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*Research Article*

# $\alpha$-Well-Posedness for Mixed Quasi Variational-Like Inequality Problems

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The concepts of $\alpha$-well-posedness, $\alpha$-well-posedness in the generalized sense, L-$\alpha$-well-posedness and L-$\alpha$-well-posedness in the generalized sense for mixed quasi variational-like inequality problems are investigated. We present some metric characterizations for these well-posednesses.

## 1. Introduction

Well-posedness plays a crucial role in the stability theory for optimization problems, which guarantees that, for an approximating solution sequence, there exists a subsequence which converges to a solution. The study of well-posedness for scalar minimization problems started from Tykhonov [1] and Levitin and Polyak [2]. Since then, various notions of well-posedness for scalar minimization problems have been defined and studied in [3–8] and the references therein. It is worth noting that the recent study for various types of well-posedness has been generalized to variational inequality problems [9–13], generalized variational inequality problems [14, 15], quasi variational inequality problems [16], generalized quasi variational inequality problems [17], generalized vector variational inequality problems [18], vector quasi variational inequality problems [19], mixed quasi variational-like inequality problems [20], and many other problems.

In this paper, we are interested in investigating four classes of well-posednesses for a mixed quasi variational-like inequality problem. The paper is organized as follows. In Section 2, we introduce the definitions of $\alpha$-well-posedness, $\alpha$-well-posedness in the generalized sense, L-$\alpha$-well-posedness and L-$\alpha$-well-posedness in the generalized sense for a mixed quasi variational-like inequality problem. In Section 3, some characterizations of $\alpha$-well-posedness, and L-$\alpha$-well-posedness for a mixed quasi variational-like inequality problem are obtained. In Section 4, some characterizations of $\alpha$-well-posedness in the generalized sense and L-$\alpha$-well-posedness in the generalized sense for a mixed quasi variational-like inequality problem are presented.

## 2. Preliminaries

Throughout this paper, without other specification, let $E$ be a real Banach space with the dual $E^*$, let $K$ be a nonempty closed convex subset of $E$, and let $S : K \to 2^K$ be a set-valued map. Let $F : K \to 2^{E^*}$ be a set-valued map with nonempty values, let $\eta : K \times K \to E$ be a single-valued map, and let $f : K \to R$ be a real-valued function. Ceng et al. [20] introduced the following mixed quasi variational-like inequality problem, which is to find a point $x_0 \in K$ such that, for some $u_0 \in F(x_0)$,

$$\text{(MQVLI)} \quad x_0 \in S(x_0), \quad \langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) \le 0, \quad \forall y \in S(x_0). \tag{2.1}$$

Denote by $\Gamma$ the solution set of (MQVLI). Let $\alpha > 0$; we introduce the notions of several classes of $\alpha$-well-posednesses for (MQVLI).

*Definition 2.1.* A sequence $(x_n)_n$ in $K$ is an $\alpha$-approximating sequence for (MQVLI) if

(i) there exists a sequence $(u_n)_n$ in $E^*$, with $u_n \in F(x_n)$, for all $n \in N$;

(ii) there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that

$$\begin{gathered} d(x_n, S(x_n)) \le \varepsilon_n, \quad \forall n \in N, \\ \langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2 \le \varepsilon_n, \quad \forall y \in S(x_n), \forall n \in N. \end{gathered} \tag{2.2}$$

*Definition 2.2.* (MQVLI) is said to be $\alpha$-well-posed (resp., $\alpha$-well-posed in the generalized sense) if it has a unique solution $x_0$ and every $\alpha$-approximating sequence $(x_n)_n$ strongly converges to $x_0$ (resp., if the solution set $\Gamma$ of (MQVLI) is nonempty and for every $\alpha$-approximating sequence $(x_n)_n$ has a subsequence which strongly converges to a point of $\Gamma$).

*Definition 2.3.* A sequence $(x_n)_n$ is an L-$\alpha$-approximating sequence for (MQVLI) if there exists a real number sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that

$$\begin{gathered} d(x_n, S(x_n)) \le \varepsilon_n, \quad \forall n \in N, \\ \langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2 \le \varepsilon_n, \quad \forall y \in S(x_n), v \in F(y), n \in N. \end{gathered} \tag{2.3}$$

*Definition 2.4.* (MQVLI) is said to be L-$\alpha$-well-posed (resp., L-$\alpha$-well-posed in the generalized sense) if it has a unique solution $x_0$ and every L-$\alpha$-approximating sequence $(x_n)_n$ strongly converges to $x_0$ (resp., if the solution set $\Gamma$ of (MQVLI) is nonempty and for every L-$\alpha$-approximating sequence $(x_n)_n$ has a subsequence which strongly converges to a point of $\Gamma$).

It is worth noting that if $\alpha = 0$, then the definitions of $\alpha$-well-posedness, $\alpha$-well-posedness in the generalized sense, L-$\alpha$-well-posedness, and L-$\alpha$-well-posedness in the generalized sense for (MQVLI), respectively, reduce to those of the well-posedness, well-posedness in the generalized sense, L-well-posedness, and L-well-posedness in the generalized sense for (MQVLI) in [20]. We also note that Definition 2.2 generalizes and extends $\alpha$-well-posedness and $\alpha$-well-posedness in the generalized sense of variational inequalities in [10] which are related to the continuously differentiable gap function of variational inequalities introduced by Fukushima [21].

In order to investigate the $\alpha$-well-posedness for (MQVLI), we need the following definitions.

We recall the notion of Mosco convergence [22]. A sequence $(H_n)_n$ of subsets of $E$ Mosco converges to a set $H$ if

$$H = \liminf_n H_n = w - \limsup_n H_n, \tag{2.4}$$

where $\liminf_n H_n$ and $w - \limsup_n H_n$ are, respectively, the Painlevé-Kuratowski strong limit inferior and weak limit superior of a sequence $(H_n)_n$, that is,

$$\begin{gathered}\liminf_n H_n = \{y \in E : \exists y_n \in H_n,\ n \in N,\ \text{with } y_n \to y\}, \\ w - \limsup_n H_n = \{y \in E : \exists n_k \uparrow +\infty,\ n_k \in N,\ \exists y_{n_k} \in H_{n_k},\ k \in N,\ \text{with } y_{n_k} \rightharpoonup y\},\end{gathered} \tag{2.5}$$

where "$\rightharpoonup$" means weak convergence, and "$\to$" means strong convergence.

If $H = \liminf_n H_n$, we call the sequence $(H_n)_n$ of subsets of $E$ Lower Semi-Mosco convergent to a set $H$.

It is easy to see that a sequence $(H_n)_n$ of subsets of $E$ Mosco converges to a set $H$ implies that the sequence $(H_n)_n$ also Lower Semi-Mosco converges to the set $H$, but the converse is not true in general.

We will use the usual abbreviations usc and lsc for "upper semicontinuous" and "lower semicontinuous", respectively. For any $x, y \in E$, $[x, y]$ will denote the line segment $\{tx + (1-t)y : t \in [0,1]\}$, while $[x, y)$ and $(x, y)$ are defined analogously. We will frequently use $s, w$, and $w^*$ to denote, respectively, the norm topology on $E$, the weak topology on $E$, and the weak* topology on $E$. Given a convex set $K$, a multivalued map $F : K \to 2^{E^*}$ will be called upper hemicontinuous, if its restriction on any line segment $[x, y] \subseteq K$ is usc with respect to the $w^*$ topology on $E^*$. $F : K \to 2^{E^*}$ will be called $\eta$-monotone if, for any $x, y \in K$, for all $u \in F(x), v \in F(y)$, $\langle u - v, \eta(x, y)\rangle \geq 0$. We refer the reader to [23, 24] for basic facts about multivalued maps.

**Lemma 2.5** (see [25]). *Let $(H_n)_n$ be a sequence of nonempty subsets of a Banach space $E$ such that*

(i) *$H_n$ is convex for every $n \in N$;*

(ii) *$H_0 \subseteq \liminf_n H_n$;*

(iii) *there exists $m \in N$ such that $\operatorname{int} \bigcap_{n \geq m} H_n \neq \emptyset$.*

*Then, for every $u_0 \in \operatorname{int} H_0$, there exists a positive real number $\delta$ such that*

$$\operatorname{int} B(u_0, \delta) \subseteq H_n, \quad \forall n \geq m, \tag{2.6}$$

*where $B(u_0, \delta)$ is a closed ball with a center $u_0$ and radius $\delta$. If $E$ is a finite dimensional space, then assumption (iii) can be replaced by*

(iii)′ int $H_0 \neq \emptyset$.

The following lemmas play important role in this paper.

**Lemma 2.6.** *Let $E$ be a real separable Banach space with the dual $E^*$, let $S_0$ be a nonempty convex subset of $E$, and let $F : S_0 \to 2^{E^*}$ be a set-valued map with nonempty, weakly* compact convex valued, $\eta$-monotone, and upper hemicontinuous. Let $\eta : S_0 \times S_0 \to E$ be a single-valued map with $\eta(x, x) = 0$, for all $x \in S_0$, and let $f : S_0 \to R$ be a convex lsc function. Assume that the map $y \mapsto \langle u, \eta(x, y)\rangle$ is concave for each $(u, x) \in F(S_0) \times S_0$ and usc. If $S_1$ is a convex subset of $S_0$ with the property that, for each $x \in S_0$ and each $y \in S_1$, $(x, y] \subseteq S_1$, then for each $x_0 \in S_0$, the following conditions are equivalent:*

(i) *There exists $u_0 \in F(x_0)$, such that for all $y \in S_0$, $\langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - (\alpha/2)\|x_0 - y\|^2 \leq 0$,*

(ii) *for all $y \in S_1$, there exists $v \in F(y)$, such that $\langle v, \eta(x_0, y)\rangle + f(x_0) - f(y) - (\alpha/2)\|x_0 - y\|^2 \leq 0$.*

*Proof.* According to the $\eta$-monotonicity of $F$, (i) $\Rightarrow$ (ii) is obvious.

Next prove (ii) $\Rightarrow$ (i). Suppose that (ii) holds. Given any $y \in S_1$, let $y_n = (1/n)y + (1 - (1/n))x_0$, for $n \in N$. By the assumptions of $S_1$, $y_n \in S_1$ for each $n \in N$. It follows from the condition (ii) that for each $n \in N$, there exists $v_n \in F(y_n)$ such that

$$\langle v_n, \eta(x_0, y_n)\rangle + f(x_0) - f(y_n) - \frac{\alpha}{2}\|x_0 - y_n\|^2 \leq 0. \tag{2.7}$$

Then

$$\begin{aligned}
0 &\geq \langle v_n, \eta(x_0, y_n)\rangle + f(x_0) - f(y_n) - \frac{\alpha}{2}\|x_0 - y_n\|^2 \\
&= \left\langle v_n, \eta\left(x_0, \frac{1}{n}y + \left(1 - \frac{1}{n}\right)x_0\right)\right\rangle + f(x_0) - f\left(\frac{1}{n}y + \left(1 - \frac{1}{n}\right)x_0\right) - \frac{\alpha}{2}\left\|x_0 - \frac{1}{n}y - \left(1 - \frac{1}{n}\right)x_0\right\|^2 \\
&\geq \frac{1}{n}\langle v_n, \eta(x_0, y)\rangle + \left(1 - \frac{1}{n}\right)\langle v_n, \eta(x_0, x_0)\rangle + f(x_0) - \frac{1}{n}f(y) - \left(1 - \frac{1}{n}\right)f(x_0) - \frac{\alpha}{2}\frac{1}{n^2}\|x_0 - y\|^2 \\
&= \frac{1}{n}\langle v_n, \eta(x_0, y)\rangle + \frac{1}{n}f(x_0) - \frac{1}{n}f(y) - \frac{\alpha}{2n^2}\|x_0 - y\|^2,
\end{aligned} \tag{2.8}$$

which implies that

$$\langle v_n, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2n}\|x_0 - y\|^2 \leq 0, \quad \forall n \in N. \tag{2.9}$$

It follows that for each $n \in N$,

$$\exists v_n \in F(y_n), \quad \langle v_n, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \leq 0. \tag{2.10}$$

Since $F$ is a weak* compact valued and $(s,w^*)$-usc on the line segment $[x_0,y]$, $F$ is $(s,w^*)$-closed, and $(s,w^*)$-subcontinuous on $[x_0,y]$, it follows from $\lim_n y_n = x_0$ and $v_n \in F(y_n)$ that $\{v_n\}$ has a subsequence weak* converging to some $v \in F(x_0)$. By taking the limit of subsequence in (2.10) we get

$$\forall y \in S_1,\ \exists v \in F(x_0), \quad \langle v, \eta(x_0,y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \le 0. \tag{2.11}$$

Define the bifunction $\phi(v,y)$ on $F(x_0) \times S_0$ by

$$\phi(v,y) = \langle v, \eta(x_0,y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2. \tag{2.12}$$

For each $y \in S_0$, $\phi(\cdot,y)$ is weakly* lsc and quasiconvex on the weakly* compact convex set $F(x_0)$ while for each $v \in F(x_0), \phi(v,\cdot)$ is usc and quasiconcave on the convex set $S_1$. Hence, according to the Sion Minimax Theorem [26],

$$\sup_{y\in S_1}\min_{v\in F(x_0)} \phi(v,y) = \min_{v\in F(x_0)}\sup_{y\in S_1} \phi(v,y). \tag{2.13}$$

By (2.11), we have $\sup_{y\in S_1}\min_{v\in F(x_0)}\phi(v,y) \le 0$; hence, $\min_{v\in F(x_0)}\sup_{y\in S_1}\phi(v,y) \le 0$, which implies that there exists $v_0 \in F(x_0)$, such that

$$\langle v_0, \eta(x_0,y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \le 0, \quad \forall y \in S_1. \tag{2.14}$$

Finally, for each $y \in S_0$, choose $z \in S_1$, and a sequence $(y_n)_n$ in $(y,z] \subseteq S_1$ converging to $y$. The function $\phi(v,\cdot)$ is usc and concave on $S_0$; hence its restriction on any line segment is continuous [27, Theorem 2.35]. Accordingly, (2.14) implies there exists $v_0 \in F(x_0)$, for all $y \in S_0$,

$$\begin{aligned} &\langle v_0, \eta(x_0,y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \\ &\quad = \lim_n \left[\langle v_0, \eta(x_0,y_n)\rangle + f(x_0) - f(y_n) - \frac{\alpha}{2}\|x_0 - y_n\|^2\right] \le 0. \end{aligned} \tag{2.15}$$

Hence, (i) holds. □

**Lemma 2.7.** *Let $E$ be a real Banach space with the dual $E^*$, let $K$ be a nonempty convex subset of $E$, and let $S$ be a convex-valued set-valued map from $K$ to $2^K$. Let $F : K \to 2^{E^*}$ be a set-valued map with nonempty values, let $\eta : K \times K \to E$ be a single-valued map with $\eta(x,x) = 0$, for all $x \in K$, and let $f : K \to R$ be a convex function. Assume that the function $y \mapsto \langle u, \eta(x,y)\rangle$ is concave, for each $(u,x) \in F(K) \times K$. Then $x_0 \in \Gamma$ if and only if the following condition holds:*

$$\exists u_0 \in F(x_0),\ x_0 \in S(x_0), \quad \langle u_0, \eta(x_0,y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \le 0, \quad \forall y \in S(x_0). \tag{2.16}$$

*Proof.* The necessity is easy to get; next we start to prove the sufficiency. Let for all $y \in S(x_0)$, for all $t \in (0,1)$, $y_t = ty + (1-t)x_0$. Since $u_0 \in F(x_0)$, $x_0 \in S(x_0)$, and $S$ is convex-valued, $y_t \in S(x_0)$, it follows that

$$\langle u_0, \eta(x_0, y_t)\rangle + f(x_0) - f(y_t) - \frac{\alpha}{2}\|x_0 - y_t\|^2 \le 0, \quad \forall t \in (0,1). \tag{2.17}$$

Thus,

$$\begin{aligned} 0 &\ge \langle u_0, \eta(x_0, y_t)\rangle + f(x_0) - f(y_t) - \frac{\alpha}{2}\|x_0 - y_t\|^2 \\ &\ge t\langle u_0, \eta(x_0, y)\rangle + (1-t)\langle u_0, \eta(x_0, x_0)\rangle + f(x_0) - tf(y) - (1-t)f(x_0) - \frac{\alpha}{2}t^2\|x_0 - y\|^2 \\ &= t\left[\langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}t\|x_0 - y\|^2\right], \end{aligned} \tag{2.18}$$

which implies that

$$u_0 \in F(x_0), \ x_0 \in S(x_0), \quad \langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}t\|x_0 - y\|^2 \le 0 \quad \forall y \in S(x_0), \ \forall t \in (0,1). \tag{2.19}$$

The above inequality implies, for $t$ converging to zero, that $x_0$ is a solution of (MQVLI). This completes the proof. □

## 3. The Characterizations of Well-Posedness for (MQVLI)

In this section, we investigate some metric characterizations of $\alpha$-well-posedness and L -$\alpha$-well-posedness for (MQVLI).

For any $\varepsilon > 0$, we consider the sets

$$\begin{aligned} &Q_\varepsilon \\ &= \left\{x \in K : d(x, S(x)) \le \varepsilon, \ \exists u \in F(x) : \langle u, \eta(x, y)\rangle + f(x) - f(y) - \frac{\alpha}{2}\|x - y\|^2 \le \varepsilon, \ \forall y \in S(x)\right\}, \\ &L_\varepsilon \\ &= \left\{x \in K : d(x, S(x)) \le \varepsilon, \ \langle v, \eta(x, y)\rangle + f(x) - f(y) - \frac{\alpha}{2}\|x - y\|^2 \le \varepsilon, \ \forall y \in S(x), \ \forall v \in F(y)\right\}. \end{aligned} \tag{3.1}$$

**Theorem 3.1.** *Let the same assumptions be as in Lemma 2.7. Then, one has the following.*

(a) *(MQVLI) is $\alpha$-well-posed if and only if the solution set $\Gamma$ of (MQVLI) is nonempty and* $\lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon = 0$.

(b) *Moreover, if $F$ is $\eta$-monotone, then (MQVLI) is L-$\alpha$-well-posed if and only if the solution set $\Gamma$ of (MQVLI) is nonempty and* $\lim_{\varepsilon \to 0} \operatorname{diam} L_\varepsilon = 0$.

*Proof.* We only prove (a). The proof of (b) is similar and is omitted here. Suppose that (MQVLI) is $\alpha$-well-posed; then $\Gamma \neq \emptyset$. It follows from Lemma 2.7 that $Q_\varepsilon \neq \emptyset$. Suppose by contradiction that exists a real number $\beta$, such that $\lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon > \beta > 0$; then there exists $\varepsilon_n > 0$, with $\varepsilon_n \searrow 0$, and $(w_n)_n$, $(z_n)_n \in Q_{\varepsilon_n}$, such that $\|w_n - z_n\| > \beta$, for all $n \in N$. Since the sequences $(w_n)_n$, and $(z_n)_n$ are both $\alpha$-approximating sequences for (MQVLI), $(w_n)_n$ and $(z_n)_n$ strongly converge to the unique solution $u_0$, and this gives a contradiction. Therefore, $\lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon = 0$.

Conversely, let $(x_n)_n \subset K$ be an $\alpha$-approximating sequence for (MQVLI). Then there exists a sequence $(u_n)_n$ in $E^*$ with $u_n \in F(x_n)$ and a sequence $(\varepsilon_n)_n$ in $R^+$ with $\varepsilon_n \to 0$, such that

$$d(x_n, S(x_n)) \leq \varepsilon_n, \quad \langle u_n, \eta(x_n, y) \rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2 \leq \varepsilon_n, \quad \forall y \in S(x_n),\ \forall n \in N. \tag{3.2}$$

That is, $x_n \in Q_{\varepsilon_n}$, for all $n \in N$. It is easy to see $\lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon = 0$ and $\Gamma \neq \emptyset$ imply that $\Gamma$ is a singleton point set. Indeed, if there exist two different solutions $z_1$, $z_2$, then from Lemma 2.7, we know that $z_1, z_2 \in Q_\varepsilon$, for all $\varepsilon > 0$. Thus, $\lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon \geq \|z_1 - z_2\| \neq 0$, a contradiction. Let $x_0$ be the unique solution of (MQVLI). It follows from Lemma 2.7 that $x_0 \in Q_{\varepsilon_n}$. Thus, $\lim_{n \to \infty}\|x_n - x_0\| \leq \lim_{n \to \infty} \operatorname{diam} Q_{\varepsilon_n} = 0$. So $(x_n)_n$ strongly converges to $x_0$. Therefore, (MQVLI) is $\alpha$-well-posed. □

**Theorem 3.2.** *Let $E$ be a real separable Banach space with the dual $E^*$, let $K$ be a nonempty closed convex subset of $E$, and let $\eta : K \times K \to E$ be a single-valued map with $\eta(x, x) = 0$, for all $x \in K$, which is $(s, w)$-continuous in each of its variables separately. And let $f : K \to R$ be a convex lsc function; let $S : K \to 2^K$ and $F : K \to 2^{E^*}$ be two set-valued maps. Assume the following conditions hold:*

(i) *$S$ is nonempty convex-valued and, for each sequence $(x_n)_n$ in $K$ converging to $x_0$, the sequence $(S(x_n))_n$ Lower Semi-Mosco converging to $S(x_0)$;*

(ii) *for every converging sequence $(w_n)_n$, there exists $m \in N$, such that $\operatorname{int} \bigcap_{n \geq m} S(w_n) \neq \emptyset$;*

(iii) *$F : K \to 2^{E^*}$ is nonempty, weak* compact convex valued, $\eta$-monotone, and upper hemicontinuous;*

(iv) *the map $y \mapsto \langle u, \eta(x, y) \rangle$ is concave for each $(u, x) \in F(K) \times K$.*

*Then, (MQVLI) is $\alpha$-well-posed if and only if*

$$Q_\varepsilon \neq \emptyset,\ \forall \varepsilon \geq 0, \quad \lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon = 0. \tag{3.3}$$

The proof of the above theorem relies on the following lemma.

**Lemma 3.3.** *Let the same assumptions be made as in Theorem 3.2. Let $(x_n)_n$ in $K$ be an $\alpha$-approximating sequence. If $(x_n)_n$ converges to some $x_0 \in K$, then $x_0$ is a solution of (MQVLI).*

*Proof.* Since $(x_n)_n$ is an $\alpha$-approximating sequence for (MQVLI), there exists a sequence $(u_n)_n$ in $E^*$ with $u_n \in F(x_n)$ and a sequence $(\varepsilon_n)_n$ in $R^+$ with $\varepsilon_n \to 0$, such that

$$d(x_n, S(x_n)) \le \varepsilon_n, \quad \langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2 \le \varepsilon_n, \quad \forall y \in S(x_n),\ \forall n \in N. \tag{3.4}$$

For each $n \in N$, choose $x'_n \in S(x_n)$, such that $\|x_n - x'_n\| < d(x_n, S(x_n)) + \varepsilon_n \le 2\varepsilon_n$. It follows from $x_n \to x_0$ and $\varepsilon_n \to 0$ that $x'_n \to x_0$. It follows from the assumption (i) that $\liminf_n S(x_n) = S(x_0)$. Thus, $x_0 \in S(x_0)$.

Assumption (ii) applied to the constant sequence $w_n = x_0$, for all $n \in N$, implies that $\operatorname{int} S(x_0) \neq \emptyset$. For every $y \in \operatorname{int} S(x_0)$, it follows from assumptions (i) and (ii) and Lemma 2.5 that there exist $m \in N$ and $\delta > 0$ such that $\operatorname{int} B(y, \delta) \subseteq S(x_n)$, for all $n > m$. Therefore, for $n$ sufficiently large, we have $y \in S(x_n)$. Notice that $\eta(\cdot, y)$ is $(s, w)$-continuous, $f$ is lsc, $F$ is $\eta$-monotone, and $(x_n)_n$ is an approximating sequence; we have, for every $v \in F(y)$

$$\begin{aligned}
\langle v, \eta(x_0, y)\rangle + f(x_0) - f(y) &\le \liminf_n \{\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y)\} \\
&\le \liminf_n \{\langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y)\} \\
&\le \liminf_n \left[\varepsilon_n + \frac{\alpha}{2}\|x_n - y\|^2\right] \\
&= \frac{\alpha}{2}\|x_0 - y\|^2.
\end{aligned} \tag{3.5}$$

Thus, for every $y \in \operatorname{int} S(x_0)$ and every $v \in F(y)$, we get $\langle v, \eta(x_0, y)\rangle + f(x_0) - f(y) - (\alpha/2)\|x_0 - y\|^2 \le 0$. Let $S_0 = S(x_0)$ and $S_1 = \operatorname{int} S(x_0)$; it follows from Lemma 2.6 that there exists $u_0 \in F(x_0)$ such that for all $y \in S(x_0), \langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - (\alpha/2)\|x_0 - y\|^2 \le 0$. According to Lemma 2.7, $x_0$ is a solution of (MQVLI). □

*Proof of Theorem 3.2.* The necessity follows from Theorem 3.1 and Lemma 2.7. Now we prove the sufficiency. Suppose that (3.3) holds. Let us show that there exists at most one solution of (MQVLI). Indeed, if there exist two different solutions $z_1, z_2$, then from Lemma 2.7, we know that $z_1, z_2 \in Q_\varepsilon$, for all $\varepsilon > 0$. Thus, $\lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon \ge \|z_1 - z_2\| \neq 0$, a contradiction. Note also that there exist $\alpha$-approximate sequences for (MQVLI); indeed, for any sequence $(\varepsilon_n)_n$ in $R^+$ with $\varepsilon_n \to 0$, and any choice of $x_n \in Q_{\varepsilon_n}$ (which is nonempty by assumption), $(x_n)_n$ is an $\alpha$-approximate sequence.

Let $(x_n)_n$ be an $\alpha$-approximating sequence for (MQVLI); then $x_n \in Q_{\varepsilon_n}$, for all $n \in N$. In light of (3.3), $(x_n)_n$ is a Cauchy sequence and strongly converging to a point $x_0 \in K$. Applying Lemma 3.3, we get that $x_0$ is a solution of (MQVLI) and so (MQVLI) is $\alpha$-well-posed. □

Now, we present a result in which assumption (ii) and the monotonicity of $F$ are dropped, while the continuity requirements are strengthened.

**Theorem 3.4.** *Let $E$ be a real separable Banach space with the dual $E^*$, let $K$ be a nonempty closed convex subset of $E$, and let $\eta : K \times K \to E$ be a single-valued map with $\eta(x, x) = 0$, for all $x \in K$, which is $(s, s)$-continuous. And let $f : K \to R$ be a convex and continuous function, let $S : K \to 2^K$ and $F : K \to 2^{E^*}$ be two set-valued maps. Assume the following assumptions hold:*

(i) *the multifunction $S$ is nonempty convex-valued and for each sequence $(x_n)_n$ in $K$ converging to $x_0$, the sequence $(S(x_n))_n$ Lower Semi-Mosco converging to $S(x_0)$;*

(ii) $F : K \to 2^{E^*}$ *is nonempty, weak* compact, and convex valued, $(s, w^*)$-usc;*

(iii) *the map $y \mapsto \langle u, \eta(x, y)\rangle$ is concave for each $(u, x) \in F(K) \times K$.*

*Then, (MQVLI) is $\alpha$-well-posed if and only if* (3.3) *holds.*

The proof of the above theorem relies on the following lemma.

**Lemma 3.5.** *Let the assumptions be as in Theorem 3.4. Let $(x_n)_n$ in $K$ be an $\alpha$-approximating sequence. If $(x_n)_n$ converges to some $x_0 \in K$, then $x_0$ is a solution of (MQVLI).*

*Proof.* Since $(x_n)_n$ is an $\alpha$-approximating sequence for (MQVLI), there exist a sequence $(u_n)_n$ in $E^*$ with $u_n \in F(x_n)$ and a sequence $(\varepsilon_n)_n$ in $R^+$, $\varepsilon_n \to 0$, such that

$$d(x_n, S(x_n)) \le \varepsilon_n, \quad \langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2 \le \varepsilon_n, \quad \forall y \in S(x_n),\ \forall n \in N. \tag{3.6}$$

As in Lemma 3.3, we infer $x_0 \in S(x_0)$. Since $S(x_n)$ Lower Semi-Mosco converges to $S(x_0)$, for every $y \in S(x_0)$, there exists a sequence $y_n \in S(x_n)$, for all $n \in N$, such that $\lim_n y_n = y$ in the strongly topology. Since $\eta$ is $(s, s)$-continuous, the sequence $(\eta(x_n, y_n))_n$ converges strongly to $\eta(x_0, y)$. It follows from (ii) and Proposition 2.19 in [24] that there exists a subsequence $(u_{n_j})_j$ of $(u_n)_n$ weak* converging to some $u_0 \in E^*$. It follows from (ii) and Proposition 2.17 in [24] that $F$ is $(s, w^*)$-closed, and so $u_0 \in F(x_0)$. Thus, we have

$$\begin{aligned}
&\left|\left\langle u_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle - \langle u_0, \eta(x_0, y)\rangle\right| \\
&\quad \le \left|\left\langle u_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right) - \eta(x_0, y)\right\rangle\right| + \left|\left\langle u_0 - u_{n_j}, \eta(x_0, y)\right\rangle\right| \\
&\quad \le \left\|u_{n_j}\right\|\left\|\eta\left(x_{n_j}, y_{n_j}\right) - \eta(x_0, y)\right\| - \left|\left\langle u_0 - u_{n_j}, \eta(x_0, y)\right\rangle\right| \\
&\quad \longrightarrow 0.
\end{aligned} \tag{3.7}$$

Hence, $\langle u_{n_j}, \eta(x_{n_j}, y_{n_j})\rangle \to \langle u_0, \eta(x_0, y)\rangle$ and so

$$\begin{aligned}
&\langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \\
&\quad = \lim_j \left[\left\langle u_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle + f\left(x_{n_j}\right) - f\left(y_{n_j}\right) - \frac{\alpha}{2}\left\|x_{n_j} - y_{n_j}\right\|^2\right] \\
&\quad \le \lim_j \varepsilon_{n_j} = 0.
\end{aligned} \tag{3.8}$$

Applying Lemma 2.7, $x_0$ is a solution of (MQVLI). □

*Proof of Theorem 3.4.* The necessity follows from Theorem 3.1 and Lemma 2.7. Now we prove the sufficiency. Suppose that (3.3) holds. It follows from the proof of Theorem 3.2 that there exists at most one solution of (MQVLI) and there exist $\alpha$-approximate sequences for (MQVLI). Let $(x_n)_n$ be an $\alpha$-approximating sequence for (MQVLI); then $x_n \in Q_{\varepsilon_n}$, for all $n \in N$. In light of (3.3), $(x_n)_n$ is a Cauchy sequence and strongly converging to a point $x_0 \in K$. Applying Lemma 3.5, we get that $x_0$ is a solution of (MQVLI) and so (MQVLI) is $\alpha$-well-posed. □

We have analogous results for L-$\alpha$-well-posedness.

**Theorem 3.6.** *Let $E$ be a real separable Banach space with the dual $E^*$, let $K$ be a nonempty closed convex subset of $E$, and let $\eta : K \times K \to E$ be a single-valued map with $\eta(x,x) = 0$, for all $x \in K$, which is $(s,w)$-continuous in each of its variables separately. And let $f : K \to R$ be a convex lsc function; let $S : K \to 2^K$ and $F : K \to 2^{E^*}$ be two set-valued maps. Assume the following assumptions hold:*

(i) *the multifunction $S$ is nonempty convex-valued and for each sequence $(x_n)_n$ in $K$ converging to $x_0$, the sequence $(S(x_n))_n$ Lower Semi-Mosco converging to $S(x_0)$;*

(ii) *for every converging sequence $(w_n)_n$, there exists $m \in N$, such that $\operatorname{int} \bigcap_{n\geq m} S(w_n) \neq \emptyset$;*

(iii) *$F : K \to 2^{E^*}$ is a set-valued map with nonempty, weak* compact convex valued, $\eta$-monotone and upper hemicontinuous;*

(iv) *the map $y \mapsto \langle u, \eta(x,y)\rangle$ is concave for each $(u,x) \in F(K) \times K$.*

*Then (MQVLI) is L-$\alpha$-well-posed if and only if*

$$L_\varepsilon \neq \emptyset,\ \forall \varepsilon \geq 0, \quad \lim_{\varepsilon\to 0} \operatorname{diam} L_\varepsilon = 0. \tag{3.9}$$

**Lemma 3.7.** *Let the same assumptions be as in Theorem 3.6. Let $(x_n)_n$ in $K$ be an L-$\alpha$-approximating sequence. If $(x_n)_n$ converges to some $x_0 \in K$, then $x_0$ is a solution of (MQVLI).*

*Proof.* Since $(x_n)_n$ is an L-$\alpha$-approximating sequence for (MQVLI), there exists a sequence $(\varepsilon_n)_n$ in $R^+$, $\varepsilon_n \to 0$, such that $d(x_n, S(x_n)) \leq \varepsilon_n$, and

$$\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2 \leq \varepsilon_n, \quad \forall y \in S(x_n),\ v \in F(y), n \in N. \tag{3.10}$$

From the proof of Lemma 3.3, (i) and (ii), we can obtain $x_0 \in S(x_0)$, $\operatorname{int} S(x_0) \neq \emptyset$, and for each $y \in \operatorname{int} S(x_0)$, one has $y \in S(x_n)$ for $n$ sufficiently large. It follows from (iii) that for every $y \in \operatorname{int} S(x_0)$ and every $v \in F(y)$, we have

$$\begin{aligned} &\langle v, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \\ &\quad\leq \liminf_n \left[\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2\right] \\ &\quad\leq \liminf_n \varepsilon_n = 0. \end{aligned} \tag{3.11}$$

Let $S_0 = S(x_0)$ and $S_1 = \operatorname{int} S(x_0)$; it follows from Lemma 2.6 that there exists $u_0 \in F(x_0)$ such that for all $y \in S(x_0)$, $\langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - (\alpha/2)\|x_0 - y\|^2 \leq 0$. According to Lemma 2.7, $x_0$ is a solution of (MQVLI). □

*Proof of Theorem 3.6.* Assume that (3.9) holds. Let $(x_n)_n$ in $K$ be an L-$\alpha$-approximating sequence for (MQVLI); then there exists a sequence $(\varepsilon_n)_n$ in $R^+$, such that $x_n \in L_{\varepsilon_n}$. It is easy to see that $\lim_{\varepsilon\to 0} \operatorname{diam} L_\varepsilon = 0$ and $\Gamma \neq \emptyset$ imply that $\Gamma$ is a singleton point set. Indeed, if there exist two different solutions $z_1, z_2$, then from Lemma 2.7 and the $\eta$-monotonicity of $F$, we know that $z_1, z_2 \in L_\varepsilon$, for all $\varepsilon > 0$. Thus, $\lim_{\varepsilon\to 0} \operatorname{diam} L_\varepsilon \geq \|z_1 - z_2\| \neq 0$, a contradiction. Let $x_0$ be the unique solution of (MQVLI). It follows from Lemma 2.7 and the $\eta$-monotonicity of $F$ that $x_0 \in L_{\varepsilon_n}$. Thus, $\lim_{n\to\infty}\|x_n - x_0\| \leq \lim_{n\to\infty} \operatorname{diam} L_{\varepsilon_n} = 0$. So $(x_n)_n$ strongly converge to $x_0$. It follows from Lemma 3.7 that $x_0 \in \Gamma$. Therefore, (MQVLI) is L-$\alpha$-well-posed.

Conversely, assume that the problem is L-$\alpha$-well-posed, It follows from the $\eta$-monotonicity of $F$ that $\emptyset \neq \Gamma \subset L_\varepsilon, \forall \varepsilon > 0$. Suppose by contradiction that a real number $\beta$ exists, such that $\lim_{\varepsilon\to 0} \operatorname{diam} L_\varepsilon > \beta > 0$; then there exists $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, and $(w_n)_n, (z_n)_n \in L_{\varepsilon_n}$, such that $\|w_n - z_n\| > \beta, \forall n \in N$. Since the sequences $(w_n)_n$ and $(z_n)_n$ are both L-$\alpha$-approximating sequences for (MQVLI), $(w_n)_n$ and $(z_n)_n$ strongly converge to the unique solution $u_0$, and this gives a contradiction. Therefore, $\lim_{\varepsilon\to 0} \operatorname{diam} L_\varepsilon = 0$. □

**Theorem 3.8.** *Let $E$ be a real separable Banach space, let $K$ be a nonempty closed convex subset of $E$, and let $\eta : K \times K \to E$ be a single-valued map with $\eta(x,x) = 0$, for all $x \in K$, which is $(s,s)$-continuous. And let $f : K \to R$ be a convex continuous function; $S$ be a set-valued map from $K$ to $2^K$. Assume the following assumptions hold:*

(i) *the multifunction $S$ is nonempty convex-valued and for each sequence $(x_n)_n$ in $K$ converging to $x_0$, the sequence $(S(x_n))_n$ Lower Semi-Mosco converging to $S(x_0)$;*

(ii) *$F : K \to 2^{E^*}$ is a set-valued map with nonempty, weak* compact convex-valued, $(s,w^*)$-usc, and $\eta$-monotone;*

(iii) *the map $y \mapsto \langle u, \eta(x,y)\rangle$ is concave for each $(u,x) \in F(K) \times K$.*

*Then (MQVLI) is L-$\alpha$-well-posed if and only if* (3.9) *holds.*

**Lemma 3.9.** *Let the same assumptions be as in Theorem 3.8. Let $(x_n)_n$ in $K$ be an L-$\alpha$-approximating sequence. If $(x_n)_n$ converges to some $x_0 \in K$, then $x_0$ is a solution of (MQVLI).*

*Proof.* Since $(x_n)_n$ is an L-$\alpha$-approximating sequence for (MQVLI), there exists a sequence $(\varepsilon_n)_n$ in $R^+$, $\varepsilon_n \to 0$, such that $d(x_n, S(x_n)) \leq \varepsilon_n$, and

$$\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) - \frac{\alpha}{2}\|x_n - y\|^2 \leq \varepsilon_n, \quad \forall y \in S(x_n), \forall v \in F(y), n \in N. \tag{3.12}$$

It follows from the Lower Semi-Mosco convergence of $S$ and the proof of Lemma 3.3 that $x_0 \in S(x_0)$. Since $S(x_n)$ Lower Semi-Mosco converges to $S(x_0)$, for every $y \in S(x_0)$, there exists a sequence $y_n \in S(x_n)$, for all $n \in N$, strongly converging to $y$. For each $n \in N$ select $v_n \in F(y_n)$. It follows from (ii) and Proposition 2.19 in [24] that there exists a subsequence $(v_{n_j})_j$ of $(v_n)_n$ weak* converging to some $v \in E^*$. It follows from (ii) and Proposition 2.17 in [24] that $F$ is $(s,w^*)$-closed, and so $v \in F(y)$. By the continuity of $\eta$ and similar argument with the proof of Lemma 3.5, we know that

$$\left\langle v_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle \longrightarrow \langle v, \eta(x_0, y)\rangle. \tag{3.13}$$

It follows from (3.12) that

$$\left\langle v_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle + f\left(x_{n_j}\right) - f\left(y_{n_j}\right) - \frac{\alpha}{2}\left\|x_{n_j} - y_{n_j}\right\|^2 \leq \varepsilon_{n_j}. \tag{3.14}$$

We deduce from the above inequality that

$$\forall y \in S(x_0),\ \exists v \in F(y), \quad \langle v, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \leq 0. \tag{3.15}$$

Let $S_0 = S_1 = S(x_0)$; by Lemma 2.6 we know that there exists $u_0 \in F(x_0)$, such that

$$\forall y \in S(x_0), \quad \langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y\ ) - \frac{\alpha}{2}\|x_0 - y\|^2 \leq 0. \tag{3.16}$$

Then using Lemma 2.7, $x_0$ is a solution of (MQVLI). □

*Proof of Theorem 3.8.* Assume that (3.9) holds. If $(x_n)_n$ in $K$ is an L-$\alpha$-approximating sequence, then from the proof of Theorem 3.6, we know that $(x_n)_n$ converges to some $x_0 \in K$. By Lemma 3.9, $x_0$ is a solution of (MQVLI) and so (MQVLI) is L-$\alpha$-well-posed. The converse is exactly same as that in the proof of Theorem 3.6. □

## 4. The Characterizations of $\alpha$-Well-Posed in the Generalized Sense for (MQVLI)

In this section, we investigate some metric characterizations of $\alpha$-well-posedness in the generalized sense for (MQVLI).

*Definition 4.1* (see [8]). Let $A$ be a nonempty subset of $X$. The measure of noncompactness $\mu$ of the set $A$ is defined by

$$\mu(A) = \inf\left\{\varepsilon > 0 : A \subseteq \bigcup_{i=1}^{n} A_i,\ \operatorname{diam} A_i < \varepsilon,\ i = 1, 2, \ldots, n\right\}. \tag{4.1}$$

*Definition 4.2* (see [8]). Let $(X, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $X$. The Hausdorff distance $H(\cdot, \cdot)$ between $A$ and $B$ is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\}, \tag{4.2}$$

where $e(A, B) = \sup_{a\in A} d(a, B)$ with $d(a, B) = \inf_{b\in B}\|a - b\|$.

**Theorem 4.3.** *Let the same assumptions be as in Lemma 2.7. Then, one has the following.*

(a) *(MQVLI) is $\alpha$-well-posed in the generalized sense if and only if the solution set $\Gamma$ of (MQVLI) is nonempty compact and $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$.*

(b) *Moreover, if $F$ is $\eta$-monotone, then (MQVLI) is L-$\alpha$-well-posed in the generalized sense if and only if the solution set $\Gamma$ of (MQVLI) is nonempty compact and $e(L_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$.*

*Proof.* We only prove (a). The proof of (b) is similar and is omitted here. Assume that (MQVLI) is $\alpha$-well-posed in the generalized sense; then $\Gamma$ is nonempty and compact. It follows from Lemma 2.7 that $Q_\varepsilon \neq \emptyset$. Now we show that $e(Q_\varepsilon, \Gamma) \to 0, as\ \varepsilon \to 0$. Suppose by contradiction that there exists $\beta > 0,\ \varepsilon_n \to 0$ and $w_n \in Q_{\varepsilon_n}$, such that $d(w_n, \Gamma) > \beta$. It follows from $w_n \in Q_{\varepsilon_n}$ that $(w_n)_n$ is an $\alpha$-approximating sequence for (MQVLI). Since (MQVLI) is $\alpha$-well-posedness in the generalized sense, there exists a subsequence $(w_{n_k})_k$ of $(w_n)_n$ strongly converging to a point of $\Gamma$. This contradicts $d(w_n, \Gamma) > \beta$. Thus $e(Q_\varepsilon, \Gamma) \to 0, as\ \varepsilon \to 0$.

For the converse, let $(x_n)_n$ be an $\alpha$-approximating sequence for (MQVLI); then $x_n \in Q_{\varepsilon_n}$. It follows from $e(Q_{\varepsilon_n}, \Gamma) \to 0$ that there exists a sequence $z_n \subset \Gamma$, such that $d(x_n, z_n) \to 0$. Since $\Gamma$ is compact, there exists a subsequence $(z_{n_k})_k$ of $(z_n)_n$ strongly converging to $x_0 \in \Gamma$. Thus the corresponding subsequence $(x_{n_k})_k$ of $(x_n)_n$ is strongly converging to $x_0$. Therefore, (MQVLI) is $\alpha$-well-posed in the generalized sense. □

**Theorem 4.4.** *Let the same assumptions be as in Theorem 3.4. Then (MQVLI) is $\alpha$-well-posed in the generalized sense if and only if*

$$Q_\varepsilon \neq \emptyset,\ \forall \varepsilon > 0, \qquad \lim_{\varepsilon \to \infty} \mu(Q_\varepsilon) = 0. \tag{4.3}$$

*Proof.* Assume that (MQVLI) is $\alpha$-well-posed in the generalized sense; so $Q_\varepsilon \neq \emptyset$, for all $\varepsilon > 0$. By Theorem 4.3(a), $\Gamma$ is nonempty compact and $\lim_{\varepsilon \to 0} e(Q_\varepsilon, \Gamma) \to 0$. For any $\varepsilon > 0$, we have

$$H(Q_\varepsilon, \Gamma) = \max\{e(Q_\varepsilon, \Gamma), e(\Gamma, Q_\varepsilon)\} = e(Q_\varepsilon, \Gamma), \tag{4.4}$$

since $\Gamma$ is compact, $\mu(\Gamma) = 0$. For every $\varepsilon > 0$, the following relation holds (see, e.g., [13])

$$\mu(Q_\varepsilon) \leq 2H(Q_\varepsilon, \Gamma) + \mu(\Gamma) = 2H(Q_\varepsilon, \Gamma) = 2e(Q_\varepsilon, \Gamma). \tag{4.5}$$

It follows from $\lim_{\varepsilon \to 0} e(Q_\varepsilon, \Gamma) \to 0$ that $\lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0$.

Conversely, assume that (4.3) holds. Then, for any $\varepsilon > 0$, $\mathrm{cl}(Q_\varepsilon)$ is nonempty closed and increasing with $\varepsilon > 0$. By (4.3), $\lim_{\varepsilon \to 0} \mu(\mathrm{cl}(Q_\varepsilon)) = \lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0$, where $\mathrm{cl}(Q_\varepsilon)$ is the closure of $Q_\varepsilon$. By the generalized Cantor theorem [23, page 412], we know that

$$\lim_{\varepsilon \to 0} H(\mathrm{cl}(Q_\varepsilon), \Delta) = 0, \quad \text{as } \varepsilon \longrightarrow 0, \tag{4.6}$$

where $\Delta = \bigcap_{\varepsilon > 0} \mathrm{cl}(Q_\varepsilon)$ is nonempty compact.

Now we show that

$$\Gamma = \Delta. \tag{4.7}$$

It follows from Lemma 2.7 that $\Gamma \subseteq \Delta$. So we need to prove that $\Delta \subseteq \Gamma$. Indeed, let $x_0 \in \Delta$. Then $d(x_0, Q_\varepsilon) = 0$ for every $\varepsilon > 0$. Given $\varepsilon_n > 0$, $\varepsilon_n \to 0$, for every $n$ there exists $x_n \in Q_{\varepsilon_n}$ such that $d(x_0, x_n) < \varepsilon_n$. Hence, $x_n \to x_0$ and

$$d(x_n, S(x_n)) \leq \varepsilon_n, \tag{4.8}$$

$$\exists u_n \in F(x_n), \quad \langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n + \frac{\alpha}{2}\|x_n - y\|^2, \quad \forall y \in S(x_n). \tag{4.9}$$

It follows from (4.8), $x_n \to x_0$, and the proof of Lemma 3.3 that $x_0 \in S(x_0)$.

Since $S(x_n)$ Lower Semi-Mosco converges to $S(x_0)$, for every $y \in S(x_0)$, there exists a sequence $y_n \in S(x_n)$, for all $n \in N$, such that $\lim_n y_n = y$ in the strong topology.

Since $\eta$ is $(s,s)$-continuous, the sequence $(\eta(x_n, y_n))_n$ converges strongly to $\eta(x_0, y)$. It follows from (ii) and Proposition 2.19 in [24] that there exists a subsequence $(u_{n_j})_j$ of $(u_n)_n$ weak* converging to some $u_0 \in E^*$. It follows from (ii) and Proposition 2.17 in [24] that $F$ is $(s, w^*)$-closed, and so $u_0 \in F(x_0)$. It follows from the proof of Lemma 3.5 that

$$\left\langle u_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle \longrightarrow \langle u_0, \eta(x_0, y)\rangle. \tag{4.10}$$

Hence,

$$\begin{aligned}
&\langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \\
&\quad = \lim_j \left[\left\langle u_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle + f\left(x_{n_j}\right) - f\left(y_{n_j}\right) - \frac{\alpha}{2}\left\|x_{n_j} - y_{n_j}\right\|^2\right] \\
&\quad \le \lim_j \varepsilon_{n_j} = 0,
\end{aligned} \tag{4.11}$$

that is,

$$\exists u_0 \in F(x_0), \quad \langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \le 0, \quad \forall y \in S(x_0). \tag{4.12}$$

By Lemma 2.7, we know that $x_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.6) and (4.7) that $\lim_{\varepsilon \to 0} e(Q_\varepsilon, \Gamma) = 0$. It follows from the compactness of $\Gamma$ and Theorem 4.3(a) that (MQVLI) is $\alpha$-well-posed in the generalized sense. The proof is completed. □

**Theorem 4.5.** *Let the same assumptions be as in Theorem 3.8. Then (MQVLI) is L-$\alpha$-well-posed in the generalized sense if and only if*

$$L_\varepsilon \ne \emptyset, \ \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0. \tag{4.13}$$

*Proof.* Assume that (MQVLI) is L-$\alpha$-well-posed in the generalized sense. It follows from Lemma 2.7 and the $\eta$-monotonicity of $F$ that $\Gamma \subset L_\varepsilon$, for all $\varepsilon > 0$. And so $L_\varepsilon \ne \emptyset$, for each $\varepsilon > 0$. By similar argument with that in the proof of Theorem 4.3(a), we can get $e(L_\varepsilon, \Gamma) \to 0$ as $\varepsilon \to 0$. From the proof of Theorem 4.4, we also obtain

$$\mu(L_\varepsilon) \le 2H(L_\varepsilon, \Gamma) + \mu(\Gamma) = 2H(L_\varepsilon, \Gamma) = 2e(L_\varepsilon, \Gamma). \tag{4.14}$$

Thus, $\lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0$.

Conversely, assume that (4.13) holds. Then, for any $\varepsilon > 0$, $\mathrm{cl}(L_\varepsilon)$ is nonempty closed and increasing with $\varepsilon > 0$. By (4.13), $\lim_{\varepsilon\to 0}\mu(\mathrm{cl}(L_\varepsilon)) = \lim_{\varepsilon\to 0}\mu(L_\varepsilon) = 0$, where $\mathrm{cl}(L_\varepsilon)$ is the closure of $L_\varepsilon$. By the generalized Cantor theorem [23, Page 412], we know that

$$\lim_{\varepsilon\to 0} H(\mathrm{cl}(L_\varepsilon), \Delta) = 0, \quad \text{as } \varepsilon \longrightarrow 0, \tag{4.15}$$

where $\Delta = \bigcap_{\varepsilon>0} \mathrm{cl}(L_\varepsilon)$ is nonempty compact.

Now we show that

$$\Gamma = \Delta. \tag{4.16}$$

It follows from Lemma 2.7 and the monotonicity of $F$ that $\Gamma \subseteq \Delta$. So we need to prove that $\Delta \subseteq \Gamma$. Indeed, let $x_0 \in \Delta$. Then $d(x_0, L_\varepsilon) = 0$ for every $\varepsilon > 0$. Given $\varepsilon_n > 0$, $\varepsilon_n \to 0$, for every $n$ there exists $x_n \in L_{\varepsilon_n}$ such that $d(x_0, x_n) < \varepsilon_n$. Hence, $x_n \to x_0$ and

$$d(x_n, S(x_n)) \le \varepsilon_n, \tag{4.17}$$

$$\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n + \frac{\alpha}{2}\|x_n - y\|^2, \quad \forall y \in S(x_n),\ \forall v \in F(y). \tag{4.18}$$

It follows from (4.17), $x_n \to x_0$, and the proof of Lemma 3.3 that $x_0 \in S(x_0)$.

Since $S(x_n)$ Lower Semi-Mosco converges to $S(x_0)$, for every $y \in S(x_0)$, there exists a sequence $y_n \in S(x_n)$, for all $n \in N$, such that $\lim_n y_n = y$ in the strong topology.

For each $n \in N$ select $v_n \in F(y_n)$. Since $F$ is $(s, w^*)$-usc with weak* compact convex values, we can find a subsequence $(v_{n_j})_j$ of $(v_n)_n$ weak* converging to some $v \in F(y)$. By the continuity of $\eta$ and similar argument with the proof of Lemma 3.5, we know that

$$\left\langle v_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle \longrightarrow \langle v, \eta(x_0, y)\rangle. \tag{4.19}$$

Hence,

$$\left\langle v_{n_j}, \eta\left(x_{n_j}, y_{n_j}\right)\right\rangle + f\left(x_{n_j}\right) - f\left(y_{n_j}\right) - \frac{\alpha}{2}\left\|x_{n_j} - y_{n_j}\right\|^2 \le \varepsilon_{n_j}. \tag{4.20}$$

We deduce from the above inequality that

$$\forall y \in S(x_0),\ \exists v \in F(y), \quad \langle v, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \le 0. \tag{4.21}$$

By Lemma 2.6 we know that there exist $u_0 \in F(x_0)$, such that

$$\forall y \in S(x_0), \quad \langle u_0, \eta(x_0, y)\rangle + f(x_0) - f(y) - \frac{\alpha}{2}\|x_0 - y\|^2 \le 0. \tag{4.22}$$

It follows from Lemma 2.7 that $x_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.15) and (4.16) that $\lim_{\varepsilon\to 0} e(L_\varepsilon, \Gamma) = 0$. It follows from the compactness of $\Gamma$ and Theorem 4.3(b) that (MQVLI) is L-$\alpha$-well-posed in the generalized sense. The problem is completed. □

*Remark 4.6.* (i) It is easy to see that if $\alpha = 0$, then by the main results in our paper, we can recover the corresponding results in [20] with the weaker condition $S(x_n)$ Lower Semi-Mosco converging to $S(x_0)$ instead of the condition $S(x_n)$ Mosco converging to $S(x_0)$.

(ii) The proof methods of Theorems 4.4 and 4.5 are different from those ofTheorems 4.1 and 4.2 in [20].

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