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*Research Article*

# A Note on the Generalized $q$-Bernoulli Measures with Weight $\alpha$

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We discuss a new concept of the $q$-extension of Bernoulli measure. From those measures, we derive some interesting properties on the generalized $q$-Bernoulli numbers with weight $\alpha$ attached to $\chi$.

## 1. Introduction

Let $p$ be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$ (see [1–14]).

When we talk of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. Throughout this paper we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, and we use the notation of $q$-number as

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.1}$$

(see [1–14]). Thus, we note that $\lim_{q \to 1}[x]_q = x$.

In [2], Carlitz defined a set of numbers $\xi_k = \xi_k(q)$ inductively by

$$\xi_0 = 1, \qquad (q\xi+1)^k - \xi_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.2}$$

with the usual convention of replacing $\xi^k$ by $\xi_k$.

These numbers are $q$-extension of ordinary Bernoulli numbers $B_k$. But they do not remain finite when $q = 1$. So he modified (1.2) as follows:

$$\beta_{0,q} = 1, \qquad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.3}$$

with the usual convention of replacing $\beta^k$ by $\beta_{k,q}$.

The numbers $\beta_{k,q}$ are called the $k$-th Carlitz $q$-Bernoulli numbers.

In [1], Carlitz also considered the extended Carlitz's $q$-Bernoulli numbers as follows:

$$\beta^h_{0,q} = \frac{h}{[h]_q}, \qquad q^h\left(q\beta^h + 1\right)^k - \beta^h_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.4}$$

with the usual convention of replacing $(\beta^h)^k$ by $\beta^h_{k,q}$.

Recently, Kim considered $q$-Bernoulli numbers, which are different extended Carlitz's $q$-Bernoulli numbers, as follows: for $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,

$$\widetilde{\beta}^{(\alpha)}_{0,q} = 1, \quad q\left(q^\alpha \widetilde{\beta}^{(\alpha)} + 1\right)^n - \widetilde{\beta}^{(\alpha)}_{n,q} = \begin{cases} \dfrac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.5}$$

with the usual convention of replacing $(\widetilde{\beta}^{(\alpha)})^k$ by $\widetilde{\beta}^{(\alpha)}_{k,q}$ (see [3]).

The numbers $\widetilde{\beta}^{(\alpha)}_{k,q}$ are called the $k$-th $q$-Bernoulli numbers with weight $\alpha$.

For fixed $d \in \mathbb{Z}_+$ with $(p, d) = 1$, we set

$$\begin{gathered} X = X_d = \varprojlim_N \left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \qquad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\bmod\ dp^N\right)\right\}, \end{gathered} \tag{1.6}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \tag{1.7}$$

(see [3, 4, 15, 16]). By (1.5) and (1.7), the Witt's formula for the $q$-Bernoulli numbers with weight $\alpha$ is given by

$$\int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x) = \widetilde{\beta}_{n,q}^{(\alpha)}, \quad \text{where } n \in \mathbb{Z}_+. \tag{1.8}$$

The $q$-Bernoulli polynomials with weight $\alpha$ are also defined by

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \widetilde{\beta}_{l,q}^{(\alpha)}. \tag{1.9}$$

From (1.7), (1.8), and (1.9), we can derive the Witt's formula for $\widetilde{\beta}_{n,q}^{(\alpha)}(x)$ as follows:

$$\int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(y) = \widetilde{\beta}_{n,q}^{(\alpha)}(x), \quad \text{where } n \in \mathbb{Z}_+. \tag{1.10}$$

For $n \in \mathbb{Z}_+$ and $d \in \mathbb{N}$, the distribution relation for the $q$-Bernoulli polynomials with weight $\alpha$ are known that

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} q^a \widetilde{\beta}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right), \tag{1.11}$$

(see [3]). Recently, several authors have studied the $p$-adic $q$-Euler and Bernoulli measures on $\mathbb{Z}_p$ (see [8, 9, 11, 13, 14]). The purpose of this paper is to construct $p$-adic $q$-Bernoulli distribution with weight $\alpha$ (= $p$-adic $q$-Bernoulli unbounded measure with weight $\alpha$) on $\mathbb{Z}_p$ and to study their integral representations. Finally, we construct the generalized $q$-Bernoulli numbers with weight $\alpha$ and investigate their properties related to $p$-adic $q$-$L$-functions.

## 2. $p$-Adic $q$-Bernoulli Distribution with Weight $\alpha$

Let $X$ be any compact-open subset of $\mathbb{Q}_p$, such as $\mathbb{Z}_p$ or $\mathbb{Z}_p^*$. A $p$-adic distribution $\mu$ on $X$ is defined to be an additive map from the collection of compact open set in $X$ to $\mathbb{Q}_p$:

$$\mu\left(\bigcup_{k=1}^{n} U_k\right) = \sum_{k=1}^{n} \mu(U_k) (\text{additivity}), \tag{2.1}$$

where $\{U_1, U_2, \dots, U_n\}$ is any collection of disjoint compact opensets in $X$.

The set $\mathbb{Z}_p$ has a topological basis of compact open sets of the form $a + p^n \mathbb{Z}_p$.

Consequently, if $U$ is any compact open subset of $\mathbb{Z}_p$, it can be written as a finite disjoint union of sets

$$U = \bigcup_{j=1}^{k} (a_j + p^n \mathbb{Z}_p), \tag{2.2}$$

where $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_k \in \mathbb{Z}$ with $0 \le a_i < p^n$ for $i = 1, 2, \dots k$.

Indeed, the $p$-adic ball $a+p^n\mathbb{Z}_p$ can be represented as the union of smaller balls

$$a+p^n\mathbb{Z}_p=\bigcup_{b=0}^{p-1}\left(a+bp^n+p^{n+1}\mathbb{Z}_p\right). \tag{2.3}$$

**Lemma 2.1.** *Every map $\mu$ from the collection of compact-open sets in $X$ to $\mathbb{Q}_p$ for which*

$$\mu\left(a+p^N\mathbb{Z}_p\right)=\bigcup_{b=0}^{p-1}\left(a+bp^N+dp^{N+1}\mathbb{Z}_p\right) \tag{2.4}$$

*holds whenever $a+p^N\mathbb{Z}_p\subset X$, extends to a $p$-adic distribution (= $p$-adic unbounded measure) on $X$.*

Now we define a map $\mu_{k,q}^{(\alpha)}$ on the balls in $\mathbb{Z}_p$ as follows:

$$\mu_{k,q}^{(\alpha)}(a+p^n\mathbb{Z}_p)=\frac{[p^n]_{q^\alpha}^k}{[p^n]_q}q^a f_{k,q^{p^n}}^{(\alpha)}\left(\frac{\{a\}_n}{p^n}\right), \tag{2.5}$$

where $\{a\}_n$ is the unique number in the set $\{0,1,2,\ldots,p^n-1\}$ such that $\{a\}_n\equiv a(\text{mod }p^n)$.

If $a\in\{0,1,2,\ldots,p^n-1\}$, then

$$\begin{aligned}\sum_{b=0}^{p-1}\mu_{k,q}^{(\alpha)}\left(a+bp^n+p^{n+1}\mathbb{Z}_p\right)&=\sum_{b=0}^{p-1}\frac{[p^{n+1}]_{q^\alpha}^k}{[p^{n+1}]_q}q^{a+bp^n}f_{k,q^{p^{n+1}}}^{(\alpha)}\left(\frac{a+bp^n}{p^{n+1}}\right)\\&=q^a\frac{[p^n]_{q^\alpha}^k}{[p^n]_q}\frac{[p]_{(q^{p^n})^\alpha}^k}{[p]_{q^{p^n}}}\sum_{b=0}^{p-1}q^{bp^n}f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{(a/p^n)+b}{p}\right).\end{aligned} \tag{2.6}$$

From (2.6), we note that $\mu_{k,q}^{(\alpha)}$ is $p$-adic distribution on $\mathbb{Z}_p$ if and only if

$$\frac{[p]_{(q^{p^n})^\alpha}^k}{[p]_{q^{p^n}}}\sum_{b=0}^{p-1}q^{bp^n}f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{(a/p^n)+b}{p}\right)=f_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right). \tag{2.7}$$

**Theorem 2.2.** *Let $\alpha\in\mathbb{N}$ and $k\in\mathbb{Z}_+$. Then we see that $\mu_{k,q}^{(\alpha)}(a+p^n\mathbb{Z}_p)$ is $p$-adic distribution on $\mathbb{Z}_p$ if and only if*

$$\frac{[p]_{(q^{p^n})^\alpha}^k}{[p]_{q^{p^n}}}\sum_{b=0}^{p-1}q^{bp^n}f_{k,(q^{p^n})^p}^{(\alpha)}\left(\frac{(a/p^n)+b}{p}\right)=f_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right). \tag{2.8}$$

*One sets*

$$f_{k,q^{p^n}}^{(\alpha)}(x)=\widetilde{\beta}_{k,q^{p^n}}^{(\alpha)}(x). \tag{2.9}$$

*From* (2.5) *and* (2.9), *one gets*

$$\mu_{k,q}^{(\alpha)}(a+p^n\mathbb{Z}_p)=\frac{[p^n]_{q^\alpha}^k}{[p^n]_q}q^a\widetilde{\beta}_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right). \tag{2.10}$$

By (1.11), (2.10), and Theorem 2.2, we obtain the following theorem.

**Theorem 2.3.** *Let* $\mu_{k,q}^{(\alpha)}$ *be given by*

$$\mu_{k,q}^{(\alpha)}\left(a+dp^N\mathbb{Z}_p\right)=\frac{[dp^N]_{q^\alpha}^k}{[dp^N]_q}q^a\widetilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{a}{dp^N}\right). \tag{2.11}$$

*Then* $\mu_{k,q}^{(\alpha)}$ *extends to a* $\mathbb{Q}(q)$*-valued distribution on the compact open sets* $U\subset X$.

*From* (2.11), *one notes that*

$$\begin{aligned}\int_X d\mu_{k,q}^{(\alpha)}(x)&=\lim_{N\to\infty}\sum_{x=0}^{dp^N-1}\mu_{k,q}^{(\alpha)}\left(x+dp^N\mathbb{Z}_p\right)\\&=\lim_{N\to\infty}\frac{[dp^N]_{q^\alpha}^k}{[dp^N]_q}\sum_{a=0}^{dp^N-1}q^a\widetilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{a}{dp^N}\right).\end{aligned} \tag{2.12}$$

*By* (1.11) *and* (2.12), *one gets*

$$\int_X d\mu_{k,q}^{(\alpha)}(x)=\widetilde{\beta}_{k,q}^{(\alpha)}. \tag{2.13}$$

Therefore, we obtain the following theorem.

**Theorem 2.4.** *For* $\alpha\in\mathbb{N}$ *and* $k\in\mathbb{Z}_+$, *one has*

$$\int_X d\mu_{k,q}^{(\alpha)}(x)=\widetilde{\beta}_{k,q}^{(\alpha)}. \tag{2.14}$$

*Let* $\chi$ *be Dirichlet character with conductor* $d\in\mathbb{N}$. *Then one defines the generalized* $q$*-Bernoulli numbers attached to* $\chi$ *as follows:*

$$\begin{aligned}\widetilde{\beta}_{n,\chi,q}^{(\alpha)}&=\int_X\chi(x)[x]_{q^\alpha}^n d\mu_q(x)\\&=\frac{[d]_{q^\alpha}^n}{[d]_q}\sum_{a=0}^{d-1}q^a\chi(a)\widetilde{\beta}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right).\end{aligned} \tag{2.15}$$

*From* (2.11) *and* (2.15), *one can derive the following equation;*

$$\begin{aligned}
\int_X \chi(x)\,d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N\to\infty}\sum_{x=0}^{dp^N-1}\chi(x)\mu_{k,q}^{(\alpha)}\left(x+dp^N\mathbb{Z}_p\right) \\
&= \lim_{N\to\infty}\frac{[dp^N]_{q^\alpha}^k}{[dp^N]_q}\sum_{x=0}^{dp^N-1}\chi(x)q^x\widetilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{x}{dp^N}\right) \\
&= \frac{[d]_{q^\alpha}^k}{[d]_q}\sum_{a=0}^{d-1}q^a\chi(a)\left\{\lim_{N\to\infty}\frac{[p^N]_{q^{\alpha d}}^k}{[p^N]_{q^d}}\sum_{x=0}^{p^N-1}q^{dx}\widetilde{\beta}_{k,q^{dp^N}}\left(\frac{(a/d)+x}{p^N}\right)\right\} \\
&= \frac{[d]_{q^\alpha}^k}{[d]_q}\sum_{a=0}^{d-1}q^a\chi(a)\widetilde{\beta}_{k,q^d}^{(\alpha)}\left(\frac{a}{d}\right) = \widetilde{\beta}_{k,\chi,q}^{(\alpha)}, \\
\int_{pX} \chi(x)\,d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N\to\infty}\frac{[dp^{N+1}]_{q^\alpha}^k}{[dp^{N+1}]_q}\sum_{x=0}^{dp^N-1}\chi(px)q^{px}\widetilde{\beta}_{k,q^{dp^{N+1}}}^{(\alpha)}\left(\frac{px}{dp^{N+1}}\right) \\
&= \frac{[p]_{q^\alpha}^k}{[p]_q}\frac{[d]_{q^{p\alpha}}^k}{[d]_{q^p}}\sum_{a=0}^{d-1}\chi(pa)q^{pa}\lim_{N\to\infty}\frac{[p^N]_{q^{dp\alpha}}^k}{[p^N]_{q^{dp}}}\sum_{x=0}^{p^N-1}q^{pdx}\widetilde{\beta}_{k,q^{pdp^N}}^{(\alpha)}\left(\frac{p(xd+a)}{pdp^N}\right) \\
&= \frac{[p]_{q^\alpha}^k}{[p]_q}\frac{[d]_{q^{\alpha p}}^k}{[d]_{q^p}}\sum_{a=0}^{d-1}\chi(p)\chi(a)q^{pa}\widetilde{\beta}_{k,q^{pd}}^{(\alpha)}\left(\frac{a}{d}\right) = \chi(p)\frac{[p]_{q^\alpha}^k}{[p]_q}\widetilde{\beta}_{k,\chi,q^p}^{(\alpha)}.
\end{aligned} \tag{2.16}$$

*For* $\beta(\neq 1)\in X^*$, *one has*

$$\begin{aligned}
\int_{pX}\chi(x)\,d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{p}{\beta}\right)\frac{[p]_{q^{\alpha/\beta}}^k}{[p]_{q^{1/\beta}}}\widetilde{\beta}_{k,\chi,q^{p/\beta}}^{(\alpha)}, \\
\int_X\chi(x)\,d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{1}{\beta}\right)\widetilde{\beta}_{k,\chi,q^{1/\beta}}^{(\alpha)}.
\end{aligned} \tag{2.17}$$

Therefore, we obtain the following theorem.

**Theorem 2.5.** *For* $\beta(\neq 1)\in X^*$, *one has*

$$\int_X\chi(x)\,d\mu_{k,q}^{(\alpha)}(x) = \widetilde{\beta}_{k,\chi,q}^{(\alpha)},$$

$$\int_{pX}\chi(x)\,d\mu_{k,q}^{(\alpha)}(x) = \chi(p)\frac{[p]_{q^\alpha}^k}{[p]_q}\widetilde{\beta}_{k,\chi,q^p}^{(\alpha)},$$

$$\int_{pX}\chi(x)d\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x)=\chi\left(\frac{p}{\beta}\right)\frac{[p]^k_{q^{\alpha/\beta}}}{[p]_{q^{1/\beta}}}\widetilde{\beta}^{(\alpha)}_{k,\chi,q^{p/\beta}},$$
$$\int_{X}\chi(x)d\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x)=\chi\left(\frac{1}{\beta}\right)\widetilde{\beta}^{(\alpha)}_{k,\chi,q^{1/\beta}}. \tag{2.18}$$

*Define*

$$\mu^{(\alpha)}_{k,\beta,q}(U)=\mu^{(\alpha)}_{k,q}(U)-\beta^{-1}\frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]_q}\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta U). \tag{2.19}$$

*By a simple calculation, one gets*

$$\int_{X^*}\chi(x)d\mu^{(\alpha)}_{k,\beta,q}(x)=\int_{X}\chi(x)d\mu^{(\alpha)}_{k,q}(x)-\beta^{-1}\frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]_q}\int_{pX}\chi(x)\mu^{(\alpha)}_{k,q^{1/\beta}}(x)$$
$$=\widetilde{\beta}^{(\alpha)}_{k,\chi,q}-\chi(p)\frac{[p]^k_{q^\alpha}}{[p]_q}\widetilde{\beta}^{(\alpha)}_{k,\chi,q^p},$$
$$\frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]^k_q}\int_{X^*}\chi(x)d\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x)=\frac{[1/\beta]^k_{q^\alpha}}{[1/\beta]_q}\chi\left(\frac{1}{\beta}\right)\widetilde{\beta}^{(\alpha)}_{k,\chi,q^{1/\beta}}$$
$$-\chi\left(\frac{p}{\beta}\right)\frac{[p/\beta]^k_{q^\alpha}}{[p/\beta]_q}\widetilde{\beta}^{(\alpha)}_{k,\chi,q^{p/\beta}}. \tag{2.20}$$

*By* (2.19) *and* (2.20), *one gets*

$$\int_{X^*}\chi(x)d\mu^{(\alpha)}_{k,\beta,q}(\beta x)=\int_{X}\chi(x)d\mu^{(\alpha)}_{k,q}(x)-\beta^{-1}\frac{[\beta^{-1}]^k_{q^\alpha}}{[\beta^{-1}]_q}\int_{pX}\chi(x)\mu^{(\alpha)}_{k,q^{1/\beta}}(\beta x)$$
$$=\widetilde{\beta}^{(\alpha)}_{k,\chi,q}-\chi(p)\frac{[p]^k_{q^\alpha}}{[p]_q}\widetilde{\beta}^{(\alpha)}_{k,\chi,q^p}-\frac{1}{\beta}\frac{[1/\beta]^k_{q^\alpha}}{[1/\beta]_q}\chi\left(\frac{1}{\beta}\right)\widetilde{\beta}^{(\alpha)}_{k,\chi,q^{1/\beta}} \tag{2.21}$$
$$+\chi\left(\frac{p}{\beta}\right)\frac{[p/\beta]^k_{q^\alpha}}{[p/\beta]_q}\widetilde{\beta}^{(\alpha)}_{k,\chi,q^{p/\beta}}.$$

*Now one defines the operator* $\chi^y=\chi^{y,k,\alpha:q}$ *on* $f(q)$ *by*

$$\chi^y f(q)=\chi^{y,k,\alpha:q}f(q)=\frac{[y]^k_{q^\alpha}}{[y]_q}\chi(y)f(q^y). \tag{2.22}$$

*Thus, by* (2.22), *one gets*

$$\begin{aligned}\chi^{x,k,\alpha:q}\circ\chi^{y,k,\alpha:q}f(q)&=\chi^{x,k,\alpha:q}\frac{[y]_{q^\alpha}^k}{[y]_q}\chi(y)f(q^y)\\&=\frac{[y]_{q^\alpha}^k}{[y]_q}\chi(y)\chi(x)\frac{[y]_{q^{\alpha y}}^k}{[y]_{q^y}}\chi(y)f(q^{xy})\\&=\frac{[xy]_{q^\alpha}^k}{[xy]_q}\chi(xy)f(q^{xy})\\&=\chi^{xy,k,\alpha:q}f(q)\\&=\chi^{xy}f(q).\end{aligned}\tag{2.23}$$

*Let us define* $\chi^x\chi^y=\chi^{x,k,\alpha:q}\circ\chi^{y,k,\alpha:q}$. *Then one has*

$$\chi^x\chi^y=\chi^{xy}.\tag{2.24}$$

*From the definition of* $\chi^x$, *one can easily derive the following equation;*

$$(1-\chi^p)\left(1-\frac{1}{\beta}x^{1/\beta}\right)=1-\frac{1}{\beta}x^{1/\beta}-\chi^p+\frac{1}{\beta}x^{p/\beta}.\tag{2.25}$$

*Let* $f(q)=\widetilde{\beta}_{k,\chi,q}^{(\alpha)}$. *Then one gets*

$$\begin{aligned}(1-\chi^p)\left(1-\frac{1}{\beta}x^{1/\beta}\right)\widetilde{\beta}_{k,\chi,q}^{(\alpha)}&=\widetilde{\beta}_{k,\chi,q}^{(\alpha)}-\frac{1}{\beta}\frac{[1/\beta]_{q^\alpha}^k}{[1/\beta]_q}\chi\left(\frac{1}{\beta}\right)\widetilde{\beta}_{k,\chi,q}^{(\alpha)}-\frac{[p]_{q^\alpha}^k}{[p]_q}\chi(p)\widetilde{\beta}_{k,\chi,q^p}^{(\alpha)}\\&\quad+\frac{1}{\beta}\frac{[p/\beta]_{q^\alpha}^k}{[p/\beta]_q}\chi\left(\frac{p}{\beta}\right)\widetilde{\beta}_{k,\chi,q^{p/\beta}}^{(\alpha)}.\end{aligned}\tag{2.26}$$

*By* (2.21) *and* (2.26), *one obtains the following equation:*

$$\int_{X^*}\chi(x)d\mu_{k,\beta,q}^{(\alpha)}(\beta x)=(1-\chi^p)\left(1-\frac{1}{\beta}x^{1/\beta}\right)\widetilde{\beta}_{k,\chi,q}^{(\alpha)},\tag{2.27}$$

*where* $\beta(\neq 1)\in X^*$.

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