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Research Article

Supercyclicity and Hypercyclicity of an Isometry Plus a Nilpotent

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Suppose that X is a separable normed space and the operators A and Q are bounded on X. In this paper, it is shown that if $AQ = QA$, A is an isometry, and Q is a nilpotent then the operator $A + Q$ is neither supercyclic nor weakly hypercyclic. Moreover, if the underlying space is a Hilbert space and A is a co-isometric operator, then we give sufficient conditions under which the operator $A+Q$ satisfies the supercyclicity criterion.

1. Introduction

Let x be a vector in a separable normed space $\mathcal{X}$ and T an operator on $\mathcal{X}$. The orbit of x under T is defined by

$$\operatorname{orb}(T,x) = \{T^n x : n = 0,1,2,\dots\}. \tag{1.1}$$

We recall that a vector x in $\mathcal{X}$ is cyclic for an operator T on $\mathcal{X}$ if the closed linear span of $\operatorname{orb}(T,x)$ is $\mathcal{X}$; it is supercyclic, if the set of all scalar multiples of the elements of $\operatorname{orb}(T,x)$ is dense in $\mathcal{X}$; also it is said to be (weakly) hypercyclic if $\operatorname{orb}(T,x)$ is (weakly) dense in $\mathcal{X}$. An operator T is called cyclic, supercyclic, or (weakly) hypercyclic operator, respectively, if it has a cyclic, supercyclic, or (weakly) hypercyclic vector. Recently, the cyclicity of operators has attracted much attention from operator theorists. For a good source on this topic, see [1]. Hilden and Wallen in [2] proved that isometries on Hilbert spaces with dimension more than one are not supercyclic. Ansari and Bourdon in [3] and Miller in [4] independently proved this fact on Banach spaces. Moreover, recently it is shown in [5] that m-isometric operators on Hilbert spaces, which forms a larger class than isometries, are neither supercyclic nor weakly hypercyclic. In this paper, it is shown that an isometry plus a nilpotent on normed spaces are neither supercyclic nor weakly hypercyclic if they commute. We also discuss this fact when the underlying space is a Hilbert space and the isometry is replaced by a co-isometry. We begin with some elementary properties of such operators. In what follows, as usual, for an operator T, $\sigma_{ap}(T)$, $\sigma_p(T)$, and $\sigma(T)$ are denoted, respectively, the approximate point spectrum, point spectrum, and spectrum of T. Also, $\mathbb{D}$ denotes the open unit disc. Recall that an operator Q on a normed space $\mathcal{X}$ is a nilpotent operator of order $N \geq 1$ if $Q^N = 0$ and $Q^{N-1} \neq 0$. From now on, we assume that Q is a nilpotent operator of order $N \geq 1$ unless stated otherwise.

Proposition 1.1. *Suppose that $\mathcal{X}$ is a normed space, and $A \in \mathcal{B}(\mathcal{X})$ is an isometry such that $AQ = QA$. If $T = A + Q$, then*

(i) $\sigma(T) = \sigma(A)$,

(ii) $\sigma_p(T) = \sigma_p(A)$,

(iii) $\sigma_{ap}(T) = \sigma_{ap}(A)$.

Proof. (i) Suppose that $\lambda \notin \sigma(A)$. Then it is easily seen that

$$(T-\lambda)^{-1} = \sum_{k=1}^{N} (-1)^{k-1} (A-\lambda)^{-k} Q^{k-1} \tag{1.2}$$

which implies that $\lambda \notin \sigma(T)$. Consequently, $\sigma(T) \subseteq \sigma(A)$. Since $A = T - Q$, a similar argument shows that $\sigma(A) \subseteq \sigma(T)$.

(ii) If $\lambda \in \sigma_p(A)$, there exits $x \neq 0$ such that $Ax = \lambda x$. Therefore,

$$TQ^{N-1}x = AQ^{N-1}x = \lambda Q^{N-1}x. \tag{1.3}$$

Now, if $Q^{N-1}x \neq 0$, then $\lambda \in \sigma_p(T)$; otherwise,

$$TQ^{N-2}x = AQ^{N-2}x = \lambda Q^{N-2}x. \tag{1.4}$$

Also, if $Q^{N-2}x \neq 0$ then $\lambda \in \sigma_p(T)$; otherwise, consider $Q^{N-3}x$ and continue this process to conclude that $Tx = Ax = \lambda x$ which implies that $\lambda \in \sigma_p(T)$. Hence, $\sigma_p(A) \subseteq \sigma_p(T)$. Moreover, since $A = T - Q$, using a similar method, we get $\sigma_p(T) \subseteq \sigma_p(A)$.

(iii) Let $\lambda \in \sigma_{ap}(T)$; then there exists a sequence $(x_n)_n$ in $\mathcal{X}$ such that $\|x_n\| = 1$ and

$$Tx_n - \lambda x_n \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{1.5}$$

Therefore,

$$AQ^{N-1}x_n - \lambda Q^{N-1}x_n \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{1.6}$$

Suppose that there is $c_1 > 0$ so that

$$\left\| Q^{N-1} x_n \right\| > c_1 \tag{1.7}$$

for all $n \geq 1$; then $Ay_n - \lambda y_n \to 0$ as $n \to +\infty$ where

$$y_n = \frac{Q^{N-1} x_n}{\| Q^{N-1} x_n \|} \tag{1.8}$$

which, in turn, implies that $\lambda \in \sigma_{ap}(A)$.

Now, if (1.7) does not hold, then we can assume, without loss of generality, that $(x_n)_n$ satisfies

$$Q^{N-1} x_n \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{1.9}$$

So by (1.5),

$$AQ^{N-2} x_n - \lambda Q^{N-2} x_n \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{1.10}$$

Now, if there is a constant $c_2 > 0$ such that

$$\left\| Q^{N-2} x_n \right\| > c_2 \tag{1.11}$$

for all n, then $Az_n - \lambda z_n \to 0$ where

$$z_n = \frac{Q^{N-2} x_n}{\| Q^{N-2} x_n \|} \tag{1.12}$$

which implies that $\lambda \in \sigma_{ap}(A)$. Otherwise, we can assume, without loss of generality, that $Q^{N-2} x_n \to 0$ as $n \to \infty$ and by (1.5)

$$AQ^{N-3} x_n - \lambda Q^{N-3} x_n \longrightarrow 0 \tag{1.13}$$

as $n \to \infty$. The procedure continues to conclude that $\lambda \in \sigma_{ap}(A)$. Since $A = T - Q$, by a similar method $\sigma_{ap}(A) \subseteq \sigma_{ap}(T)$. □

In the remaining results of this section, the operators A and T are as in Proposition 1.1.

Corollary 1.2. *Suppose that $\mathcal{X}$ is a normed space. Then $T - \lambda I$ is bounded below where $|\lambda| \neq 1$.*

Proof. Since A is an isometry, $\sigma_{ap}(T) = \sigma_{ap}(A) \subseteq \partial\mathbb{D}$. In fact, let $\lambda \in \sigma_{ap}(A)$; then $|\lambda| \leq \|A\| = 1$; moreover, there exists a sequence $(x_n)_n$ in $\mathcal{X}$ with $\|x_n\| = 1$ and so $(A - \lambda I)(x_n) \to 0$ if $n \to \infty$. Therefore,

$$0 \leq 1 - |\lambda| \leq \|(A - \lambda I)(x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{1.14}$$

and so $|\lambda| = 1$. Now, if $|\lambda| \neq 1$, then $\lambda \notin \sigma_{ap}(T)$ and so $T - \lambda$ is bounded below. □

Corollary 1.3. *Suppose that $\mathcal{X}$ is an infinite dimensional Banach space. Then the operator T on $\mathcal{X}$ is not a compact operator.*

Proof. If T is a compact operator, then $0 \in \sigma(T) = \sigma(A)$. Thus $\overline{\mathbb{D}} \subseteq \sigma(T)$ which contradicts the fact that the spectrum of a compact operator is at most countable. □

Proposition 1.4. *If the operators T and A are defined on a normed space $\mathcal{X}$, then* $\ker(T - \lambda) \subseteq \ker(A - \lambda)$ *for every scalar λ.*

Proof. Fix $\lambda \in \mathbb{C}$ and suppose that $Tx = \lambda x$ for some nonzero vector x. By Proposition 1.1, $\lambda \in \sigma_p(A)$ which implies that $|\lambda| = 1$. Therefore, if $n > N - 1$, we have

$$\begin{aligned} \|x\|^2 &= \|T^n x\|^2 \\ &= \left\| A^{n-(N-1)} \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{N-1-k} x \right\|^2 \\ &= \left\| \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{N-1-k} x \right\|^2 \\ &= \binom{n}{N-1}^2 \left\| \sum_{k=0}^{N-1} \frac{(N-1)!(n-N+1)!}{k!(n-k)!} Q^k A^{N-1-k} x \right\|^2. \end{aligned} \tag{1.15}$$

Consequently,

$$\|x\| \geq \binom{n}{N-1} \left[\left\| Q^{N-1} x \right\| - \sum_{k=0}^{N-2} \frac{(N-1)!(n-N+1)!}{k!(n-k)!} \left\| Q^k A^{N-1-k} x \right\| \right]. \tag{1.16}$$

Since

$$\lim_{n \to \infty} \frac{(N-1)!(n-N+1)!}{k!(n-k)!} = 0 \tag{1.17}$$

for every $0 \leq k \leq N - 2$, we conclude that $\|Q^{N-1}x\| = 0$. Continue the above process to get $Qx = 0$, and so $Ax = \lambda x$. □

Corollary 1.5. *If $\mathcal{X}$ is a Hilbert space, then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

Proof. Let x and y be eigenvectors of T corresponding to distinct eigenvalues λ_1 and λ_2. So, $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$. By Proposition 1.4, $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ which implies that $|\lambda_1| = |\lambda_2| = 1$. Suppose that $\langle \cdot \rangle$ denotes the inner product of $\mathcal{X}$. Then

$$0 = \left\| \lambda_1 x + \lambda_2 y \right\|^2 - \left\| x + y \right\|^2 = 2 \operatorname{Re}\left(\lambda_1 \overline{\lambda_2} - 1 \right) \langle x, y \rangle. \tag{1.18}$$

Replacing y by iy, we obtain $\operatorname{Im}(\lambda_1\overline{\lambda_2}-1)\langle x,y\rangle = 0$; consequently,

$$\left(\frac{\lambda_1}{\lambda_2}-1\right)\langle x,y\rangle = \left(\lambda_1\overline{\lambda_2}-1\right)\langle x,y\rangle = 0. \tag{1.19}$$

But $\lambda_1 \neq \lambda_2$, and so $\langle x,y\rangle = 0$. □

Recall that an operator T is power bounded if there exists some constant $c > 0$ such that $\|T^n\| \leq c$ for all $n = 1,2,3,\ldots$.

Proposition 1.6. *Let $\mathcal{X}$ be a normed space and $x \in \mathcal{X}$. If there is a constant $c > 0$ such that $\|T^n x\| \leq c$ for all $n \geq 1$, then $Qx = 0$. In particular, if T is power bounded, then $Q = 0$.*

Proof. Since the sequence $(\|T^n x\|)_n$ is bounded, an argument similar to the proof of the Proposition 1.4 shows that $Qx = 0$. □

2. Supercyclicity and Hypercyclicity

We begin this section with a useful lemma.

Lemma 2.1. *Let $\mathcal{X}$ be a normed space. For nonnegative integers k, n, if*

$$P_k(n) = x_0 + x_1 n + x_2 n^2 + \cdots + x_k n^k \tag{2.1}$$

is a polynomial in n with coefficients in $\mathcal{X}$ of degree k, then the sequence $(\|P_k(n)\|)_n$ is eventually increasing.

Proof. We prove the lemma by induction on k, the degree of the polynomial $P_k(n)$. For $k = 1$, let $P_1(n) = x_0 + x_1 n$. It is easily seen that for every $n \geq 1$

$$\|P_1(n+1)\| \leq \frac{1}{2}(\|P_1(n)\| + \|P_1(n+2)\|). \tag{2.2}$$

Since $\lim_{n\to\infty}\|P_1(n)\| = +\infty$, there is a positive integer i such that

$$\|P_1(i)\| < \|P_1(i+1)\|. \tag{2.3}$$

This fact coupled with (2.2) implies that

$$0 < \|P_1(i+1)\| - \|P_1(i)\| \leq \|P_1(n+1)\| - \|P_1(n)\| \tag{2.4}$$

for every $n \geq i$. Therefore, the sequence $(\|P_1(n)\|)_{n\geq i}$ is increasing. Suppose that $(\|P_k(n)\|)_n$ is eventually increasing and let

$$P_{k+1}(n) = x_0 + x_1 n + \cdots + x_{k+1} n^{k+1}, \tag{2.5}$$

where $x_{k+1} \neq 0$. Since

$$\lim_{n\to\infty} \left\| x_1 + x_2(n+1) + \cdots + x_{k+1}(n+1)^k \right\| = +\infty, \tag{2.6}$$

using the induction hypothesis there is a positive integer j such that for every $n \geq j$

$$\left\| x_1 + x_2(n+1) + \cdots + x_{k+1}(n+1)^k \right\| \geq \max\left\{ 2\|x_0\|, \left\| x_1 + x_2 n + \cdots + x_{k+1} n^k \right\| \right\}. \tag{2.7}$$

Therefore,

$$\frac{\|P_{k+1}(n+1)\|}{\|P_{k+1}(n)\|} \geq \frac{(n+1)\left\| x_1 + x_2(n+1) + \cdots + x_{k+1}(n+1)^k \right\| - \|x_0\|}{n\left\| x_1 + x_2 n + \cdots + x_{k+1} n^k \right\| + \|x_0\|} \geq 1 \tag{2.8}$$

for every $n \geq j$. Hence, the sequence $(\|p_{k+1}(n)\|)_{n\geq j}$ is increasing. □

Theorem 2.2. *Suppose that $\mathcal{X}$ is a normed space, and $A \in B(\mathcal{X})$ is an isometry such that $AQ = QA$. If $T = A + Q$, then the operator T is neither supercyclic nor weakly hypercyclic.*

Proof. Let $\widetilde{\mathcal{X}}$ be the completion of $\mathcal{X}$ and $\widetilde{T}$, $\widetilde{A}$, and $\widetilde{Q}$ the extensions of T, A, and Q on $\widetilde{\mathcal{X}}$, respectively. Thus, $\widetilde{T} = \widetilde{A} + \widetilde{Q}$ where $\widetilde{A}$ is an isometry and $\widetilde{Q}$ is a nilpotent operator; moreover, $\widetilde{A}\widetilde{Q} = \widetilde{Q}\widetilde{A}$. Also, note that the supercyclicity of the operator T implies the supercyclicity of $\widetilde{T}$. So we can assume, without loss of generality, that $\mathcal{X}$ is a Banach space.

As we have seen in the proof of Proposition 1.4, if $x \in \mathcal{X}$ then

$$\|T^n x\| = \left\| \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{N-1-k} x \right\|, \tag{2.9}$$

and so by Lemma 2.1, the sequence $(\|T^n x\|)_n$ is eventually increasing. Suppose that x_0 is a supercyclic vector for T. Thus, for any $x \in \mathcal{X}$ there is a sequence $(n_i)_i$ of positive integers and a sequence $(\alpha_i)_i$ of scalars such that $\alpha_i T^{n_i} x_0 \to x$. Moreover, since the sequence $(\|T^n x_0\|)_n$ is eventually increasing, we have $\|\alpha_i T^{n_i} x_0\| \leq \|\alpha_i T^{n_i+1} x_0\|$ for large i. So letting $i \to \infty$, we conclude that $\|x\| \leq \|Tx\|$, for all x in $\mathcal{X}$. On the other hand, the supercyclicity of T implies that it has a dense range and so is invertible. Thus, in light of Proposition 1.1 we see that A is invertible. It is easy to see that

$$T^{-1} = A^{-1} + P, \tag{2.10}$$

where

$$P = \sum_{k=1}^{N-1} (-1)^k A^{-(k+1)} Q^k. \tag{2.11}$$

Since $P^N = 0$, by a similar argument the sequence $(\|T^{-n}x\|)_n$ is eventually increasing for every $x \in \mathcal{X}$. But T^{-1} is also supercyclic (see [1, Theorem 1.12]); therefore,

$$\|x\| \leq \left\|T^{-1}x\right\| \tag{2.12}$$

for every $x \in \mathcal{X}$. Thus, T is an isometry which implies that it is not a supercyclic operator.

To show that the operator T is not weakly hypercyclic, note that

$$\|T^{*n}x^*\| = \left\|\sum_{k=0}^{N-1}\binom{n}{k}Q^{*k}A^{*n-k}x^*\right\| \tag{2.13}$$

for every $x^* \in \mathcal{X}^*$ and every positive integer n. If $\ker Q^* \neq \{0\}$, then there is a nonzero $x^* \in \mathcal{X}^*$ such that $\|T^{*n}x^*\| = \|A^{*n}x^*\| \leq \|x^*\|$ because $\|A^*\| = \|A\| = 1$. Now, suppose that x_0 is a weakly hypercyclic vector for T. Since $\mathrm{orb}(T, x_0)$ is weakly dense in $\mathcal{X}$ and x^* is nonzero, the set $\{x^*(T^n x_0) : n \geq 0\}$ is dense in $\mathbb{C}$. But

$$\|x^*(T^n x_0)\| = \|(T^{*n}x^*)(x_0)\| \leq \|T^{*n}x^*\|\|x_0\| \leq \|x^*\|\|x_0\| \tag{2.14}$$

for all $n \geq 0$, which is a contradiction. If $\ker Q^* = \{0\}$, then $Q^* = 0$ and so $T = A$ is not a weakly hypercyclic operator. □

We remark that there are Banach space isometries which are also weakly supercyclic. Indeed, the unweighted bilateral weighted shift on the space $l^p(\mathbb{Z})$ where $p > 2$ is weakly supercyclic (see [1, Corollary 10.32]). However, the question that whether an isometry plus a nonzero nilpotent which commute with each other, are weakly supercyclic or not is still an open question.

The following examples show that the commutativity of A and Q is essential in the preceding theorem.

Example 2.3. *Let $(e_n)_{n=-\infty}^{+\infty}$ be the standard orthonormal basis for $l^2(\mathbb{Z})$. Define the isometric operator A by $Ae_n = e_{n+1}$ for all $n \in \mathbb{Z}$ and the weighted shift operator Q by $Qe_n = w_n e_{n+1}$, where $w_{2n} = 0$ for all integers n, $w_{2n-1} = 1/(2n-1)^2$ for all $n \geq 1$, and $w_{2n-1} = 1/(1-2n)$ for all $n \leq 0$. Note that $Q^2 = 0$ and $AQ \neq QA$. Moreover, since $1 \leq \inf_n(1 + w_n) \leq \sup_n(1 + w_n) \leq 2$, the weighted shift operator $T = A + Q$ is invertible. To see that T is supercyclic by Theorem 3.4 of [6], it is sufficient to show that*

$$\lim_{n\to\infty}\prod_{j=1}^{n}(1 + w_j)\prod_{j=1}^{n}\frac{1}{1 + w_{-j}} = 0. \tag{2.15}$$

But $\prod_{j=1}^{\infty}(1+w_j)$ *is finite, because* $\sum_{j=1}^{\infty} w_j < \infty$. *Furthermore,* $\prod_{j=1}^{\infty} 1/(1+w_{-j}) = 0$, *because*

$$\begin{aligned}\sum_{j=1}^{\infty}\left(1-\frac{1}{1+w_{-j}}\right) &= \sum_{j=1}^{\infty}\frac{w_{-j}}{1+w_{-j}} = \sum_{j=1}^{\infty}\frac{w_{-(2j-1)}}{1+w_{-(2j-1)}} \\ &= \sum_{j=1}^{\infty}\frac{1/(2j-1)}{1+1/(2j-1)} = \sum_{j=1}^{\infty}\frac{1}{2j} = \infty\end{aligned} \tag{2.16}$$

(see [7, pages 299 and 300]). Therefore, (2.15) *holds.*

Example 2.4. *Consider the isometric operator* A *on* $l^2(\mathbb{Z})$ *defined by* $Ae_n = e_{n-1}$ *and the weighted shift operator* Q *defined by* $Qe_n = w_n e_{n-1}$, *where* $w_{2n} = 0$ *for all* $n \in \mathbb{Z}$, $w_{2n-1} = 1/(2n-1)$, *for* $n \geq 1$, *and* $w_{2n-1} = 1/(2n-1)^2$ *for* $n \leq 0$. *Note that* $Q^2 = 0$ *and* $AQ \neq QA$. *Also, since*

$$(1+w_1)(1+w_2)\cdots(1+w_n) \geq w_1 + w_2 + \cdots + w_n \tag{2.17}$$

for all $n \geq 1$, *and* $\sum_{n=1}^{\infty} w_n = \infty$, *we conclude that*

$$\lim_{n\to\infty}(1+w_1)(1+w_2)\cdots(1+w_n) = \infty. \tag{2.18}$$

Furthermore,

$$\lim_{n\to\infty}(1+w_{-1})(1+w_{-2})\cdots(1+w_{-n}) < \infty, \tag{2.19}$$

because

$$\sum_{n=1}^{\infty} w_{-n} < \infty. \tag{2.20}$$

Hence, using Corollary 10.27 of [1], we observe that the operator $A+Q$ *is weakly hypercyclic.*

3. A Co-isometry Plus a Nilpotent

From now on, we assume that $\mathscr{H}$ is a separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Recall that the unilateral shift operator $S : \mathscr{H} \to \mathscr{H}$ is given by $Se_n = e_{n+1}$ for all n and the backward shift operator $B : \mathscr{H} \to \mathscr{H}$ is defined by $Be_0 = 0$ and $Be_n = e_{n-1}$ for all $n \geq 1$. It is known that the operator B is supercyclic (see [1, page 9]). It follows that a co-isometry can be supercyclic. In this section, we give sufficient conditions such that the sum of a co-isometry and a nilpotent is supercyclic on $\mathscr{H}$.

Theorem 3.1. *Suppose that* A *is a co-isometric operator on a Hilbert space* $\mathscr{H}$. *Then* A *is supercyclic if and only if* $\cap_{n\geq 0} A^{*n}\mathscr{H} = (0)$.

Proof. First assume that $\cap_{n\geq 0} A^{*n}\mathscr{H} = (0)$. Then by the von Neumann-Wold decomposition, $A^* = S^m$ for some positive integer m (see [8]). Therefore, $A = B^m$ which is a positive power of a supercyclic operator and so is supercyclic [9]. For the converse, assume that $M = \cap_{n\geq 0} A^{*n}\mathscr{H} \neq (0)$, and let P_M denote the orthogonal projection on M. By the von Neumann-Wold decomposition, M is a reducing subspace for A and $A^*|_M$ is unitary. Since $(A|_M)^* = P_M A^*|_M$ is also unitary, the operator $A|_M$ is not supercyclic. Assume that A is supercyclic, and $h = g \oplus k$ is a supercyclic vector for A, where $g \in M$ and $k \in M^{\perp}$. If $g = 0$, then $\mathscr{H} = \mathscr{M}^{\perp}$ which is impossible; so $g \neq 0$. Take $f \in M$, and let $\varepsilon > 0$ be arbitrary. Then there is a nonnegative integer n and a scalar $\alpha \in \mathbb{C}$ such that

$$\|\alpha A^n g - f\| \leq \|\alpha A^n (g \oplus k) - (f \oplus 0)\| < \varepsilon. \tag{3.1}$$

Hence, g is a supercyclic vector for $A|_M$ which is impossible. □

To prove the next theorem, we need the supercyclicity criterion due to H. N. Salas (see [10], or more generally [11]).

Supercyclicity Criterion

Suppose that X is a separable Banach space and T is a bounded operator on X. If there is an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$, and two dense sets Y and Z of X such that

(1) there exists a function $S : Z \to Z$ satisfying $TSx = x$ for all $x \in Z$,

(2) $\|T^{n_k}x\| \cdot \|S^{n_k}y\| \to 0$ for every $x \in Y$ and $y \in Z$,

then T is supercyclic.

Theorem 3.2. *Suppose that A is a co-isometry on a Hilbert space $\mathscr{H}$ such that $\cap_{n\geq 0} A^{*n}\mathscr{H} = (0)$. If $AQ = QA$, then the operator $T = A + Q$ satisfies the supercyclicity criterion.*

Proof. By Corollary 1.2, the operator T^* is bounded below and so is left invertible. Consequently, T is a right invertible operator. Let $x \in \cap_{n\geq 0} T^{*n}\mathscr{H}$. For every $i \geq 0$, there is a vector x_{N+i} in $\mathscr{H}$ such that $T^{*N+i}x_{N+i} = x$. Since $Q^N = 0$, we have

$$\begin{aligned} x &= T^{*N+i}x_{N+i} \\ &= \sum_{k=0}^{N+i} \binom{N+i}{k} \left(A^{*k}Q^{*N+i-k}\right)(x_{N+i}) \\ &= \sum_{k=i+1}^{N+i} \binom{N+i}{k} \left(A^{*k}Q^{*N+i-k}\right)(x_{N+i}) \end{aligned} \tag{3.2}$$

which implies that $x \in A^{*i+1}\mathscr{H}$. Hence, $x \in \cap_{n\geq 0} T^{*n}\mathscr{H} = (0)$ and so the operator T admits a complete set of eigenvectors that is, $\mathscr{H} = \vee_{\mu\in\mathbb{D}_r} \ker(T - \mu)$ for every positive real number r, where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ (see [12], Part (A) of the lemma). Since T^* is bounded below, TT^* is invertible. Take $S = T^*(TT^*)^{-1}$. Choose $r > 0$ so that $r < 1/\|S\|$, and let

$$Y = Z = \operatorname{span}\{\ker(T - \mu) : \mu \in \mathbb{D}_r\}. \tag{3.3}$$

Now, if $h \in Y = Z$, then

$$\|T^n h\| \|S^n h\| \leq |\mu|^n \|S\|^n \|h\| \leq (r\|S\|)^n \|h\| \longrightarrow 0 \tag{3.4}$$

as $n \to \infty$. Finally, $T^n S^n h = h$ for every $h \in \mathcal{H}$ and every $n \geq 0$. Thus, the operator T satisfies the supercyclicity criterion. □

The Hilbert-Schmidt class, $C_2(\mathcal{H})$, is the class of all bounded operators S defined on a Hilbert space $\mathcal{H}$, satisfying

$$\|S\|_2^2 = \sum_{n=1}^{\infty} \|Se_n\|^2 < +\infty, \tag{3.5}$$

where $\|\cdot\|$ is the norm on $\mathcal{H}$ induced by its inner product. We recall that $C_2(\mathcal{H})$ is a Hilbert space equipped with the inner product $\langle S, T\rangle = \operatorname{tr}(ST^*)$ in which $\operatorname{tr}(S^*T)$ denotes the trace of S^*T. Furthermore, $C_2(\mathcal{H})$ is an ideal of the algebra of all bounded operators on $\mathcal{H}$, see [8]. For any bounded operator B on a Hilbert space $\mathcal{H}$, the left multiplication operator L_B and the right multiplication operator R_B on $C_2(\mathcal{H})$ are defined by $L_B(S) = BS$ and $R_B(S) = SB$ for all $S \in C_2(\mathcal{H})$. It is known that an operator B satisfies the supercyclicity criterion if and only if L_B is supercyclic on the space $C_2(\mathcal{H})$ (see [13, page 37]). In the following proposition, we see that an operator T may satisfy the supercyclicity criterion although R_T is not a supercyclic operator on $C_2(\mathcal{H})$.

Proposition 3.3. *Suppose that $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ is a co-isometry such that $\cap_{n\geq 0} A^{*n}\mathcal{H} = (0)$ and $AQ = QA$. Then the operator $T = A + Q$ satisfies the supercyclicity criterion but the operator R_T is not supercyclic on $C_2(\mathcal{H})$.*

Proof. By Theorem 3.2, the operator T satisfies the supercyclicity criterion. Moreover, for every $S \in C_2(\mathcal{H})$ we have

$$\begin{aligned} \|R_A(S)\|_2^2 &= \|SA\|_2^2 = \|(SA)^*\|_2^2 = \|A^*S^*\|_2^2 \\ &= \sum_{n=1}^{\infty} \|A^*S^*e_n\|^2 = \sum_{n=1}^{\infty} \|S^*e_n\|^2 = \|S\|_2^2, \end{aligned} \tag{3.6}$$

which implies that R_A is an isometry. Also, if $S \in C_2(\mathcal{H})$, then $R_Q^N(S) = 0$. Since $R_T(S) = R_A(S) + R_Q(S)$, Theorem 2.2 implies that R_T is not supercyclic. □

The proof of the following proposition is similar to the proof of the second part of Theorem 2.2, and we omit it.

Proposition 3.4. *Suppose that $\mathcal{X}$ is a normed space and $A \in \mathcal{B}(\mathcal{X})$ is a co-isometry such that $AQ = QA$. Then the operator $T = A + Q$ is not weakly hypercyclic.*

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