Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 856896, 15 pages
doi:10.1155/2011/856896

Research Article

L^r-L^p Stability of the Incompressible Flows with Nonzero Far-Field Velocity

Jaiok Roh

Department of Mathematics, Hallym University, Chuncheon 200-702, Republic of Korea

Correspondence should be addressed to Jaiok Roh, joroh@dreamwiz.com

Received 16 June 2011; Accepted 15 September 2011

Academic Editor: Ibrahim Sadek

We consider the stability of stationary solutions $\mathbf{w}$ for the exterior Navier-Stokes flows with a nonzero constant velocity $\mathbf{u}_\infty$ at infinity. For $\mathbf{u}_\infty = 0$ with nonzero stationary solution $\mathbf{w}$, Chen (1993), Kozono and Ogawa (1994), and Borchers and Miyakawa (1995) have studied the temporal stability in L^p spaces for $1 < p$ and obtained good stability decay rates. For the spatial direction, we recently obtained some results. For $\mathbf{u}_\infty \neq 0$, Heywood (1970, 1972) and Masuda (1975) have studied the temporal stability in L^2 space. Shibata (1999) and Enomoto and Shibata (2005) have studied the temporal stability in L^p spaces for $p \geq 3$. Then, Bae and Roh recently improved Enomoto and Shibata's results in some sense. In this paper, we improve Bae and Roh's result in the spaces L^p for $p > 1$ and obtain L^r-L^p stability as Kozono and Ogawa and Borchers and Miyakawa obtained for $\mathbf{u}_\infty = 0$.

1. Introduction

The motion of nonstationary flow of an incompressible viscous fluid past an isolated rigid body is formulated by the following initial boundary value problem of the Navier-Stokes equations:

$$\begin{gathered}\frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}|_{\partial\Omega} = 0, \qquad \lim_{|x|\to\infty}\mathbf{u}(x,t) = \mathbf{u}_\infty,\end{gathered} \tag{1.1}$$

where Ω is an exterior domain in R^n with a smooth boundary $\partial\Omega$, and $\mathbf{u}_\infty$ denotes a given constant vector describing the velocity of the fluid at infinity. In this paper, we consider a nonzero constant $\mathbf{u}_\infty$. The physical model of the exterior Navier-Stokes equations with a nonzero constant $\mathbf{u}_\infty$ can be considered as the motion of water in the sea when a boat is moving with the speed $-\mathbf{u}_\infty$, while the one with zero constant $\mathbf{u}_\infty$ can be considered when a boat is stopped. There are few known results for the case $\mathbf{u}_\infty \neq 0$, while, with $\mathbf{u}_\infty = 0$, many results were obtained for the temporal decay and weighted estimates of solutions of (1.1) (refer [1–12]).

Now, we set $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$ in (1.1) and have

$$\begin{gathered} \frac{\partial}{\partial t}\mathbf{v} - \Delta\mathbf{v} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p_1 = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega \times (0,\infty), \\ \mathbf{v}|_{t=0} = \mathbf{u}_0 - \mathbf{u}_\infty, \qquad \mathbf{v}|_{\partial\Omega} = -\mathbf{u}_\infty, \qquad \lim_{|x|\to\infty} \mathbf{v}(x,t) = 0. \end{gathered} \tag{1.2}$$

Consider the following linear problem:

$$\begin{gathered} \frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}|_{\partial\Omega} = 0, \qquad \lim_{|x|\to\infty} \mathbf{u}(x,t) = 0, \end{gathered} \tag{1.3}$$

which is referred to as the Oseen equations; see [13].

In order to formulate the problem (1.3), Enomoto and Shibata [14] used the Helmholtz decomposition:

$$L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega), \tag{1.4}$$

where $1 < p < \infty$,

$$\begin{aligned} L_p(\Omega)^n &= \left\{u = (u_1,\ldots,u_n) : u_j \in L_p(\Omega),\ j = 1,\ldots,n\right\}, \\ C_{0,\sigma}^\infty &= \left\{u = (u_1,\ldots,u_n) \in C_0^\infty(\Omega)^n : \nabla \cdot u = 0 \text{ in } \Omega\right\}, \\ J_p(\Omega) &= \text{the completion of } C_{0,\sigma}^\infty(\Omega), \text{ in } L_p(\Omega)^n, \\ G_p(\Omega) &= \left\{\nabla\pi \in L_p(\Omega)^n : \pi \in L_{p,\mathrm{loc}}\left(\overline{\Omega}\right)\right\}. \end{aligned} \tag{1.5}$$

The Helmholtz decomposition of $L_p(\Omega)^n$ was proved by Fujiwara and Morimoto [15], Miyakawa [16], and Simader and Sohr [17]. Let P be a continuous projection from $L_p(\Omega)^n$ onto $J_p(\Omega)^n$.

By applying P into (1.3) and setting $\mathcal{O}_{\mathbf{u}_\infty} = P(-\Delta + \mathbf{u}_\infty \cdot \nabla)$, one has

$$\mathbf{u}_t + \mathcal{O}_{\mathbf{u}_\infty}\mathbf{u} = 0, \quad \text{for } t > 0,\ \mathbf{u}(0) = \mathbf{u}_0, \tag{1.6}$$

where the domain of $\mathcal{O}_{\mathbf{u}_\infty}$ is given by

$$\mathfrak{D}_p(\mathcal{O}_{\mathbf{u}_\infty}) = \left\{u \in J_p(\Omega) \cap W_p^2(\Omega)^n : u|_{\partial\Omega} = 0\right\}. \tag{1.7}$$

Then, Enomoto and Shibata [14] proved that $\mathcal{O}_{\mathbf{u}_\infty}$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ which is called the Oseen semigroup (one can also refer to [16, 18]) and obtained the following properties.

Proposition 1.1. *Let* $\sigma_0 > 0$ *and assume that* $|\mathbf{u}_\infty| \leq \sigma_0$. *Let* $1 \leq r \leq q \leq \infty$, *then*

$$\|T(t)a\|_{L^q(\Omega)} \leq C_{r,q,\sigma_0} t^{-(3/2)(1/r-1/q)} \|a\|_{L^r(\Omega)}, \quad t > 0, \tag{1.8}$$

where $(r,q) \neq (1,1)$ *and* (∞,∞),

$$\|\nabla T(t)a\|_{L^q(\Omega)} \leq C_{r,q,\sigma_0} t^{-(3/2)(1/r-1/q)-1/2} \|a\|_{L^r(\Omega)}, \quad t > 0, \tag{1.9}$$

where $1 \leq r \leq q \leq 3$ *and* $(r,q) \neq (1,1)$.

The main purpose of this paper is to discuss the temporal stability of stationary solution $\mathbf{w}$ of the nonlinear Navier-Stokes equation (1.2). One can note that $\mathbf{w}$ satisfies the following equations:

$$\begin{gathered} -\Delta \mathbf{w} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p_2 = \mathbf{f}, \qquad \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}|_{\partial\Omega} = -\mathbf{u}_\infty, \qquad \lim_{|x|\to\infty} \mathbf{w}(x) = 0. \end{gathered} \tag{1.10}$$

For suitable $\mathbf{f}$, Shibata [19] proved that, for any given $0 < \delta < 1/4$, there exists ϵ such that if $0 < |\mathbf{u}_\infty| \leq \epsilon$, then one has

$$\|\mathbf{w}\|_{p,2} + |||\mathbf{w}|||_\delta + \|p_2\|_{p,1} \leq |\mathbf{u}_\infty|^\beta, \tag{1.11}$$

where

$$\begin{gathered} \|\mathbf{u}\|_{p,m} = \|\partial^m \mathbf{u}\|_{L^p(\Omega)}, \\ |||\mathbf{u}|||_\delta = \sup_{x\in\Omega}(1+|x|)(1+s_{\mathbf{u}_\infty}(x))^\delta |\mathbf{u}(x)| + \sup_{x\in\Omega}(1+|x|)^{3/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+\delta}|\nabla \mathbf{u}(x)|, \\ s_{\mathbf{u}_\infty}(x) = |x| - x^T \cdot \frac{\mathbf{u}_\infty}{|\mathbf{u}_\infty|} \qquad \delta < \beta < 1-\delta. \end{gathered} \tag{1.12}$$

Throughout this paper, we assume that $\mathbf{f}$ satisfies the assumption in Shibata [19]. Now, we consider the polar coordinate system

$$y_1 = r\cos\theta, \qquad y_2 = r\sin\theta\,\cos\phi, \qquad y_3 = r\sin\theta\,\sin\phi, \tag{1.13}$$

for $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq r < \infty$. Let S be an orthogonal matrix such that $S\mathbf{u}_\infty = |\mathbf{u}_\infty|(1,0,0)^T$ and put $s(y) = |y| - y_1$. By a change of variable $y = Sx$,

$$|x| = |y| = r, \qquad s_{\mathbf{u}_\infty}(x) = s(y) = r(1-\cos\theta). \tag{1.14}$$

See Shibata [19] for the detail. Now, by using the above change of variable, we can see easily that $\mathbf{w}$ satisfies

$$\|\mathbf{w}\|_{L^{3/(1+\delta_1)}(\Omega)} + \|\mathbf{w}\|_{L^{3/(1-\delta_2)}(\Omega)} + \|\nabla\mathbf{w}\|_{L^{3/(2+\delta_1)}(\Omega)} + \|\nabla\mathbf{w}\|_{L^{3/(2-\delta_2)}(\Omega)} \leq C|\mathbf{u}_\infty|^{1/2}, \tag{1.15}$$

for small δ_1, δ_2, where C is independent on $\mathbf{u}_\infty$.

One can also refer to [20] for more general cases of the existence and regularity of stationary Navier-Stokes equations.

For the stability of stationary solutions $\mathbf{w}$, by setting $\mathbf{u} = \mathbf{v} - \mathbf{w}$ and $p = p_1 - p_2$ for $\mathbf{v}$, p_1, $\mathbf{w}$, p_2 in (1.2) and (1.10), we have the following equations in Ω:

$$\begin{gathered}
\frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0, \\
\nabla \cdot \mathbf{u} = 0, \\
\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{for } x \in \Omega, \\
\mathbf{u}(x,t) = 0 \quad \text{for } x \in \partial\Omega, \qquad \lim_{|x|\to\infty} \mathbf{u}(x,t) = 0.
\end{gathered} \tag{1.16}$$

Here, in fact, the initial data should be $\mathbf{u}_0 - \mathbf{u}_\infty - \mathbf{w}$, but for our convenience, we denote by $\mathbf{u}_0$ for $\mathbf{u}_0 - \mathbf{u}_\infty - \mathbf{w}$ if there is no confusion.

First, Heywood [21, 22] and Masuda [23] have studied the temporal stability in L^2 space. Shibata [19] proved that there exists small ϵ such that if $0 < |\mathbf{u}_\infty| \leq \epsilon$ and $\|\mathbf{u}_0\|_3 \leq \epsilon$, then a unique solution $\mathbf{u}(x,t)$ of (1.16) has the following properties: for any $3 < p < \infty$,

$$\begin{gathered}
[\mathbf{u}]_{3,0,t} + [\mathbf{u}]_{p,\mu(p),t} + [\nabla\mathbf{u}]_{3,1/2,t} \leq \sqrt{\epsilon}, \\
\lim_{t\to 0^+}\left[\|\mathbf{u}(t) - \mathbf{u}_0\|_3 + [\mathbf{u}]_{p,\mu(p),t} + [\nabla\mathbf{u}]_{3,1/2,t}\right] = 0,
\end{gathered} \tag{1.17}$$

where

$$[\mathbf{z}]_{p,\rho,t} = \sup_{0<s<t} s^\rho \|\mathbf{z}(s,\cdot)\|_p, \qquad \mu(p) = \frac{1}{2} - \frac{3}{2p}. \tag{1.18}$$

After that, Enomoto and Shibata [14] considered the stability for arbitrary $\mathbf{u}_\infty$ by deleting the smallness condition of $|\mathbf{u}_\infty|$. But in this case, all constants in their results depend on σ_0 when $|\mathbf{u}_\infty| \leq \sigma_0$. Also, they assumed the existence of stationary solution $\mathbf{w}$ with

$$\|\mathbf{w}\|_{L^{3/(1+\delta_1)}(\Omega)} + \|\mathbf{w}\|_{L^{3/(1-\delta_2)}(\Omega)} + \|\nabla\mathbf{w}\|_{L^{3/(2+\delta_1)}(\Omega)} + \|\nabla\mathbf{w}\|_{L^{3/(2-\delta_2)}(\Omega)} \leq \alpha, \tag{1.19}$$

for small δ_1, δ_2 and α. Then, as a result, they proved (1.16) has a unique solution $\mathbf{u}(x,t)$ with

$$\begin{gathered}
\lim_{t\to 0^+}\left\{\|\mathbf{u}(t) - \mathbf{u}_0\|_3 + t^{1/2}(\|\mathbf{u}(t)\|_{L^\infty} + \|\nabla\mathbf{u}(t)\|_{L^3})\right\} = 0, \\
\|\mathbf{u}(t)\|_{\mathbf{u}(t)} = o\left(t^{-((1/2)-(3/2p))}\right), \quad \text{for any } 3 \leq p \leq \infty, \\
\|\nabla\mathbf{u}(t)\|_3 = o\left(t^{-1/2}\right)
\end{gathered} \tag{1.20}$$

as $t \to \infty$ when $\mathbf{u}_0$ is small enough in the space $L^3(\Omega)$.

Also, Bae and Roh [24] improved Enomoto-Shibata's result in some sense. But their result is limited in the space L^p for $3/2 < p$, while we consider all $1 < p$. Moreover, their result depends on s and r, while ours only depends on r, where $\mathbf{w} \in L^s$ and $\mathbf{u}_0 \in L^r$. Also, their optimal decay rate is $2/3 + \delta$, while ours is $3/2 + \delta$.

Now, in the next main Theorem, we settle the temporal stability of stationary solutions for the Navier-Stokes equations with a nonzero constant vector at infinity. The idea of the proof is initiated by Kato [25] for $\mathbf{w} = 0$ and a very well-known method. Also, for $\mathbf{w} \neq 0$ with $\mathbf{u}_\infty = 0$, Kozono and Ogawa [12] also used similar method.

Theorem 1.2. *There exists small $\epsilon(p, q, r)$ such that if $0 < |\mathbf{u}_\infty| \le \epsilon$ and $\|\mathbf{u}_0\|_{L^3(\Omega)} < \epsilon$, then a unique solution $\mathbf{u}(x, t)$ of (1.16) has the following properties:*

$$\begin{aligned} \|\mathbf{u}(t)\|_{L^p(\Omega)} &\le C_\epsilon t^{-3/2(1/r-1/p)} \|\mathbf{u}_0\|_r \quad \text{for } 1 < r < p \le \infty,\ t > 0, \\ \|\nabla \mathbf{u}(t)\|_{L^q(\Omega)} &\le C_\epsilon t^{-3/2(1/r-1/q)-1/2} \|\mathbf{u}_0\|_r \quad \text{for } 1 < r < q \le 3,\ t > 0, \end{aligned} \tag{1.21}$$

where $\mathbf{u}_0 \in L^3(\Omega) \cap L^r(\Omega)$.

2. Proof of Main Theorem

First, we consider the following linear problem:

$$\begin{gathered} \frac{\partial}{\partial t}\mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + (\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w} + \nabla p = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}|_{\partial\Omega} = 0, \qquad \lim_{|x| \to \infty} \mathbf{u}(x, t) = 0. \end{gathered} \tag{2.1}$$

By applying Helmholtz-Leray projection P and setting

$$\begin{aligned} \mathcal{L}\mathbf{u} &= P[-\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + (\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w}] \\ &= \mathcal{O}_{\mathbf{u}_\infty}\mathbf{u} + P[(\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w}], \end{aligned} \tag{2.2}$$

we have

$$\mathbf{u}_t + \mathcal{L}\mathbf{u} = 0, \quad \text{for } t > 0,\ \mathbf{u}(0) = \mathbf{u}_0. \tag{2.3}$$

And we note that the domain of $\mathcal{L}$ is

$$\mathfrak{D}_p(\mathcal{L}) = \mathfrak{D}_p(\mathcal{O}_{\mathbf{u}_\infty}) = \left\{ u \in J_p(\Omega) \cap W_p^2(\Omega)^n |u|_{\partial\Omega} = 0 \right\}. \tag{2.4}$$

Let $S(t)$ be a semigroup generated by the linear operator $\mathcal{L}$, then, by Duharmel's Principle, a solution $\mathbf{u}(x,t)$ of (2.1) can be written as in the following integral form,

$$\mathbf{u}(x,t) = S(t)\mathbf{u}_0 = T(t)\mathbf{u}_0 + \int_0^t T(t-\tau)P[(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}]\, d\tau, \tag{2.5}$$

where $T(t)$ is an analytic semigroup generated by the Oseen operator $\mathcal{O}_{\mathbf{u}_\infty}$.

Lemma 2.1. *Let $\mathbf{u}_0 \in L^3(\Omega)\cap L^r(\Omega)$ for $1 < r < 3$, then there exists a small $\epsilon(p,q,r)$ such that if $|\mathbf{u}_\infty| \le \epsilon$ and $\|\mathbf{u}_0\|_{L^3(\Omega)} < \epsilon$, then a solution $\mathbf{u}(x,t)$ represented by* (2.5) *satisfies $1 < p \le \infty$ with $1/r - 1/p < 2/3$,*

$$\|\mathbf{u}(t)\|_{L^p(\Omega)} = \|S(t)\mathbf{u}_0\|_{L^p(\Omega)} \le C_\epsilon t^{-3/2(1/r-1/p)}\|\mathbf{u}_0\|_{L^r(\Omega)}, \quad t > 0, \tag{2.6}$$

and for $1 < q \le 3$ with $1/r - 1/q < 1/3$,

$$\|\nabla\mathbf{u}(t)\|_{L^q(\Omega)} = \|\nabla S(t)\mathbf{u}_0\|_{L^q(\Omega)} \le C_\epsilon t^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_{L^r(\Omega)}, \quad t > 0. \tag{2.7}$$

Proof. Before we prove Lemma 2.1 note from (1.15) that we have

$$\|\mathbf{w}\|_{L^{3/(1+\delta_1)}(\Omega)} + \|\mathbf{w}\|_{L^{3/(1-\delta_2)}(\Omega)} + \|\nabla\mathbf{w}\|_{L^{3/(2+\delta_1)}(\Omega)} + \|\nabla\mathbf{w}\|_{L^{3/(2-\delta_2)}(\Omega)} \le C|\mathbf{u}_\infty|^{1/2}, \tag{2.8}$$

for small $\delta_1,\delta_2 > 0$. In fact, by straight calculations, we can choose any $\delta_1,\delta_2 \le 3/16$.

Step 1. Let $3 < p \le \infty$ with $1/3 \le 1/r - 1/p < 2/3$ and $3/2 < q \le 3$ with $1/r - 1/q < 1/3$. We consider the following iteration method to obtain our estimates:

$$\mathbf{u}_{k+1}(t) = T(t)\mathbf{u}_0 + \int_0^t T(t-\tau)P[(\mathbf{w}\cdot\nabla)\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{w}]d\tau. \tag{2.9}$$

We let $1/q - 1/p = 1/3$ and

$$M_p^k = \sup_{t\in[0,\infty)} t^{n/2(1/r-1/p)}\left\|u^k(t)\right\|_p, \qquad N_q^k = \sup_{t\in(0,\infty)} t^{n/2(1/r-1/q)+1/2}\left\|\nabla u^k(t)\right\|_q. \tag{2.10}$$

If $t \ge 2$, then by Proposition 1.1, for small $\delta_1,\delta_2 > 0$, we have

$$\begin{aligned}
&\int_0^t \|T(t-\tau)P[(\mathbf{w}\cdot\nabla)\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{w}]\|_p d\tau \\
&\quad \le C\left[\int_0^{t-1} (t-\tau)^{-n/2(1/r_1-1/p)}\|(\mathbf{w}\cdot\nabla)\mathbf{u}_k\|_{r_1} d\tau + \int_{t-1}^t (t-\tau)^{-n/2(1/r_2-1/p)}\|(\mathbf{w}\cdot\nabla)\mathbf{u}_k\|_{r_2} d\tau \right.\\
&\qquad \left. + \int_0^{t-1} (t-\tau)^{-n/2(1/r_1-1/p)}\|(\mathbf{u}_k\cdot\nabla)\mathbf{w}\|_{r_1} d\tau + \int_{t-1}^t (t-\tau)^{-n/2(1/r_2-1/p)}\|(\mathbf{u}_k\cdot\nabla)\mathbf{w}\|_{r_2} d\tau\right]
\end{aligned}$$

$$\begin{aligned}
&\leq C|\mathbf{u}_\infty| N_q^k \left[\int_0^{t-1} (t-\tau)^{-1-\delta_1/2} \tau^{-3/2(1/r-1/q)-1/2} d\tau + \int_{t-1}^{t} (t-\tau)^{-1+\delta_2/2} \tau^{-3/2(1/r-1/q)-1/2} d\tau \right] \\
&\quad + C|\mathbf{u}_\infty| M_p^k \left[\int_0^{t-1} (t-\tau)^{-1-\delta_1/2} \tau^{-3/2(1/r-1/p)} d\tau + \int_{t-1}^{t} (t-\tau)^{-1+\delta_2/2} \tau^{-3/2(1/r-1/p)} d\tau \right] \\
&\leq C|\mathbf{u}_\infty| \left(M_p^k + N_q^k \right) t^{-3/2(1/r-1/p)},
\end{aligned} \tag{2.11}$$

where $1/r_1 = 1/p + 2/3 + \delta_1/3$ and $1/r_2 = 1/p + 2/3 - \delta_2/3$. If $0 < t < 2$, then we have

$$\begin{aligned}
&\int_0^t \| T(t-\tau) P[(\mathbf{w}\cdot\nabla)\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{w}] \|_p d\tau \\
&\quad \leq C\left[\int_0^t (t-\tau)^{-n/2(1/r_3-1/p)} \|(\mathbf{w}\cdot\nabla)\mathbf{u}_k\|_{r_3} d\tau + \int_0^t (t-\tau)^{-n/2(1/r_3-1/p)} \|(\mathbf{u}_k\cdot\nabla)\mathbf{w}\|_{r_3} d\tau \right] \\
&\quad \leq C|\mathbf{u}_\infty| \left(M_p^k + N_q^k \right) t^{-3/2(1/r-1/p)},
\end{aligned} \tag{2.12}$$

where $1/r_3 = 1/p + 2/3 - \delta_2/3$. So, we obtain

$$\|\mathbf{u}_{k+1}(t)\|_p \leq C t^{-n/2(1/r-1/p)} \|\mathbf{u}_0\|_r + C|\mathbf{u}_\infty| t^{-3/2(1/r-1/p)} \left[M_p^k + N_q^k \right], \quad \forall t > 0, \tag{2.13}$$

which implies

$$M_p^{k+1} \leq C\|\mathbf{u}_0\|_r + C|\mathbf{u}_\infty| \left(M_p^k + N_q^k \right). \tag{2.14}$$

Similarly, we obtain for $t \geq 2$,

$$\begin{aligned}
\|\nabla \mathbf{u}_{k+1}(t)\|_q &\leq \|\nabla T(t)\mathbf{u}_0\|_q + \int_0^t \|\nabla T(t-\tau) P[(\mathbf{w}\cdot\nabla)\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{w}]\|_q d\tau \\
&\leq C t^{-n/2(1/r-1/q)-1/2} \|\mathbf{u}_0\|_r + C|\mathbf{u}_\infty| N_q^k \int_0^{t-1} (t-\tau)^{-1-\delta_1/2} \tau^{-3/2(1/r-1/q)-1/2} d\tau \\
&\quad + C|\mathbf{u}_\infty| N_q^k \int_{t-1}^{t} (t-\tau)^{-1+\delta_2/3} \tau^{-3/2(1/r-1/q)-1/2} d\tau \\
&\quad + C|\mathbf{u}_\infty| M_p^k \int_0^t (t-\tau)^{-n/2(1/r_4-1/q)-1/2} \tau^{-3/2(1/r-1/p)} d\tau \\
&\leq C t^{-n/2(1/r-1/q)-1/2} \|\mathbf{u}_0\|_r + C|\mathbf{u}_\infty| t^{-3/2(1/r-1/q)-1/2} \left[M_p^k + N_q^k \right],
\end{aligned} \tag{2.15}$$

where $1/r_4 = 2/3 + 1/p = 1/3 + 1/q$. Also, for $0 < t < 2$, we have

$$\begin{aligned}&\int_0^t \|\nabla T(t-\tau)P[(\mathbf{w}\cdot\nabla)\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{w}]\|_q d\tau \\ &\quad\le C|\mathbf{u}_\infty|\left(M_p^k + N_q^k\right)t^{-3/2(1/r-1/q)-1/2+\delta_2/2} \le C|\mathbf{u}_\infty|\left(M_p^k + N_q^k\right)t^{-3/2(1/r-1/q)-1/2}.\end{aligned} \tag{2.16}$$

Therefore, we get

$$M_p^{k+1} + N_q^{k+1} \le C\|\mathbf{u}_0\|_r + C|\mathbf{u}_\infty|\left(M_p^k + N_q^k\right). \tag{2.17}$$

So if $C|\mathbf{u}_\infty| < 1$ (the constant C is bounded as $|\mathbf{u}_\infty|$ goes to zero, so we can make $C|\mathbf{u}_\infty| < 1$ by choosing small $\mathbf{u}_\infty$), then we have some K such that

$$M_p^{k+1} + N_q^{k+1} < K, \tag{2.18}$$

for all k. Hence, by taking the limit, we complete the proof.

Step 2. Now, we want to prove $1 < r < p \le 3$. For this case, we choose $3/2 < q \le 3$ and $p_1 > 3$ such that

$$\frac{1}{r} - \frac{1}{q} < \frac{1}{3}, \qquad \frac{1}{r} - \frac{1}{p_1} < \frac{2}{3}. \tag{2.19}$$

Then, we have

$$\begin{aligned}\|\mathbf{u}(t)\|_p &\le \|T(t)\mathbf{u}_0\|_p + \int_0^t \|T(t-\tau)P[(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}]\|_p d\tau \\ &\le Ct^{-3/2(1/r-1/p)}\|\mathbf{u}_0\|_r + C\int_0^t (t-\tau)^{-3/2(1/r_1-1/p)}\|\mathbf{w}\|_3\|\nabla\mathbf{u}\|_q d\tau \\ &\quad + C\int_0^t (t-\tau)^{-3/2(1/r_2-1/p)}\|\mathbf{u}\|_{p_1}\|\nabla\mathbf{w}\|_{3/2} d\tau \\ &\le C_\epsilon t^{-3/2(1/r-1/p)}\|\mathbf{u}_0\|_r,\end{aligned} \tag{2.20}$$

where $1/r_1 = 1/3 + 1/q$ and $1/r_2 = 1/p_1 + 2/3$. One can note that $1/r_1 - 1/p < 2/3$ and $1/r_2 - 1/p < 2/3$.

Step 3. Now, we want to prove $1 < r < q \le 3/2$. For this case, we choose $3/2 < q_1 \le 3$ and $p > 3$ such that

$$\frac{1}{r} - \frac{1}{q_1} < \frac{1}{3}, \qquad \frac{1}{r} - \frac{1}{p} < \frac{2}{3}. \tag{2.21}$$

Similar to Step 2, we have

$$\begin{aligned}\|\nabla u(t)\|_q &\le \|\nabla T(t)\mathbf{u}_0\|_q + \int_0^t \|\nabla T(t-\tau)P[(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}]\|_q d\tau \\ &\le Ct^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r + C\int_0^t (t-\tau)^{-3/2(1/r_1-1/q)-1/2}\|\mathbf{w}\|_3\|\nabla\mathbf{u}\|_{q_1} d\tau \\ &\quad + C\int_0^t (t-\tau)^{-3/2(1/r_2-1/q)-1/2}\|\mathbf{u}\|_p\|\nabla\mathbf{w}\|_{3/2} d\tau \\ &\le Ct^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r,\end{aligned} \tag{2.22}$$

where $1/r_1 = 1/3 + 1/q_1$ and $1/r_2 = 1/p + 2/3$. One can note that $1/r_1 - 1/q < 1/3$ and $1/r_2 - 1/q < 1/3$.

Step 4. At last, we want to prove $3 < p < \infty$ with $1/r - 1/p < 1/3$. In this case, we can do easily, by interpolation inequality, Steps 1 and 2.

Therefore, we complete the proof by Steps 1–4. □

Now, by applying the Helmholtz-Leray projection P into (1.16), we can obtain

$$\mathbf{u}_t + \mathcal{L}\mathbf{u} + P[(\mathbf{u}\cdot\nabla)\mathbf{u}] = 0, \quad \text{for } t > 0,\ \mathbf{u}(0) = \mathbf{u}_0, \tag{2.23}$$

where

$$\begin{aligned}\mathcal{L}\mathbf{u} &= P[-\Delta\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla)\mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}] \\ &= \mathcal{O}_{\mathbf{u}_\infty}\mathbf{u} + P[(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}], \\ \mathfrak{D}_p(\mathcal{L}) &= \mathfrak{D}_p(\mathcal{O}_{\mathbf{u}_\infty}) = \left\{u \in J_p(\Omega)\cap W_p^2(\Omega)^n |u|_{\partial\Omega} = 0\right\}.\end{aligned} \tag{2.24}$$

One can note from of [14, Lemma 2.6] that for $1 < p < \infty$ and $\mathbf{u} \in \mathfrak{D}_p(\mathcal{L}) = \mathfrak{D}_p(\mathcal{O}_{\mathbf{u}_\infty})$,

$$\|\mathbf{u}\|_{W^{2,p}(\Omega)} \le C_p\left(\|\mathcal{O}_{\mathbf{u}_\infty}\mathbf{u}\|_p + \|\mathbf{u}\|_p\right). \tag{2.25}$$

Also, from (1.11), we have

$$\begin{aligned}\|(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}\|_p &\le (\|\mathbf{w}\|_\infty + \|\nabla\mathbf{w}\|_\infty)\|\mathbf{u}\|_{W^{2,p}(\Omega)} \\ &\le |\mathbf{u}_\infty|\|\mathbf{u}\|_{W^{2,p}(\Omega)} \le C_p|\mathbf{u}_\infty|\left(\|\mathcal{O}_{\mathbf{u}_\infty}\mathbf{u}\|_p + \|\mathbf{u}\|_p\right).\end{aligned} \tag{2.26}$$

Since the linear operator $\mathcal{O}_{\mathbf{u}_\infty}$ generates an analytic semigroup $T(t)$ (refer to [14, 19]), we obtain an analytic semigroup $S(t)$ generated by the linear operator $\mathcal{L}$ if $|\mathbf{u}_\infty|$ is small enough. The proof is from perturbation theory of analytic semigroup (refer to [26, Theorem 2.4, page 499]).

Remark 2.2. In Lemma 2.1, by the property of a semigroup, we can remove the conditions $1/r - 1/p < 2/3$ for $\|\mathbf{u}(t)\|_{L^p(\Omega)}$ and $1/r - 1/p < 1/3$ for $\|\nabla \mathbf{u}(t)\|_{L^p(\Omega)}$, because we have $\mathbf{u}(x,t) = S(t)\mathbf{u}_0 = S(t/2)S(t/2)\mathbf{u}_0$.

Now, we are in the position to prove Theorem 1.2. For the proof, we consider a solution $\mathbf{u}(x,t)$ (1.16) as the limit of the following usual iteration method:

$$\mathbf{u}_{k+1}(t) = S(t)\mathbf{u}_0 - \int_0^t S(t-\tau)P[(\mathbf{u}_k \cdot \nabla)\mathbf{u}_k]d\tau. \tag{2.27}$$

Here, we will prove by a similar method with the proof of Lemma 2.1. One can note that we will prove without Remark 2.2.

Step 1. We prove that, for any $p > 3$, we have

$$\|\nabla \mathbf{u}(t)\|_3 < Ct^{-1/2}, \quad \|\mathbf{u}(t)\|_p < Ct^{-1/2+3/2p}, \quad \forall t > 0. \tag{2.28}$$

Let

$$\begin{gathered} M_p^k = \sup_{t\in[0,\infty)} t^{1/2-3/2p}\left\|u^k(t)\right\|_p, \quad \text{for } p > 3, \\ N_3^k = \sup_{t\in(0,\infty)} t^{1/2}\left\|\nabla u^k(t)\right\|_3. \end{gathered} \tag{2.29}$$

By Lemma 2.1 and (2.27), we obtain

$$\begin{aligned} \|\mathbf{u}_{k+1}(t)\|_p &\le Ct^{-1/2+3/2p}\|\mathbf{u}_0\|_3 + C\int_0^t (t-\tau)^{-1/2}\|\mathbf{u}_k(t)\|_p\|\nabla \mathbf{u}_k(t)\|_3 d\tau \\ &\le Ct^{-1/2+3/2p}\|\mathbf{u}_0\|_3 + CM_p^k N_3^k \int_0^t (t-\tau)^{-1/2}\tau^{-1/2+3/2p}\tau^{-1/2}d\tau \\ &\le t^{-1/2+3/2p}\left[C\|\mathbf{u}_0\|_3 + CM_p^k N_3^k\right], \end{aligned} \tag{2.30}$$

which implies

$$M_p^{k+1} \le C\|\mathbf{u}_0\|_3 + CM_p^k N_3^k. \tag{2.31}$$

Similarly, we have

$$\|\nabla \mathbf{u}_{k+1}(t)\|_3 \le Ct^{-1/2}\|\mathbf{u}_0\|_3 + C\int_0^t (t-\tau)^{-3/2p-1/2}\|\mathbf{u}_k(t)\|_p\|\nabla \mathbf{u}_k(t)\|_3 d\tau \le t^{-1/2}\left[C\|\mathbf{u}_0\|_3 + CM_p^k N_3^k\right], \tag{2.32}$$

which implies

$$N_3^{k+1} \leq C\|\mathbf{u}_0\|_3 + CM_p^k N_3^k. \tag{2.33}$$

Hence, we have

$$M_p^{k+1} + N_3^{k+1} < C\|\mathbf{u}_0\|_3 + C\left(M_p^k + N_3^k\right)^2. \tag{2.34}$$

Now, we have a sequence of the form

$$x_{k+1} \leq \alpha + \beta x_k^2, \tag{2.35}$$

and we know that such sequence satisfies

$$x_k \leq M \equiv \frac{1 - (1 - 4\alpha\beta)^{1/2}}{2\beta} < \frac{1}{2\beta}, \quad \text{if } \alpha < \frac{1}{4\beta}. \tag{2.36}$$

Therefore, by recurrence estimates, smallness of $\|\mathbf{u}_0\|_3$ implies

$$M_p^{k+1} + N_3^{k+1} < K, \tag{2.37}$$

for some constant K. Finally, we obtain

$$\|\nabla \mathbf{u}(t)\|_3 < Ct^{-1/2}, \quad \|\mathbf{u}(t)\|_p < Ct^{-1/2+3/2p}, \quad \forall t > 0. \tag{2.38}$$

Step 2. We prove that if $3/2 < p$ with $1/r - 1/p < 1/3$ and $\mathbf{u}_0 \in L^r(\Omega) \cap L^3(\Omega)$, then we have

$$\|\mathbf{u}(t)\|_p \leq Ct^{-3/2(1/r-1/p)}, \quad \forall t > 0. \tag{2.39}$$

Let

$$M_p = \sup_{t\in(0,\infty)} t^{3/2(1/r-1/p)} \|u(t)\|_p. \tag{2.40}$$

From estimates of Step 1, one can note that we have

$$\|\nabla \mathbf{u}(t)\|_3 \leq Ct^{-1/2}\|\mathbf{u}_0\|_3, \quad \forall t > 0. \tag{2.41}$$

So, we have

$$\begin{aligned}\|\mathbf{u}(t)\|_p &\le Ct^{-3/2(1/r-1/p)}\|\mathbf{u}_0\|_r + C\int_0^t (t-\tau)^{-n/2(1/r_8-1/p)}\|\mathbf{u}(t)\|_p\|\nabla\mathbf{u}(t)\|_3 d\tau \\ &\le Ct^{-n/2(1/r-1/p)}\|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_3\int_0^t (t-\tau)^{-1/2}\tau^{-n/2(1/r-1/p)}\tau^{-1/2}d\tau \\ &\le t^{-n/2(1/r-1/p)}\left[C\|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_3 M_p\right],\end{aligned} \tag{2.42}$$

which implies

$$M_p < C\|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_3 M_p, \tag{2.43}$$

where $1/r_8 = 1/3 + 1/p$.

Hence, we complete the proof with $C\|\mathbf{u}_0\|_3 < 1$.

Step 3. We prove that if $3/2 < q \le 3$ with $1/r - 1/q < 1/3$ and $\mathbf{u}_0 \in L^r(\Omega) \cap L^3(\Omega)$, then we have

$$\|\nabla\mathbf{u}(t)\|_q \le Ct^{-3/2(1/r-1/q)-1/2}, \quad \forall t > 0. \tag{2.44}$$

Let

$$N_q = \sup_{t\in(0,\infty)} t^{n/2(1/r-1/q)+1/2}\|\nabla u(t)\|_q. \tag{2.45}$$

We choose some $p_1 \approx 3$ with $p_1 > 3$ such that

$$\begin{aligned}\|\nabla\mathbf{u}\|_q &\le Ct^{-n/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r + C\int_0^t (t-\tau)^{-n/2(1/r_7-1/q)-1/2}\|\mathbf{u}\|_{p_1}\|\nabla\mathbf{u}\|_q d\tau \\ &\le Ct^{-n/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_3 N_q\int_0^t (t-\tau)^{-1/2-3/2p}\tau^{-1/2+3/2p_1}\tau^{-n/2(1/r-1/q)-1/2}d\tau \\ &\le t^{-n/2(1/r-1/q)-1/2}\left[C\|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_3 N_q\right].\end{aligned} \tag{2.46}$$

So we complete the proof with $C\|\mathbf{u}_0\|_3 < 1$.

Step 4. We prove that if $1 < r < p < \infty$, $1 < r < 3$, and $\mathbf{u}_0 \in L^r(\Omega) \cap L^3(\Omega)$, then we have

$$\|\mathbf{u}(t)\|_p \le Ct^{-3/2(1/r-1/p)}, \quad \forall t > 0. \tag{2.47}$$

Case 1 (let $p > 3/2$). Since we proved for $1/r - 1/p < 1/3$ in Step 2, we can assume that $1/3 \le 1/r - 1/p$. One notes that we can rewrite solutions $\mathbf{u}(t)$ in the form

$$\mathbf{u}(t) = S\left(\frac{t}{2}\right)\mathbf{u}\left(\frac{t}{2}\right) - \int_{t/2}^t S(t-\tau)P[(\mathbf{u}\cdot\nabla)\mathbf{u}]d\tau. \tag{2.48}$$

For any $r > 1$, we choose $l > 3/2$ such that $1/r - 1/l < 1/3$ and $1/l - 1/p < 2/3$. Also, for any $1 < r < p \leq \infty$ with $1 < r < 3$, we choose $s_1 > 3$ and $3/2 < s_2 < 3$ such that

$$\frac{1}{r} - \frac{1}{s_2} < \frac{1}{3}, \qquad \frac{1}{s_1} + \frac{1}{s_2} - \frac{1}{p} < \frac{2}{3}. \tag{2.49}$$

Then, by Steps 1–3, we have

$$\begin{aligned} \|\mathbf{u}(t)\|_p &\leq Ct^{-3/2(1/l-1/p)} \left\| \mathbf{u}\left(\frac{t}{2}\right) \right\|_l + C\int_{t/2}^{t} (t-\tau)^{-3/2(1/s-1/p)} \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_s d\tau \\ &\leq Ct^{-3/2(1/r-1/p)} \|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_r \int_{t/2}^{t} (t-\tau)^{-3/2(1/s-1/p)} \tau^{-1/2-3/2(1/r-1/s_2)} \tau^{-1/2+3/2s_1} d\tau \\ &\leq Ct^{-3/2(1/r-1/p)} \|\mathbf{u}_0\|_r, \quad \forall t > 0. \end{aligned} \tag{2.50}$$

Case 2 (let $1 < p \leq 3/2$). By Step 1–3, we have

$$\begin{aligned} \|\mathbf{u}(t)\|_p &\leq Ct^{-3/2(1/r-1/p)} \|\mathbf{u}_0\|_r + C\int_{0}^{t} (t-\tau)^{-3/2(1/s-1/p)} \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_s d\tau \\ &\leq Ct^{-3/2(1/r-1/p)} \|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_r \int_{0}^{t} (t-\tau)^{-3/2(1/s-1/p)} \tau^{-3/2(1/r-1/s_1)} \tau^{-1/2} d\tau \\ &\leq Ct^{-3/2(1/r-1/p)} \|\mathbf{u}_0\|_r, \quad \forall t > 0, \end{aligned} \tag{2.51}$$

where $s_1 > 3/2$, $1/r - 1/s_1 < 1/3$, $1/s = 1/s_1 + 1/3$.

Step 5. We prove that if $1 < r < q \leq 3$ and $\mathbf{u}_0 \in L^r(\Omega) \cap L^3(\Omega)$, then

$$\|\nabla \mathbf{u}(t)\|_q \leq Ct^{-3/2(1/r-1/q)-1/2}. \tag{2.52}$$

Case 1 (let $3/2 < q \leq 3$). Since we proved $1/r - 1/q < 1/3$ in Step 3, we can assume that $1/3 \leq 1/r - 1/q$. Now, we choose $l > 3/2$ such that $1/r - 1/l < 1/3$ and $1/l - 1/q < 1/3$. We also can have $s_1 > 3$ and $3/2 < s_2 < 3$ with

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}, \qquad \frac{1}{r} - \frac{1}{s_2} < \frac{1}{3}, \qquad \frac{1}{s} - \frac{1}{q} < \frac{1}{3}. \tag{2.53}$$

So, by Step 1–4, we obtain

$$\begin{aligned}\|\nabla\mathbf{u}(t)\|_q &\le Ct^{-3/2(1/l-1/q)-1/2}\left\|\mathbf{u}\left(\frac{t}{2}\right)\right\|_l + C\int_{t/2}^{t}(t-\tau)^{-3/2(1/s-1/q)-1/2}\|\mathbf{u}(t)\|_{s_1}\|\nabla\mathbf{u}(t)\|_{s_2}d\tau\\ &\le Ct^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_r\\ &\quad\times\int_{t/2}^{t}(t-\tau)^{-3/2(1/s-1/q)-1/2}\tau^{-1/2+3/2s_1}\tau^{-3/2(1/r-1/s_2)-1/2}d\tau\\ &\le Ct^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r.\end{aligned}\tag{2.54}$$

Case 2 (Let $1 < q \le 3/2$). By Step 1-Step 3, we have

$$\begin{aligned}\|\nabla\mathbf{u}(t)\|_q &\le Ct^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r + C\int_0^t(t-\tau)^{-3/2(1/s-1/q)-1/2}\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_s d\tau\\ &\le Ct^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r + C\|\mathbf{u}_0\|_r\int_0^t(t-\tau)^{-3/2(1/s-1/q)-1/2}\tau^{-3/2(1/r-1/s_1)}\tau^{-1/2}d\tau\\ &\le Ct^{-3/2(1/r-1/q)-1/2}\|\mathbf{u}_0\|_r, \quad \forall t>0,\end{aligned}\tag{2.55}$$

where $s_1 > 3/2$, $1/r - 1/s_1 < 1/3$, $1/s = 1/s_1 + 1/3$, and $1/s - 1/q < 1/3$.

Therefore, by Step 1–5, we complete the proof of Theorem 1.2.

Acknowledgment

The author would like to thank Professor Hyeong-Ohk Bae for valuable comments. The author also thanks deeply Professors Cheng He and Lizhen Wang for the preprint [27] “Weighted L^p-Estimates for Stokes Flow in R^n_+, with Applications to the Non-Stationary Navier-Stokes Flow” which motivates the idea of the proof. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0023386).

References

[1] H.-O. Bae and B. J. Jin, “Asymptotic behavior for the Navier-Stokes equations in 2D exterior domains,” *Journal of Functional Analysis*, vol. 240, no. 2, pp. 508–529, 2006.
[2] H.-O. Bae and B. J. Jin, “Temporal and spatial decay rates of Navier-Stokes solutions in exterior domains,” *Bulletin of the Korean Mathematical Society*, vol. 44, no. 3, pp. 547–567, 2007.
[3] H.-O. Bae and J. Roh, “Weighted estimates for the incompressible fluid in exterior domains,” *Journal of Mathematical Analysis and Applications*, vol. 355, no. 2, pp. 846–854, 2009.
[4] H.-O. Bae and J. Roh, “Optimal weighted estimates of the flows in exterior domains,” *Nonlinear Analysis*, vol. 73, no. 5, pp. 1350–1363, 2010.
[5] W. Borchers and T. Miyakawa, “On stability of exterior stationary Navier-Stokes flows,” *Acta Mathematica*, vol. 174, no. 2, pp. 311–382, 1995.

[6] Z. M. Chen, “Solutions of the stationary and nonstationary Navier-Stokes equations in exterior domains,” *Pacific Journal of Mathematics*, vol. 159, no. 2, pp. 227–240, 1993.
[7] C. He, “Weighted estimates for nonstationary Navier-Stokes equations,” *Journal of Differential Equations*, vol. 148, no. 2, pp. 422–444, 1998.
[8] C. He and T. Miyakawa, “On weighted-norm estimates for nonstationary incompressible Navier-Stokes flows in a 3D exterior domain,” *Journal of Differential Equations*, vol. 246, no. 6, pp. 2355–2386, 2009.
[9] H. Iwashita, “L_q-L_p estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces,” *Mathematische Annalen*, vol. 285, no. 2, pp. 265–288, 1989.
[10] H. Kozono and T. Ogawa, “Some L^p estimate for the exterior Stokes flow and an application to the nonstationary Navier-Stokes equations,” *Indiana University Mathematics Journal*, vol. 41, no. 3, pp. 789–808, 1992.
[11] H. Kozono and T. Ogawa, “Two-dimensional Navier-Stokes flow in unbounded domains,” *Mathematische Annalen*, vol. 297, no. 1, pp. 1–31, 1993.
[12] H. Kozono and T. Ogawa, “On stability of Navier-Stokes flows in exterior domains,” *Archive for Rational Mechanics and Analysis*, vol. 128, no. 1, pp. 1–31, 1994.
[13] C. W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akademische, Leipzig, Germany, 1927.
[14] Y. Enomoto and Y. Shibata, “On the rate of decay of the Oseen semigroup in exterior domains and its application to Navier-Stokes equation,” *Journal of Mathematical Fluid Mechanics*, vol. 7, no. 3, pp. 339–367, 2005.
[15] D. Fujiwara and H. Morimoto, “An L_r-theorem of the Helmholtz decomposition of vector fields,” *Journal of the Faculty of Science, the University of Tokyo: Section I*, vol. IX, pp. 59–102, 1961.
[16] T. Miyakawa, “On nonstationary solutions of the Navier-Stokes equations in an exterior domain,” *Hiroshima Mathematical Journal*, vol. 12, no. 1, pp. 115–140, 1982.
[17] C. G. Simader and H. Sohr, “The Helmholtz decomposition in L^q and related topics,” in *Mathematical Problems Related to the Navier-Stokes Equations*, G. P. Galdi, Ed., vol. 11 of *Advances in Mathematics for Applied Science*, pp. 1–35, World Scientific Publishing, River Edge, NJ, USA, 1992.
[18] Y. Enomoto and Y. Shibata, “Local energy decay of solutions to the Oseen equation in the exterior domains,” *Indiana University Mathematics Journal*, vol. 53, no. 5, pp. 1291–1330, 2004.
[19] Y. Shibata, “On an exterior initial-boundary value problem for Navier-Stokes equations,” *Quarterly of Applied Mathematics*, vol. 57, no. 1, pp. 117–155, 1999.
[20] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. II: Nonlinear Steady Problem*, vol. 39 of *Springer Tracts in Natural Philosophy*, Springer, New York, NY, USA, 1994.
[21] J. G. Heywood, “On stationary solutions of the Navier-Stokes equations as limits of nonstationary solutions,” *Archive for Rational Mechanics and Analysis*, vol. 37, pp. 48–60, 1970.
[22] J. G. Heywood, “The exterior nonstationary problem for the Navier-Stokes equations,” *Acta Mathematica*, vol. 129, no. 1-2, pp. 11–34, 1972.
[23] K. Masuda, “On the stability of incompressible viscous fluid motions past objects,” *Journal of the Mathematical Society of Japan*, vol. 27, pp. 294–327, 1975.
[24] H.-O. Bae and J. Roh, “Stability for the 3D Navier-Stokes Equations with nonzero far field velocity on exterior domains,” *Journal of Mathematical Fluid Mechanics*. In press.
[25] T. Kato, “Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^n$, with applications to weak solutions,” *Mathematische Zeitschrift*, vol. 187, no. 4, pp. 471–480, 1984.
[26] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York, NY, USA, 1980.
[27] C. He and L. Wang, “Weighted L^p-estimates for Stokes flow in R^n_+ with applications to the nonstationary Navier-Stokes flow,” *Science China. Mathematics*, vol. 54, no. 3, pp. 573–586, 2011.