEs that is representative of many of those methods that are customary in the study of PDEs. In order to construct a generalized solution of a nonlinear PDE

$$T(x, D)u(x) = f(x), \quad x \in \Omega \subseteq \mathbb{R}^n \tag{4.60}$$

one considers some relatively small space X of usual, sufficiently smooth functions on Ω, and a space Y of functions on Ω such that $f \in Y$. With the partial differential operator $T(x, D)$ one associates a mapping

$$T : X \longrightarrow Y \tag{4.61}$$

in the usual way, namely, for every $u \in X$ one has

$$Tu(x) = T(x, D)u(x), \quad x \in \Omega. \tag{4.62}$$

The spaces X and Y are equipped with uniform topologies, in fact, usually metrizable locally convex linear space topologies. The mapping (4.61) is assumed to be suitably compatible with the topologies on X and Y. In particular, T is supposed to be uniformly continuous, which enables one to extend the mapping (4.61) in canonical way to the completions $X^\sharp$ and $Y^\sharp$ of the spaces X and Y with respect to their respective uniform topologies. In this case, one ends up with a mapping

$$T^\sharp : X^\sharp \longrightarrow Y^\sharp. \tag{4.63}$$

A solution $u^\sharp \in X^\sharp$ of the generalized equation

$$T^\sharp u^\sharp = f \tag{4.64}$$

is now interpreted as a generalized solution of (4.60). Showing that such a generalized solution exists is often a rather difficult task, and may involve highly nontrivial ideas from function analysis and topology. Furthermore, a method that applies to a particular equation, may fail completely if the equation is changed slightly. This is in contradistinction with the generality and type independence of the solution method presented here. Moreover, one may note that the sequence of approximating solutions obtained in the proof of Theorem 4.2 is constructed using only basic properties of continuous, real-valued functions and elementary topology of Euclidean space.

5. An Application

In this section we show how the general theory developed in Sections 3 and 4 may be applied to particular equations, in fact, *systems* of equations. Furthermore, it is also demonstrated how the techniques of the preceding sections may be adapted so as to also incorporate initial and/or boundary conditions that may be associated with a given system of PDEs. In this way, we come to appreciate yet another advantage of solving linear and nonlinear PDEs by the methods introduced in this paper, as well as in [5–7]. Namely, and in contradistinction with the customary linear functional analytic methods, initial and/or boundary value problems are solved by essentially the same techniques that apply to the free problem. Indeed, the basic theory need only be adjusted in a minimal way in order to incorporate such additional conditions.

In this regard, we consider the one-dimensional parametrically driven, damped nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + 2|\psi|^2\phi = h\psi^* e^{2it} - i\gamma\psi, \tag{5.1}$$

with ψ^* denoting complex conjugation, which, upon setting $\psi = u + iv$, may be written as a system of equations

$$\begin{aligned} -v_t + u_{xx} + 2u^3 + 2v^2u &= hu\cos(2t) + hv\sin(2t) + \gamma v, \\ u_t + v_{xx} + 2v^3 + 2u^2v &= -hv\cos(2t) + hu\sin(2t) - \gamma u, \end{aligned} \tag{5.2}$$

subject to the initial condition

$$\left.\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x), \end{aligned}\right. \quad x \in \mathbb{R}, \tag{5.3}$$

where $u_0, v_0 \in \mathcal{C}^\infty(\mathbb{R})$. We will show that the initial value problem (5.2) and (5.3) admits a generalized solution $(u^\sharp, v^\sharp) \in \mathcal{NL}^\infty(\Omega)^2$, where $\Omega = \mathbb{R} \times [0,\infty)$. Furthermore, this solution is shown to be a *strongly generic weak solution* of (5.2) in the following sense.

Definition 5.1. A pair of functions $(u,v) \in L^\infty_{\text{loc}}(\Omega)^2$ is a strongly generic weak solution of (5.2) if there exists a closed nowhere dense set $\Gamma \subset \mathbb{R} \times (0,\infty)$ so that (u,v) satisfies (5.2) weakly on $(\mathbb{R} \times (0,\infty)) \setminus \Gamma$.

The motivation for this definition comes from systems theory. In this regard, recall [26] that a property of a system defined on an open subset Ω of $\mathbb{R}^n$ is *strongly generic* if it holds on an open and dense subset of Ω.

We may note that a large variety of nonlinear resonant phenomena in various physical media is described by the system of equations (5.1). Among these, we may count the Faraday resonance in fluid dynamics [27], instabilities in plasma [28], oscillons in granular materials [29] and anisotropic XY model of ferromagnetism [30–33]. In these applications, it is often the so-called soliton solutions that are of interest, and a lot of work has been carried out on the analysis of such solutions, see for instance [34–36]. Here we are concerned with just the basic existence and regularity results for solutions of the initial value problem (5.2) to (5.3), for a large class of initial conditions. Of course, in case the initial data in (5.3) is analytic, the Cauchy-Kovalevskaia Theorem guarantees that a solution exists, at least locally, while the global version of that theorem [21] gives existence of a generalized solution, in a suitable algebra of generalized functions, which is analytic everywhere except possibly on a closed nowhere dense subset of Ω. However, in the case of arbitrary $\mathcal{C}^\infty$-smooth initial values, we are not aware of any general existence or regularity results.

Before we consider the problem of existence of generalized solutions of (5.2) and (5.3), let us express the problem in the notation of Section 4. In particular, we write the system of equations (5.2) in the form

$$\mathbf{T}(x,t,D)(u,v)(x,t)=\mathbf{0}(x,t),\quad (x,t)\in\Omega, \tag{5.4}$$

where $\mathbf{0}$ denotes the two-dimensional vector valued function which is identically $\mathbf{0}$ on Ω. As in Section 4, the operator $\mathbf{T}(x,t,D)$ is defined through a jointly continuous, $\mathcal{C}^\infty$-smooth mapping

$$\mathbf{F}:\Omega\times\mathbb{R}^{10}\longrightarrow\mathbb{R}^2 \tag{5.5}$$

as

$$\mathbf{T}(x,t,D)(u,v)(x,t)=\mathbf{F}(x,t,\ldots,D^\alpha u(x,t),\ldots,D^\alpha v(x,t),\ldots),\quad |\alpha|\le 2. \tag{5.6}$$

In particular, the mapping $\mathbf{F}$ takes the form

$$\mathbf{F}(x,t,\xi)=\begin{pmatrix}-\xi_4+\xi_9+2\xi_1^3+2\xi_1\xi_2^2-h\xi_1\cos(2t)-h\xi_2\sin(2t)-\gamma\xi_2\\ -\xi_3+\xi_{10}+2\xi_2^3+2\xi_1^2\xi_2+h\xi_2\cos(2t)-h\xi_1\sin(2t)+\gamma\xi_1\end{pmatrix}. \tag{5.7}$$

With the nonlinear operator $\mathbf{T}(x,t,D)$ we may associate a mapping

$$\mathbf{T}:\mathcal{ML}^\infty(\Omega)^2\longrightarrow\mathcal{ML}^\infty(\Omega)^2, \tag{5.8}$$

the components of which are defined as

$$T_j:\mathcal{ML}^\infty(\Omega)^2\ni(u,v)\longmapsto(I\circ S)\big(F_j(\cdot,\cdot,\ldots,\mathfrak{D}^\alpha u,\ldots,\mathfrak{D}^\alpha v,\ldots)\big)\in\mathcal{ML}^\infty(\Omega) \tag{5.9}$$

for $|\alpha|\le 2$ and $j=1,2$, where F_1 and F_2 are the components of the mapping (5.5).

Furthermore, we express the partial derivatives of $\mathbf{T}(u,v)$ through

$$\begin{aligned}&\forall\beta\in\mathbb{N}^2:\\&\forall(u,v)\in\mathcal{ML}^\infty(\Omega)^2:\\&\mathfrak{D}^\beta\big(T_j(u,v)\big)=T_j^\beta(u,v),\quad j=1,2.\end{aligned} \tag{5.10}$$

Here, for $\beta\in\mathbb{N}^2$ and $j=1,2$,

$$T_j^\beta:\mathcal{ML}^\infty(\Omega)^2\longrightarrow\mathcal{ML}^\infty(\Omega) \tag{5.11}$$

is defined as

$$T_j^\beta : \mathcal{ML}^\infty(\Omega)^2 \ni (u,v) \longmapsto (I \circ S)\Big(F_j^\beta(\cdot,\cdot,\ldots,\mathfrak{D}^\alpha u,\ldots,\mathfrak{D}^\alpha v,\ldots)\Big) \in \mathcal{ML}^\infty(\Omega), \tag{5.12}$$

where $F_j^\beta : \Omega \times \mathbb{R}^{2M_\beta} \to \mathbb{R}$ is the $\mathcal{C}^\infty$-smooth mapping such that

$$D^\beta(F_j(\cdot,\cdot,\ldots,D^\alpha u,\ldots,D^\alpha v,\ldots))(x,t) = F_j^\beta(x,t,\ldots,D^\alpha u(x,t),\ldots,D^\alpha v(x,t),\ldots), \quad (x,t) \in \Omega \tag{5.13}$$

for every $u \in \mathcal{C}^\infty(\Omega)$. Just as is done in Section 4, we may use the mappings (5.11) to obtain a representation of the the operator $\mathbf{T}$ through the mappings

$$\begin{gathered} \mathbf{T}^\infty : \mathcal{ML}^\infty(\Omega)^2 \ni (u,v) \longmapsto \Big(T_1^\beta(u,v), T_2^\beta(u,v)\Big)_{\beta\in\mathbb{N}^2} \in \mathcal{ML}^0(\Omega)^{2\mathbb{N}^2}, \\ \mathbf{D}^2 : \mathcal{ML}^\infty(\Omega)^2 \ni (u,v) \longmapsto \Big(\mathfrak{D}^\beta u, \mathfrak{D}^\rho v\Big)_{\beta,\rho\in\mathbb{N}^2} \in \mathcal{ML}^\infty(\Omega)^{2\mathbb{N}^2}. \end{gathered} \tag{5.14}$$

In particular, the diagram

$$\begin{tikzcd} \mathcal{ML}^\infty(\Omega)^2 \arrow[rr, "\mathbf{T}"] \arrow[dr, "\mathbf{T}^\infty"'] & & \mathcal{ML}^\infty(\Omega)^2 \arrow[dl, "\mathbf{D}^2"] \\ & \mathcal{ML}^0(\Omega)^{2\mathbb{N}^2} & \end{tikzcd} \tag{5.15}$$

commutes.

We may associate with the mapping (5.5) the continuous mapping

$$\mathbf{F}^\infty : \Omega \times \mathbb{R}^{2\mathbb{N}^2} \longmapsto \mathbb{R}^{2\mathbb{N}^2} \tag{5.16}$$

defined as

$$\mathbf{F}^\infty\Big(x,t,(\xi_{\alpha 1},\xi_{\rho,2})_{\beta\in\mathbb{N}^2}\Big) = \begin{pmatrix} F_1^\beta\Big(x,t,(\xi_{\alpha,1},\xi_{\rho,2})_{\alpha,\rho\in\mathbb{N}^2}\Big) \\ F_2^\beta\Big(x,t,(\xi_{\alpha,1},\xi_{\rho,2})_{\alpha,\rho\in\mathbb{N}^2}\Big) \end{pmatrix}. \tag{5.17}$$

Note that the mapping (5.8) is linear in the components corresponding to u_t, v_t, u_{xx} and v_{xx}. Furthermore, F_1 does not depend on the components corresponding to u_t and v_{xx}, while F_2

does not depend on the components corresponding to v_t and u_{xx}. From this it follows quite easily that $\mathbf{F}^\infty$ satisfies the following two-dimensional version of (4.7):

$$\begin{aligned}
&\forall (x,t) \in \Omega : \\
&\exists V \in \mathcal{V}_{(x,t)}, \xi(x,t) \in \mathbb{R}^{2\mathbb{N}^2}, \quad \mathbf{F}^\infty(x,t,\xi(x,t)) = \mathbf{0} : \\
&\exists W \in \mathcal{V}_{\xi(x,t)} : \\
&\quad \mathbf{F}^\infty : V \times W \longrightarrow \mathbb{R}^{\mathbb{N}^n} \text{ open.}
\end{aligned} \tag{5.18}$$

By the same arguments used in the proof of Theorem 4.1, which applies so single equations, we obtain the following existence result for generalized solution of the system of equations (5.2).

Theorem 5.2. *If $\mathcal{ML}^\infty(\Omega)^2$ is equipped with the product uniform convergence structure with respect to the uniform convergence structure* (3.9) *on $\mathcal{ML}^\infty(\Omega)$, then the mapping* (5.8) *is uniformly continuous.*

In view of Theorem 5.2, it follows that the mapping (5.8) extends uniquely to a uniformly continuous mapping

$$\mathbf{T}^\sharp : \left(\mathcal{ML}^\infty(\Omega)^2\right)^\sharp \longrightarrow \left(\mathcal{ML}^\infty(\Omega)^2\right)^\sharp, \tag{5.19}$$

where $(\mathcal{ML}^\infty(\Omega)^2)^\sharp$ denotes the completion of $\mathcal{ML}^\infty(\Omega)^2$. We may identify $(\mathcal{ML}^\infty(\Omega)^2)^\sharp$ in a canonical way with the product of the completion $\mathcal{NL}^\infty(\Omega)$ of $\mathcal{ML}^\infty(\Omega)$. That is, there exists a unique bijective uniformly continuous mapping

$$\iota^\sharp : \left(\mathcal{ML}^\infty(\Omega)^2\right)^\sharp \longrightarrow \mathcal{NL}^\infty(\Omega)^2 \tag{5.20}$$

which extends the identity on $\mathcal{ML}^\infty(\Omega)^2$. We will throughout use this identification, and hence write

$$\mathbf{T}^\sharp : \mathcal{NL}^\infty(\Omega)^2 \longrightarrow \mathcal{NL}^\infty(\Omega)^2 \tag{5.21}$$

instead of (5.19). As in the general case considered in Section 4, we call any solution $(u^\sharp, v^\sharp) \in \mathcal{NL}^\infty(\Omega)^2$ of the extended equation

$$\mathbf{T}^\sharp\left(u^\sharp, v^\sharp\right) = \mathbf{0} \tag{5.22}$$

a generalized solution of (5.1). Since the systems of PDEs (5.2) satisfies the two-dimensional version (5.18) of (4.7) we obtain, by the methods of Theorem 4.2, the following basic existence result.

Theorem 5.3. *There exists some* $(u^\sharp, v^\sharp) \in \mathcal{NL}^\infty(\Omega)^2$ *that satisfies* (5.22).

In order to incorporate the initial condition (5.3) into the solution method, the way in which approximations are constructed must be altered near $t = 0$. This can be done in a rather straightforward way, as is seen next.

Theorem 5.4. *For any* $u_0, v_0 \in \mathcal{C}^\infty(\mathbb{R})$*, there exists a solution* $(u^\sharp, v^\sharp) \in \mathcal{NL}^\infty(\Omega)^2$ *of* (5.22) *that satisfies*

$$
\begin{aligned}
&\forall \alpha = (\alpha_1, 0) \in \mathbb{N}^2 : \\
&\forall x \in \mathbb{R} : \\
&\quad (1)\ \mathfrak{D}^{\alpha\sharp} u^\sharp(x,0) = D_x^\alpha u_0(x), \\
&\quad (2)\ \mathfrak{D}^{\alpha\sharp} v^\sharp(x,0) = D_x^\alpha v_0(x), \\
&\quad (3)\ \mathfrak{D}^{\alpha\sharp} u^\sharp, \mathfrak{D}^{\alpha\sharp} v^\sharp \text{ are continuous at } (x,0).
\end{aligned}
\tag{5.23}
$$

Proof. We express Ω as

$$
\Omega = \bigcup_{\nu \in \mathbb{Z}\times\mathbb{N}} I_\nu, \tag{5.24}
$$

where, for $\nu = (\nu_1, \nu_2) \in \mathbb{Z} \times \mathbb{N}$

$$
I_\nu = [\nu_1, \nu_1 + 1] \times [\nu_2, \nu_2 + 1]. \tag{5.25}
$$

Consider an arbitrary but fixed set I_ν. Since the mapping $\mathbf{F}^\infty$ satisfies (5.18), it follows that

$$
\begin{aligned}
&\forall (x_0, t_0) \in I_\nu \\
&\exists \xi(x_0,t_0) = \left(\xi_i^\alpha(x_0,t_0)\right) \in \mathbb{R}^{2\mathbb{N}^2}, \quad \mathbf{F}^\infty(x_0, t_0, \xi(x_0,t_0)) = \mathbf{0} \in \mathbb{R}^{2\mathbb{N}^2} : \\
&\exists \delta = \delta_{x_0,t_0}, \epsilon = \epsilon_{x_0,t_0} > 0 : \\
&\quad (1)\ \mathbf{F}^\infty : B_\delta(x_0,t_0) \times W_\epsilon^1(\xi(x_0,t_0)) \longrightarrow \mathbb{R}^{2\mathbb{N}^2} \text{open}, \\
&\quad (2)\ \mathbf{0} \in \mathbf{F}^\infty\Big((x,t) \times W_\epsilon^1(\xi(x_0,t_0))\Big), \quad (x,t) \in B_\delta(x_0,t_0),
\end{aligned}
\tag{5.26}
$$

where $W^1_\epsilon(\xi(x_0,t_0))$ is the neighborhood of $\xi(x_0,t_0)$ defined as

$$W^1_\epsilon(\xi(x_0,t_0)) = \prod_{|\alpha|\le M_1}^{i=1,2} B_\epsilon\left(\xi_i^\alpha(x_0,t_0)\right) \times \mathbb{R}^{2\mathbb{N}^2} \tag{5.27}$$

for $M_1 > 2$ a sufficiently large integer. Note that our choice of the constant M_1 does not depend on the point (x_0,t_0), or the set I_ν. If $t_0 = 0$, we may chose $\xi(x_0,t_0)$ in such a way that

$$\begin{aligned} &\forall\alpha = (\alpha_1,0) \in \mathbb{N}^2 : \\ &\quad (1)\ \xi_1^\alpha(x_0,t_0) = D_x^{\alpha_1} u_0(x_0), \\ &\quad (2)\ \xi_2^\alpha(x_0,t_0) = D_x^{\alpha_1} v_0(x_0). \end{aligned} \tag{5.28}$$

For each (x_0,t_0), let us fix $\xi(x_0,t_0) \in \mathbb{R}^{2\mathbb{N}^2}$ in (5.26) so that (5.28) holds if $t_0 = 0$. For $(x_0,t_0) \in I_\nu$ consider $\mathcal{C}^\infty$-smooth functions $U = U_{x_0,t_0}$ and $V = V_{x_0,t_0}$ on Ω such that

$$\begin{aligned} &\forall\alpha \in \mathbb{N}^2 : \\ &\quad (1)\ D^\alpha U(x_0,t_0) = \xi_1^\alpha(x_0,t_0), \\ &\quad (2)\ D^\alpha V(x_0,t_0) = \xi_2^\alpha(x_0,t_0). \end{aligned} \tag{5.29}$$

In particular, if $t_0 = 0$ we may set

$$\begin{aligned} U(x,t) &= u_0(x) + \varphi^1_{x_0,t_0}(t) + \sum_{\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2}^{\alpha_1,\alpha_2\neq 0} \frac{1}{\alpha_1!\alpha_2!} t^{\alpha_2}(x-x_0)^{\alpha_1}\xi_1^\alpha(x_0,t_0), \\ V(x,t) &= v_0(x) + \varphi^2_{x_0,t_0}(t) + \sum_{\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2}^{\alpha_1,\alpha_2\neq 0} \frac{1}{\alpha_1!\alpha_2!} t^{\alpha_2}(x-x_0)^{\alpha_1}\xi_2^\alpha(x_0,t_0), \end{aligned} \tag{5.30}$$

where $\varphi^1_{x_0,t_0}, \varphi^2_{x_0,t_0} \in \mathcal{C}^\infty(\mathbb{R})$ satisfy

$$\begin{aligned} &\forall\alpha = (0,\alpha_2) \in \mathbb{N}^2 : \\ &\quad (1)\ D_t^{\alpha_2}\varphi^1_{x_0,t_0}(0) = \xi_1^\alpha(x_0,t_0), \\ &\quad (2)\ D_t^{\alpha_2}\varphi^2_{x_0,t_0}(0) = \xi_2^\alpha(x_0,t_0). \end{aligned} \tag{5.31}$$

From the continuity of U, V and their derivatives, as well as the components F_j^β of $\mathbf{F}^\infty$, and (5.26) it follows that

$$\begin{aligned}
&\forall (x_0, t_0) \in I_{\nu_1,\nu_2} : \\
&\exists \delta = \delta_{x_0,t_0} > 0, \quad \epsilon = \epsilon_{x_0,t_0} > 0 : \\
&\forall (x,t) \in B_\delta(x_0,t_0), \quad |\beta| \le M_1 - 2, \quad j = 1,2 : \\
&\quad (1)\ \mathbf{F}^\infty : B_\delta(x_0,t_0) \times W_\epsilon^1(x_0,t_0) \longrightarrow \mathbb{R}^{2\mathbb{N}^2} \text{open}, \\
&\quad (2)\ -1 < D^\beta T_j(x,t,D)(U,V)(x,t) < 1, \\
&\quad (3)\ (D^\alpha U(x,t), D^\rho V(x,t))_{\alpha,\rho\in\mathbb{N}^2} \in W_\epsilon^1(x_0,t_0), \\
&\quad (4)\ \mathbf{0} \in \mathbf{F}^\infty\Big((x,t) \times W_\epsilon^1(x_0,t_0)\Big)
\end{aligned} \tag{5.32}$$

Since I_ν is compact, (5.32) may be strengthened to

$$\begin{aligned}
&\exists \delta_\nu > 0 : \\
&\forall (x_0, t_0) \in I_\nu : \\
&\exists \epsilon = \epsilon_{x_0,t_0} > 0 : \\
&\forall (x,t) \in B_{\delta_\nu}(x_0,t_0), \quad |\beta| \le M_1 - 2, \quad j = 1,2 : \\
&\quad (1)\ \mathbf{F}^\infty : B_{\delta_\nu}(x_0,t_0) \times W_\epsilon^1(x_0,t_0) \longrightarrow \mathbb{R}^{2\mathbb{N}^2} \text{open}, \\
&\quad (2)\ -1 < D^\beta T_j(x,t,D)(U,V)(x,t) < 1, \\
&\quad (3)\ (D^\alpha U(x,t), D^\rho V(x,t))_{\alpha,\mathrm{rho}\in\mathbb{N}^2} \in W_\epsilon^1(x_0,t_0), \\
&\quad (4)\ \mathbf{0} \in \mathbf{F}^\infty\Big((x,t) \times W_\epsilon^1(x_0,t_0)\Big).
\end{aligned} \tag{5.33}$$

Now write I_ν as

$$I_\nu = \bigcup_{k=1}^{N_\nu}\left(\bigcup_{l=1}^{N_\nu} I_{\nu,k,l}\right), \tag{5.34}$$

where $N_\nu > \sqrt{2}/\delta_\nu$ and

$$I_{\nu,k,l} = \left[\nu_1 + \frac{(k-1)}{N_\nu}, \nu_1 + \frac{k}{N_\nu}\right] \times \left[\nu_2 + \frac{(l-1)}{N_\nu}, \nu_2 + \frac{l}{N_\nu}\right]. \tag{5.35}$$

Denote by $a_{\nu,k,l}$ the midpoint of the set $I_{\nu,k,l}$ if $l \ne 1$. Otherwise,

$$a_{\nu,k,l} = \left(\nu_1 + \frac{2k-1}{2N_{\nu_1,\nu_2}}, 0\right). \tag{5.36}$$

Then the functions $U_{\nu,k,l} = U_{a_{\nu_1,\nu_2,k,l}}$ and $V_{\nu,k,l} = V_{a_{\nu_1,\nu_2,k,l}}$ satisfy

$$
\begin{aligned}
&\exists\epsilon = \epsilon_{\nu,k,l} > 0 : \\
&\forall (x,t) \in I_{\nu_1,\nu_2,k,l}, \quad |\beta| \leq M_1 - 2, \quad j = 1,2 : \\
&\quad (1)\ \mathbf{F}^{\infty} : I_{\nu,k,l} \times W_{\epsilon}^{1}(a_{\nu,k,l}) \longrightarrow \mathbb{R}^{2\mathbb{N}^2} \text{ open}, \\
&\quad (2)\ -1 < D^{\beta} T_j(x,t,D)(U_{\nu,k,l}, V_{\nu,k,l})(x,t) < 1, \\
&\quad (3)\ (D^{\alpha} U_{\nu,k,l}(x,t), D^{\rho} V_{\nu,k,l}(x,t))_{\alpha,\mathrm{rho}\in\mathbb{N}^2} \in W_{\epsilon}^{1}(a_{\nu,k,l}), \\
&\quad (4)\ \mathbf{0} \in \mathbf{F}^{\infty}\Big((x,t) \times W_{\epsilon}^{1}(a_{\nu,k,l})\Big).
\end{aligned}
\tag{5.37}
$$

Set

$$
\begin{aligned}
\Gamma_1 &= \Omega \setminus \left(\bigcup_{\nu\in\mathbb{Z}\times\mathbb{N}} \left(\bigcup_{k=1}^{N_\nu} \left(\bigcup_{l=1}^{N_\nu} \operatorname{int} I_{\nu,k,l} \right)\right)\right), \\
U_1 &= \left(\sum_{\nu\in\mathbb{Z}\times\mathbb{N}} \left(\sum_{k=1}^{N_\nu} \left(\sum_{l=1}^{N_\nu} \chi_{\nu,k,l} U_{\nu,k,l} \right)\right)\right), \\
V_1 &= \left(\sum_{\nu\in\mathbb{Z}\times\mathbb{N}} \left(\sum_{k=1}^{N_\nu} \left(\sum_{l=1}^{N_\nu} \chi_{\nu,k,l} V_{\nu,k,l} \right)\right)\right),
\end{aligned}
\tag{5.38}
$$

where $\chi_{\nu,k,l}$ is the characteristic function of int $I_{\nu,k,l}$. Then, for $j = 1,2$ and $|\beta| \leq M_1 - 2$ we have

$$
-1 \leq D^{\beta}\big(T_j(x,t,D)(U_1,V_1)\big)(x,t) \leq 1, \quad (x,t) \in \Omega \setminus \Gamma_1. \tag{5.39}
$$

Furthermore, for $|\alpha| \leq M_1$ we have

$$
\lambda_1^{\alpha,1}(x,t) \leq D^{\alpha} U_1 \leq \mu_1^{\alpha,1}(x,t), \quad (x,t) \in \Omega \setminus \Gamma_1, \tag{5.40}
$$

$$
\lambda_1^{\alpha,2}(x,t) \leq D^{\alpha} V_1 \leq \mu_1^{\alpha,2}(x,t), \quad (x,t) \in \Omega \setminus \Gamma_1, \tag{5.41}
$$

where $\lambda_1^{\alpha,i}, \mu_1^{\alpha,i} \in C^0(\Omega \setminus \Gamma_1)$, for $i = 1,2$, are defined as

$$
\begin{aligned}
\lambda_1^{\alpha,i}(x,t) &= \xi^{\alpha,i}(a_{\nu,k,l}) - \epsilon_{\nu,k,l}, \quad (x,t) \in \operatorname{int} I_{\nu,k,l}, \\
\mu_1^{\alpha,i}(x,t) &= \xi^{\alpha,i}(a_{\nu,k,l}) + \epsilon_{\nu,k,l}, \quad (x,t) \in \operatorname{int} I_{\nu,k,l}
\end{aligned}
\tag{5.42}
$$

for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \neq 0$, otherwise

$$\lambda_1^{\alpha,1}(x,t) = \begin{cases} \xi^{\alpha,1}(a_{\nu,k,l}) - \epsilon_{\nu,k,l} & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l \neq 1, \\ D^\alpha u_0(x) - \psi_{\nu,k,l}(t) & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l = 1, \end{cases}$$
$$\lambda_2^{\alpha,1}(x,t) = \begin{cases} \xi^{\alpha,2}(a_{\nu,k,l}) - \epsilon_{\nu,k,l} & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l \neq 1, \\ D^\alpha u_0(x) - \psi_{\nu,k,l}(t) & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l = 1, \end{cases}$$
$$\mu_1^{\alpha,1}(x,t) = \begin{cases} \xi^{\alpha,1}(a_{\nu,k,l}) + \epsilon_{\nu,k,l} & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l \neq 1, \\ D^\alpha u_0(x) + \psi_{\nu,k,l}(t) & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l = 1, \end{cases}$$
$$\mu_2^{\alpha,1}(x,t) = \begin{cases} \xi^{\alpha,2}(a_{\nu,k,l}) + \epsilon_{\nu,k,l} & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l \neq 1, \\ D^\alpha u_0(x) + \psi_{\nu,k,l}(t) & \text{if } (x,t) \in \operatorname{int} I_{\nu,k,l},\ l = 1, \end{cases} \tag{5.43}$$

where $\psi_{\nu,k,l} \in \mathcal{C}^0(\mathbb{R})$ is a suitable function that satisfies

$$\psi_{\nu,k,l}(0) = 0, \quad \psi_{\nu,k,l}(t) > 0. \tag{5.44}$$

Proceeding as in the proof of Theorem 4.2, we obtain a sequence $(u_n, v_n) \subset \mathcal{ML}^\infty(\Omega)^2$ such that

$$-\frac{1}{n} \leq T_j(u_n, v_n) \leq \frac{1}{n}, \quad j = 1, 2. \tag{5.45}$$

Furthermore, for each $i = 1, 2$ and $\alpha \in \mathbb{N}^2$, we may construct sequences $(\lambda_n^{\alpha,i})$ and $(\mu_n^{\alpha,i})$ in $\mathcal{ML}^0(\Omega)$ so that, for all $n \in \mathbb{N}$,

$$\lambda_n^{\alpha,i} \leq \lambda_{n+1}^{\alpha,i} \leq \mu_{n+1}^{\alpha,i} \leq \mu_n^{\alpha,i}, \tag{5.46}$$

$$0 \leq \mu_n^{\alpha,i}(x,t) - \lambda_n^{\alpha,i}(x,t) \leq \frac{1}{n}, \quad (x,t) \in \Omega. \tag{5.47}$$

Furthermore,

$$\lambda_n^{\alpha,1} \leq \mathfrak{D}^\alpha u_n \leq \mu_n^{\alpha,1}, \tag{5.48}$$

$$\lambda_n^{\alpha,2} \leq \mathfrak{D}^\alpha v_n \leq \mu_n^{\alpha,2}. \tag{5.49}$$

Moreover, for $\alpha = (\alpha_1, 0) \in \mathbb{N}^2$, each of the functions $\lambda_n^{\alpha,i}$ and $\mu_n^{\alpha,i}$ is continuous at every point $(x, 0)$ and satisfies

$$\lambda_n^{\alpha,i}(x,0) = \mu_n^{\alpha,i}(x,0), \quad x \in \mathbb{R}. \tag{5.50}$$

Clearly the sequence (u_n, v_n) is a Cauchy sequence in $\mathcal{ML}^{\infty}(\Omega)^2$ and $(\mathbf{T}(u_n, v_n))$ converges to $\mathbf{0}$ in $\mathcal{ML}^{\infty}(\Omega)^2$. Therefore there exists some $(u^{\sharp}, v^{\sharp}) \in \mathcal{NL}^{\infty}(\Omega)^2$ that satisfies (5.22). Furthermore, (5.48) to (5.49) together with (5.50) implies that (5.23) holds. □

Let us now consider the additional regularity properties that the solution $(u^{\sharp}, v^{\sharp})$ of the initial value problem (5.2)-(5.3), constructed in Theorem 5.4, may satisfy. In particular, we will show that this solution is in fact a *strongly generic weak solution*. In this regard, we note that each approximation (u_n, v_n) of $(u^{\sharp}, v^{\sharp})$ may be approximated by a sequence of $\mathcal{C}^{\infty}$-smooth functions on Ω. In particular, if Γ_n is the closed nowhere dense subset associated with (u_n, v_n) through (3.1), then, for every $m \in \mathbb{N}$, there exists a function $\varphi_m \in \mathcal{C}^{\infty}(\Omega)$ that satisfies

$$\varphi_m(x,t) = \begin{cases} 1 & \text{if } (x,t) \in B_{n,m}, \\ 0 & \text{if } (x,t) \in C_{n,m}, \end{cases} \tag{5.51}$$

where

$$\begin{aligned} B_{n,m} &= \left\{ (x,t) \in \Omega \mid \forall (x_0, t_0) \in \Gamma_n : \|(x,t) - (x_0,t_0)\| \geq \frac{1}{m} \right\}, \\ C_{n,m} &= \mathrm{cl}\left(\left\{ (x,t) \in \Omega \mid \exists (x_0, t_0) \in \Gamma_n : \|(x,t) - (x_0,t_0)\| \leq \frac{1}{2m} \right\} \right). \end{aligned} \tag{5.52}$$

Clearly the functions $u_{n,m} = u_n \varphi_m$ and $v_{n,m} = v_n \varphi_m$ are $\mathcal{C}^{\infty}$-smooth on Ω and satisfy

$$\begin{aligned} u_{n,m}(x,t) &= u_n(x,t), (x,t) \in B_{n,m}, \\ v_{n,m}(x,t) &= v_n(x,t), (x,t) \in B_{n,m}. \end{aligned} \tag{5.53}$$

Furthermore,

$$\Omega \setminus \Gamma_n = \cup_{m \in \mathbb{N}} B_{n,m}, \quad \Gamma_n = \cap_{m \in \mathbb{N}} C_{n,m} \tag{5.54}$$

so that the sequences $(D^{\alpha} u_{n,m}, D^{\alpha} v_{n,m})$ converge uniformly on compact subsets of $\Omega \setminus \Gamma_n$ to $(\mathfrak{D}^{\alpha} u_n, \mathfrak{D}^{\alpha} v_n)$ for each $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^2$.

Theorem 5.5. *The generalized solution* $(u^{\sharp}, v^{\sharp})$ *of the initial value problem* (5.2) *and* (5.3) *belongs to* $L^{\infty}_{\mathrm{loc}}(\Omega)^2$ *and satisfies the system of equations* (5.2) *weakly on an open and dense subset of* $\mathbb{R} \times (0, \infty)$.

Proof. Let $(u^{\sharp}, v^{\sharp}) \in \mathcal{NL}^{\infty}(\Omega)^2$ be the generalized solution of the initial value problem (5.2) and (5.3) constructed in Theorem 5.4. It follows by (5.40), (5.41) and the definition of the functions $\lambda_1^{\alpha,i}, \mu_1^{\alpha,i}$, for $i = 1, 2$ and $|\alpha| \leq M_1$, that $\mathfrak{D}^{\alpha\sharp} u^{\sharp}, \mathfrak{D}^{\alpha\sharp} v^{\sharp} \in L^{\infty}_{\mathrm{loc}}(\Omega)$ for $|\alpha| \leq M_1$.

Let $((u_n, v_n)) \subset \mathcal{ML}^{\infty}(\Omega)$ be the Cauchy sequence converging to $(u^{\sharp}, v^{\sharp})$, and for each $n \in \mathbb{N}$, let $((u_{n,m}, v_{n,m})) \subset \mathcal{C}^{\infty}(\Omega)^2$ be the sequence associated with (u_n, v_n) through (5.51) to (5.54). From (5.47) to (5.49) it follows that, for each $|\alpha| \leq 2$, $(\mathfrak{D}^{\alpha} u_n(x,t))$ and $(\mathfrak{D}^{\alpha} v_n(x,t))$ converge to $\mathfrak{D}^{\alpha\sharp} u^{\sharp}(x,t)$ and $\mathfrak{D}^{\alpha\sharp} v^{\sharp}(x,t)$, respectively, for each $(x,t) \in \Omega \setminus \cup_{n \in \mathbb{N}} \Gamma_n$. We may therefore select a strictly increasing sequence of integers (m_n) so that, for each $|\alpha| \leq 2$ and $(x,t) \in \Omega \setminus \cup_{n \in \mathbb{N}} \Gamma_n$, the sequences $(D^{\alpha} u_{n,m_n}(x,t))$ and $(D^{\alpha} v_{n,m_n}(x,t))$ converge to $\mathfrak{D}^{\alpha\sharp} u^{\sharp}(x,t)$

and $\mathfrak{D}^{\alpha\sharp}v^\sharp(x,t)$, respectively. Clearly, for $|\alpha| \leq 2$, the sequences $(D^\alpha u_{n,m_n})$ and $(D^\alpha v_{n,m_n})$ are pointwise bounded on $\Omega \setminus \cup_{n\in\mathbb{N}}\Gamma_n$, which is a residual set. It then follows by (2.18) that

$$\begin{aligned}
&\exists \lambda \in \mathcal{NL}(\Omega) : \\
&\forall |\alpha| \leq 2, \quad n \in \mathbb{N} : \\
&\quad -\lambda \leq D^\alpha u_{n,m_n} \leq \lambda.
\end{aligned} \tag{5.55}$$

Note that, since λ is lower semicontinuous and finite everywhere except on a closed nowhere dense set Γ, it is locally integrable on $\Omega \setminus \Gamma$. Consider any $\varphi \in \mathcal{C}^\infty(\Omega \setminus \Gamma)$ with compact support contained in $\Omega \setminus \Gamma$. Then, for $|\alpha| \leq 2$, $(\varphi D^\alpha u_{n,m_n})$ and $(\varphi D^\alpha v_{n,m_n})$ converge pointwise to $\varphi \mathfrak{D}^{\alpha\sharp}u^\sharp$ and $\varphi \mathfrak{D}^{\alpha\sharp}v^\sharp$, respectively, on $\Omega \setminus \cup_{n\in\mathbb{N}}\Gamma_n$, which is a set of full measure. Furthermore, it follows by (5.55) that

$$\begin{aligned}
&\forall |\alpha| \leq 2, \quad n \in \mathbb{N} : \\
&\quad -M_\varphi \lambda(x,t) \leq D^\alpha u_{n,m_n}(x,t) \leq M_\varphi \lambda(x,t), \quad (x,t) \in \Omega \setminus \Gamma,
\end{aligned} \tag{5.56}$$

where

$$M_\varphi = \sup\left\{ \left| D^\alpha \varphi(x,t) \right| : |\alpha| \leq 2, \ (x,t) \in \Omega \right\}. \tag{5.57}$$

It therefore follows by the Lebesgue Dominated Convergence Theorem that, for $|\alpha| \leq 2$,

$$\begin{aligned}
\int_\Omega \varphi D^\alpha u_{n,m_n}(x,t)\,dx\,dt &\longrightarrow \int_\Omega \varphi \mathfrak{D}^{\alpha\sharp}u^\sharp(x,t)\,dx\,dt, \\
\int_\Omega \varphi D^\alpha v_{n,m_n}(x,t)\,dx\,dt &\longrightarrow \int_\Omega \varphi \mathfrak{D}^{\alpha\sharp}v^\sharp(x,t)\,dx\,dt.
\end{aligned} \tag{5.58}$$

In the same way, we obtain

$$\begin{aligned}
\int_\Omega u_{n,m_n} D^\alpha \varphi(x,t)\,dx\,dt &\longrightarrow \int_\Omega u^\sharp D^\alpha \varphi(x,t)\,dx\,dt, \\
\int_\Omega v_{n,m_n} D^\alpha \varphi(x,t)\,dx\,dt &\longrightarrow \int_\Omega v^\sharp D^\alpha \varphi(x,t)\,dx\,dt
\end{aligned} \tag{5.59}$$

so that, for $|\alpha| \leq 2$, $\mathfrak{D}^{\alpha\sharp}u^\sharp$ and $\mathfrak{D}^{\alpha\sharp}v^\sharp$ are the weak derivatives of $u^\sharp$ and $v^\sharp$, respectively, on $\Omega \setminus \Gamma$.

In the same way, it follows that $(u^\sharp, v^\sharp)$ satisfies (5.2) weakly on $\Omega \setminus \Gamma$. That is, for any $\varphi, \psi \in \mathcal{C}^\infty(\Omega)$ with compact support contained in $\Omega \setminus \Gamma$, we have

$$\begin{gathered}
\int_\Omega v^\sharp \varphi_t dxdt + \int_\Omega u^\sharp \varphi_{xx} dxdt + \int_\Omega \left(2\left(u^\sharp\right)^3 + 2\left(v^\sharp\right)^2 u^\sharp\right)\varphi dxdt \\
= \int_\Omega \left(hu^\sharp \cos(2t) + hv^\sharp \sin(2t) + \gamma v^\sharp\right)\varphi dxdt, \\
-\int_\Omega u^\sharp \varphi_t dxdt + \int_\Omega v^\sharp \varphi_{xx} dxdt + \int_\Omega \left(2\left(v^\sharp\right)^3 + 2\left(u^\sharp\right)^2 v^\sharp\right)\varphi dxdt \\
= \int_\Omega \left(-hu^\sharp \cos(2t) + hv^\sharp \sin(2t) - \gamma v^\sharp\right)\varphi dxdt.
\end{gathered} \tag{5.60}$$

Note also that $(u^\sharp, v^\sharp)$ has a trace on $\mathbb{R} \times \{0\}$, which satisfies the initial condition in the classical sense. □

It should be noted that the singularity set Γ associated with the strongly generic weak solution $(u^\sharp, v^\sharp)$ of the initial value problem (5.2) and (5.3) constructed in Theorem 5.5 may have arbitrarily large positive Lebesgue measure. As such, the solution may exhibit rather wild singularities across this set, which may represent certain chaotic phenomena in the physical systems that the equations are supposed to model.

One may also observe that the techniques for the solution of the initial value problem (5.2) and (5.3) applies also to the Schrödinger equation with different types of nonlinear terms. Furthermore, one may relax the smoothness condition on the initial data to, say,

$$u_0, v_0 \in \mathcal{C}^\infty(\mathbb{R} \setminus \Gamma_0) \tag{5.61}$$

with Γ_0 a countable and closed nowhere dense set in $\mathbb{R}$. Of course, in this case the solution would not be a strongly generic weak solution in the sense of Definition 5.1. However, the following generalization of Definition 5.1 remains valid

$$\begin{aligned}
&\exists \Gamma \subset \Omega \text{ closed nowhere dense :} \\
&(1)\ \left(u^\sharp, v^\sharp\right) \in L^\infty_{\text{loc}}(\Omega \setminus \Gamma), \\
&(2)\ \left(u^\sharp, v^\sharp\right) \text{ satisfies (5.2) weakly on } \Omega \setminus \Gamma.
\end{aligned} \tag{5.62}$$

6. Conclusion

We have constructed a space of generalized functions, the elements of which admit generalized partial derivatives of all orders. Furthermore, it has been shown that this space contains generalized solutions of a large class of nonlinear PDEs. Moreover, these generalized solutions satisfy a blanket regularity property. In particular, each such generalized function may be represented in the space $\mathcal{NL}(\Omega)^{\mathbb{N}^n}$, through its generalized derivatives, as nearly finite normal lower semicontinuous functions.

As a demonstration of how the general theory may be applied to particular systems of equations with associated initial and/or boundary values, we constructed generalized solution of a parametrically driven damped nonlinear Schrödinger equation with an initial condition. It is also shown that this solution satisfies the equations weakly on an open and dense subset of the domain of definition of the system.

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