

Research Article

Essential Norm of Operators into Weighted-Type Spaces on the Unit Ball

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The essential norm of any operator from a general Banach space of holomorphic functions on the unit ball in \mathbb{C}^n into the little weighted-type space is calculated. Some applications of the formula are given.

1. Introduction and Preliminaries

Characterizing the compactness of composition or weighted composition operators, their differences, and Toeplitz operators between Banach spaces of analytic functions has attracted attention of numerous authors, and there has been a great interest in the matter of calculating or estimating essential norms of operators, see, for example, [1–8]. Motivated by this line of investigations, the first two authors calculated in [3] the essential norm of any operator acting between weighted-type spaces or between Bloch spaces on the unit disk and also estimated it on the weighted-Bergman space $A_\alpha^1(\mathbb{D})$.

We obtain a formula for the essential norm of any operator into a weighted-type space on the unit ball in \mathbb{C}^n whose domain space belongs to a general class of Banach holomorphic function spaces, thus extending to the case of the unit ball some results in [3]. Some applications of our main results are given.

Let \mathbb{B} be the open unit ball in the euclidian complex-vector space \mathbb{C}^n and $H(\mathbb{B})$ the space of all holomorphic functions on \mathbb{B} . The *pseudohyperbolic distance* between $z, u \in \mathbb{B}$ is denoted by $\rho(z, u)$ (see [9] for more details). If X is a Banach space, by B_X we denote the closed unit ball in X .

Let v be a positive continuous function on \mathbb{B} (*weight*). The weighted-type space (or Bergman space of infinite order) $H_v^\infty(\mathbb{B}) = H_v^\infty$ is defined by

$$H_v^\infty = \left\{ f \in H(\mathbb{B}) : \|f\|_{H_v^\infty} := \sup_{z \in \mathbb{B}} v(z)|f(z)| < \infty \right\}. \quad (1.1)$$

The little weighted-type space $H_v^0(\mathbb{B}) = H_v^0$ consists of all $f \in H_v^\infty$ such that

$$\lim_{|z| \rightarrow 1} v(z)|f(z)| = 0. \quad (1.2)$$

With the norm $\|\cdot\|_{H_v^\infty}$, both are Banach spaces. The norm topology of H_v^∞ is finer than the topology τ of uniform convergence on compact subsets of \mathbb{B} . For $v_\alpha(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, the standard weighted-type spaces, which we denote by H_α^∞ and H_α^0 , are obtained. These spaces appear in the study of growth conditions of analytic functions, see, for example, [10, 11].

The Bloch-type space $\mathcal{B}_\mu(\mathbb{B}) = \mathcal{B}_\mu$ consists of all $f \in H(\mathbb{B})$ such that

$$b_\mu(f) = \sup_{z \in \mathbb{B}} \mu(z)|\Re f(z)| < \infty, \quad (1.3)$$

where μ is a weight,

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z) = \langle \nabla f(z), \bar{z} \rangle \quad (1.4)$$

is the radial derivative of f , and ∇f is the complex gradient of f .

The little Bloch-type space $\mathcal{B}_{\mu,0}(\mathbb{B}) = \mathcal{B}_{\mu,0}$ consists of all $f \in \mathcal{B}_\mu$ such that

$$\lim_{|z| \rightarrow 1} \mu(z)|\Re f(z)| = 0. \quad (1.5)$$

Both are Banach spaces with the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$. For the standard weight v_p , $p > 0$, we get the p -Bloch space \mathcal{B}^p and the little p -Bloch space \mathcal{B}_0^p .

The standard weighted-Bergman space $A_\beta^p(\mathbb{B}) = A_\beta^p$, $\beta > -1$, $p \geq 1$, is the set of all analytic functions on \mathbb{B} such that

$$\|f\|_{A_\beta^p}^p = \int_{\mathbb{B}} |f(z)|^p c_\beta (1 - |z|^2)^\beta dV(z) < \infty, \quad (1.6)$$

where $dV(z)$ is the normalized volume measure on \mathbb{B} and $c_\beta = \Gamma(n + \beta + 1) / n! \Gamma(\beta + 1)$.

A weight v is radial if it satisfies $v(z) = v(|z|)$ for every $z \in \mathbb{B}$. Throughout this paper we assume that all weights v are *typical*, that is, they are radial, nonincreasing with respect to $|z|$ and such that $\lim_{|z| \rightarrow 1} v(z) = 0$. Many results on weighted-type spaces of analytic functions and on operators between them are given in terms of the so-called *associated weights*

(see [10]) and in terms of the weights. For a weight v the associated weight \tilde{v} is defined as follows

$$\tilde{v}(z) := \left(\sup \left\{ |f(z)| : f \in H_v^\infty, \|f\|_{H_v^\infty} \leq 1 \right\} \right)^{-1}. \quad (1.7)$$

For a typical weight v the associated weight \tilde{v} is also typical. Furthermore, for each $z \in \mathbb{B}$ there is an $f_z \in H_v^\infty$, $\|f_z\|_{H_v^\infty} \leq 1$, such that $f_z(z) = 1/\tilde{v}(z)$, and the same holds for the space H_v^0 . It is known that $H_v^\infty \cong H_{\tilde{v}}^\infty$ and $H_v^0 \cong H_{\tilde{v}}^0$, that is, they are isometrically isometric [10].

We say that a weight v satisfies condition (L1) if it is radial and

$$\inf_{k \in \mathbb{N}} \frac{v(1 - 2^{-(k+1)})}{v(1 - 2^{-k})} > 0. \quad (L1)$$

For radial weights v satisfying condition (L1), we have that v and \tilde{v} are equivalent, that is, there is a $C \geq 1$ such that $v \leq \tilde{v} \leq Cv$. Recently Lusky and Taskinen [12] have shown, among other results, that H_α^0 is isomorphic to c_0 .

Since the closed unit ball $B_{H_v^\infty}$ is a compact subset of $(H(\mathbb{B}), \tau)$, a result of Dixmier-Ng [13] gives that the subspace of $(H_v^\infty)^*$

$$G_v^\infty := \{l \in (H_v^\infty)^* : l|_{B_{H_v^\infty}} \text{ is } \tau\text{-continuous}\} \quad (1.8)$$

is a predual of H_v^∞ , that is, $(G_v^\infty)^* \cong H_v^\infty$. Clearly the evaluation functional at $z \in \mathbb{B}$, defined by $\delta_z(f) = f(z)$, belongs to G_v^∞ . The norm of δ_z is denoted by $\|\delta_z\|_v$. Moreover, the set $\{\delta_z : z \in \mathbb{B}\}$ is a total set, that is, its linear span is norm dense in G_v^∞ . More precisely, the next isomorphism result is due to Bierstedt and Summers.

Lemma 1.1 (see [11]). *The map $f \mapsto [l \mapsto \langle l, f \rangle]$ is an onto isometric isomorphism between H_v^∞ and $(G_v^\infty)^*$ and the restriction map $l \mapsto l|_{H_v^0}$ gives rise to an isometric isomorphism between G_v^∞ and $(H_v^0)^*$.*

Similarly to the corresponding result in the one variable [14], one can prove the following.

Lemma 1.2. *Suppose $\beta > -1$ and $\gamma > 0$. Then $(A_\beta^1)^*$ is isomorphic to H_γ^0*

$$\langle f, g \rangle_{\beta, \gamma} = \int_{\mathbb{B}} f(z) \overline{g(z)} c_{\beta+\gamma} v_{\beta+\gamma}(z) dV(z), \quad f \in H_\gamma^\infty, \quad g \in A_\beta^1. \quad (1.9)$$

Moreover, under the pairing $\langle f, g \rangle_{\beta, \gamma}$ with $g \in H_\gamma^0$ and $f \in A_\beta^1$, we also have that $(H_\gamma^0)^*$ is isomorphic to A_β^1 .

For $z, w \in \mathbb{B}$, let

$$K_z^\beta(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+\beta+1}}. \quad (1.10)$$

Then the kernel function $K_z^{\beta+\gamma}$ clearly belongs to H_Y^0 and to A_β^1 . It has the reproducing property

$$f(z) = \int_{\mathbb{B}} f(w) \overline{K_z^\beta(w)} c_\beta v_\beta(w) dV(w), \quad z \in \mathbb{B}, \quad (1.11)$$

for every function $f \in A_\beta^1$ (see [9, Theorem 2.2]). As a direct application we get that $g(w) = \langle K_w^{\beta+\gamma}, g \rangle_{\beta,\gamma}$ for all $g \in A_\beta^1$ and $f(w) = \langle f, K_w^{\beta+\gamma} \rangle_{\beta,\gamma}$ for all $f \in H_Y^\infty$.

Let E be a Banach space of analytic functions on \mathbb{B} containing the constant functions. We denote its norm by $\|\cdot\|_E$. With $\|\cdot\|_{E^*}$ we denote the norm of its Banach dual space E^* . Consider the following conditions on E .

(C1) There are positive constants s and C such that

$$|f(z)| \leq C \frac{\|f\|_E}{(1-|z|^2)^s}, \quad \text{for every } f \in E, \text{ and for each } z \in \mathbb{B}. \quad (1.12)$$

(C2) The analytic polynomials are dense on E .

(C3) $\|\delta_z\|_{E^*} \rightarrow \infty$, when $|z| \rightarrow 1$.

(C4) The linear span of the set $\{\delta_z : z \in \mathbb{B}\}$ is dense in E^* .

(C5) There is a $C > 0$ such that $\|f\|_E \leq C\|f\|_\infty$ for all $f \in E$, where $\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|$, the supremum taken in the extended real line.

It follows from (C1) that the closed unit ball B_E is τ -bounded, hence it is an equicontinuous and a τ -relatively compact set. Moreover, we have the following.

Proposition 1.3. *Assume that E satisfies conditions (C1) and (C5). Then the contraction operators $K_r : E \rightarrow E$ given by*

$$K_r(f)(z) := f(rz), \quad 0 < r < 1, \quad (1.13)$$

are well defined and compact.

Proof. Each given $f \in E$ may be approximated uniformly on compact subsets by the sequence $(S_k)_{k \in \mathbb{N}_0}$ of its Taylor polynomials at 0. Therefore, $S_k(rz) \rightarrow f(rz)$ uniformly on \mathbb{B} as $k \rightarrow \infty$. Further, $(S_k(rz))_{k \in \mathbb{N}_0} \subset E$ is a Cauchy sequence by (C5), so it converges to some element in E . Hence $K_r(f) \in E$.

Moreover K_r is compact. Indeed, any sequence $(f_m)_{m \in \mathbb{N}} \subset B_E$ has a τ -convergent subsequence, say itself, to an element $g \in H(\mathbb{B})$. Therefore $(f_m(rz))_{m \in \mathbb{N}}$ converges uniformly on \mathbb{B} to $g(rz)$, and again (C5) yields that $(K_r(f_m))_{m \in \mathbb{N}}$ is a Cauchy sequence in E that converges to $K_r(g)$. \square

We consider another condition on E .

(C6) For every $f \in E$, we have $\|K_r(f)\|_E \leq \|f\|_E$ for $0 < r < 1$.

Remarks. (a) Note that (C6) implies that $\|K_r\|_{E \rightarrow E} = \sup_{\|f\|_E \leq 1} \|K_r(f)\|_E \leq 1$.

(b) By (C1) every functional at $z \in \mathbb{B}_n$, $\delta_z : E \rightarrow \mathbb{C}$, is bounded, and therefore

$$|\delta_z(f)| \leq \|\delta_z\|_{E^*} \|f\|_{E'} \quad \text{for every } f \in E. \quad (1.14)$$

(c) Spaces like H_v^0 , the little Bloch space, Hardy spaces H^p and weighted-Bergman spaces A_β^p fulfill conditions (C1)–(C6). For H^p , $s = n/p$, and for A_β^p , we have that $s = (n + \beta + 1)/p$ (see [9]).

Let X and Y be Banach spaces. The essential norm of a bounded operator T is the distance in the operator norm from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T + K\|_{X \rightarrow Y} : K \text{ is compact}\}. \quad (1.15)$$

We write $A \leq B$ if there is a positive constant C , not depending on properties of A and B , such that $A \leq CB$. We also write $A \simeq B$ whenever $A \leq B$ and $B \leq A$.

2. The Essential Norm of Operators

The next proposition is an extension of Proposition 2.1 in [5].

Proposition 2.1. *Assume that E satisfies conditions (C1)–(C6). Then there exists a sequence $(L_m)_{m \in \mathbb{N}}$ of compact operators on E such that*

(i) *for any $0 < t < 1$,*

$$\lim_{m \rightarrow \infty} \sup_{\|f\|_E \leq 1} \sup_{|z| \leq t} |(I - L_m)(f)(z)| = 0; \quad (2.1)$$

(ii)

$$\limsup_{m \rightarrow \infty} \|I - L_m\|_{E \rightarrow E} \leq 1. \quad (2.2)$$

Proof. We prove that for every $0 < t < 1$ and $\varepsilon > 0$ there is a compact operator $L : E \rightarrow E$ such that

$$\|I - L\|_{E \rightarrow E} < 1 + 2\varepsilon, \quad (2.3)$$

and

$$\sup_{\|f\|_E \leq 1} \sup_{|z| \leq t} |(I - L)f(z)| < \varepsilon. \quad (2.4)$$

When this is done, a standard diagonal argument by taking a sequence $(t_n) \uparrow 1$ and a sequence of positive numbers $(\varepsilon_n) \downarrow 0$ will give the result.

The operator L will be constructed as a suitable finite convex combination of the operators K_r and therefore by Proposition 1.3, it will be compact.

The operators $I - K_r : (H(\mathbb{B}), \tau) \rightarrow (H(\mathbb{B}), \tau)$ are continuous, and for each $f \in H(\mathbb{B})$, we have that $(I - K_r)f \rightarrow 0$ in $(H(\mathbb{B}), \tau)$ when $r \rightarrow 1$. By the Banach-Steinhaus theorem for Fréchet spaces $(I - K_r) \rightarrow 0$ uniformly on relatively compact subsets of $(H(\mathbb{B}), \tau)$ as $r \rightarrow 1$; thus in particular on B_E . Hence, if we pick s_1 such that $t < s_1 < 1$, we have

$$\lim_{r \rightarrow 1} \sup_{\|f\|_E \leq 1} \sup_{|z| \leq s_1} |(I - K_r)f(z)| = 0. \quad (2.5)$$

Since B_E is an equicontinuous set, the operator $\delta : \mathbb{B} \rightarrow E^*$, $z \mapsto \delta_z$ is continuous, so we can find $z_0 \in \mathbb{B}$ such that $\sup_{|z| \leq s_1} \|\delta_z\|_{E^*} = \|\delta_{z_0}\|_{E^*}$. Moreover we have the inequalities

$$\frac{1}{\|1\|_E} = \frac{|\delta_z(1)|}{\|1\|_E} \leq \|\delta_z\|_{E^*}. \quad (2.6)$$

Therefore, we find an $r_1 \in (0, 1)$ such that

$$\sup_{\|f\|_E \leq 1} \sup_{|z| \leq s_1} \frac{|(I - K_{r_1})f(z)|}{\|\delta_z\|_{E^*}} < \min \left\{ \varepsilon, \frac{\varepsilon}{\|\delta_{z_0}\|_{E^*}} \right\}. \quad (2.7)$$

Since the bounded set $\{\delta_z / \|\delta_z\|_{E^*} : z \in \mathbb{B}\}$ is equicontinuous in E^* , the weak*-topology coincides on it with the coarser one of the convergence on the dense subset of the polynomials, and also with the finer one of the uniform convergence on relatively compact sets in E . Moreover, observe that for each polynomial P ,

$$\lim_{|z| \rightarrow 1} \left\langle \frac{\delta_z}{\|\delta_z\|_{E^*}}, P \right\rangle = \lim_{|z| \rightarrow 1} \frac{P(z)}{\|\delta_z\|_{E^*}} = 0, \quad (2.8)$$

since condition (C3) holds and P is bounded on \mathbb{B} .

This means that $\lim_{|z| \rightarrow 1} \delta_z / \|\delta_z\|_{E^*} = 0$ on relatively compact sets in E , in particular on $K_{r_1}(B_E)$. Hence there is an $s_2 > s_1$ such that for each $f \in B_E$, we have

$$\sup_{|z| \geq s_2} \frac{|\delta_z(K_{r_1}f)|}{\|\delta_z\|_{E^*}} \leq \varepsilon. \quad (2.9)$$

Therefore,

$$\sup_{\|f\|_E \leq 1} \sup_{|z| \geq s_2} \frac{|(I - K_{r_1})f(z)|}{\|\delta_z\|_{E^*}} < 1 + \varepsilon. \quad (2.10)$$

We can continue in this way and find two strictly increasing sequences (s_k) and (r_k) satisfying

$$\sup_{\|f\|_E \leq 1} \sup_{|z| \leq s_k} \frac{|(I - K_{r_k})f(z)|}{\|\delta_z\|_{E^*}} < \min \left\{ \varepsilon, \frac{\varepsilon}{\|\delta_{z_0}\|_{E^*}} \right\}, \quad (2.11)$$

$$\sup_{\|f\|_E \leq 1} \sup_{|z| \geq s_{k+1}} \frac{|(I - K_{r_k})f(z)|}{\|\delta_z\|_{E^*}} < 1 + \varepsilon. \quad (2.12)$$

Let $m > 1/\varepsilon$ and put

$$L = \frac{1}{m} \sum_{k=1}^m K_{r_k}. \quad (2.13)$$

Then by (2.11) and the fact that $\|\delta_{z_0}\|_{E^*} \geq \|\delta_z\|_{E^*}$ for $|z| \leq s_1$, we have that (2.4) holds since

$$\sup_{\|f\|_E \leq 1} \sup_{|z| \leq t} |(I - L)f(z)| \leq \frac{1}{m} \sum_{k=1}^m \sup_{\|f\|_E \leq 1} \sup_{|z| \leq s_1} |(I - K_{r_k})f(z)| < \varepsilon. \quad (2.14)$$

Now we show that (2.3) holds. Similarly to above, we have that

$$\sup_{\|f\|_E \leq 1} \sup_{|z| < t} \frac{|(I - L)f(z)|}{\|\delta_z\|_{E^*}} < \varepsilon. \quad (2.15)$$

Moreover, by (2.12), if $s_l < |z| \leq s_{l+1}$ and $\|f\|_E \leq 1$, then

$$\frac{|(I - K_{r_k})f(z)|}{\|\delta_z\|_{E^*}} < 1 + \varepsilon, \quad (2.16)$$

except possibly for $k = l$, in which case

$$\frac{|(I - K_{r_l})f(z)|}{\|\delta_z\|_{E^*}} \leq \|(I - K_{r_l})f\|_E \leq \|I - K_{r_l}\|_{E \rightarrow E} \leq 2. \quad (2.17)$$

Hence, we get for all $z \in \mathbb{B}$ and $\|f\|_E \leq 1$ that

$$\frac{|(I - L)f(z)|}{\|\delta_z\|_{E^*}} \leq \frac{1}{m} \sum_{k \neq l} \frac{|(I - K_{r_k})f(z)|}{\|\delta_z\|_{E^*}} + \frac{1}{m} \frac{|(I - K_{r_l})f(z)|}{\|\delta_z\|_{E^*}} < 1 + 2\varepsilon. \quad (2.18)$$

In light of condition (C4), we have

$$\|I - L\|_{E \rightarrow E} = \sup_{z \in \mathbb{B}} \left\| (I - L)^* \frac{\delta_z}{\|\delta_z\|_{E^*}} \right\|_{E^*} = \sup_{z \in \mathbb{B}} \sup_{\|f\|_E \leq 1} \frac{|(I - L)f(z)|}{\|\delta_z\|_{E^*}} < 1 + 2\varepsilon. \quad (2.19)$$

Thus we also have (2.3), and the statement is proved. \square

Next we state a formula for the norm of any operator from E to either H_v^∞ or H_v^0 . We omit its proof.

Theorem 2.2. *Suppose $T : E \rightarrow H_v^\infty$ is a bounded operator. Then*

$$\|T\|_{E \rightarrow H_v^\infty} = \sup_{z \in \mathbb{B}} \frac{\|T^*(\delta_z)\|_{E^*}}{\|\delta_z\|_v}. \quad (2.20)$$

Since many operators can be written as weighted composition operators we state the following useful result. As usual, $uC_\varphi(f) = u \cdot (f \circ \varphi)$ where u is an analytic function on \mathbb{B} and φ is an analytic self-map of \mathbb{B} .

Corollary 2.3. *If $uC_\varphi : E \rightarrow H_v^\infty$ is a bounded weighted composition operator, then*

$$\|uC_\varphi\|_{E \rightarrow H_v^\infty} = \sup_{z \in \mathbb{B}} |u(z)| \|\tilde{v}(z)\|_{E^*}. \quad (2.21)$$

Example 2.4. Let $E = A_\beta^p$, $\beta > -1$, $p > 1$ with $1/p + 1/q = 1$ and $\tilde{v}_\alpha(z) = v_\alpha(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$. Then, since for all $f \in A_\beta^p$ and $z \in \mathbb{B}$,

$$|f(z)| \leq \frac{\|f\|_{A_\beta^p}}{(1 - |z|^2)^{(n+1+\beta)/p}}, \quad (2.22)$$

and since equality is attained at

$$f_z(w) = \frac{(1 - |z|^2)^{(n+1+\beta)/p}}{(1 - \langle w, z \rangle)^{2(n+1+\beta)/p}}, \quad (2.23)$$

we get

$$\|\delta_{\varphi(z)}\|_{A_\beta^q} = \frac{1}{(1 - |\varphi(z)|^2)^{(n+1+\beta)/p}}. \quad (2.24)$$

So we get

$$\|uC_\varphi\|_{A_\beta^p \rightarrow H_v^\infty} = \sup_{z \in \mathbb{B}} \frac{(1 - |z|^2)^\alpha |u(z)|}{(1 - |\varphi(z)|^2)^{(n+1+\beta)/p}}. \quad (2.25)$$

Now we are ready for our main result. Compare it with Theorem 2.2.

Theorem 2.5. *Assume that E satisfies conditions (C1)–(C6). If $T : E \rightarrow H_v^0$ is a bounded operator, then*

$$\|T\|_e = \limsup_{|z| \rightarrow 1^-} \frac{\|T^*(\delta_z)\|_{E^*}}{\|\delta_z\|_v}. \quad (2.26)$$

Proof. If

$$\varrho := \limsup_{|z| \rightarrow 1^-} \frac{\|T^*(\delta_z)\|_{E^*}}{\|\delta_z\|_v}, \quad (2.27)$$

then the same reasoning as in the first part of [3, Theorem 3.1] shows that $\|T\|_e \geq \varrho$.

To prove the reverse inequality, let $(L_m)_{m \in \mathbb{N}}$ be the sequence provided by Proposition 2.1. Then

$$\|T\|_e \leq \|T - T \circ L_m\|_{E \rightarrow H_v^0}, \quad (2.28)$$

for all $m \in \mathbb{N}$.

Next, we estimate

$$\limsup_{m \rightarrow \infty} \|T - T \circ L_m\|_{E \rightarrow H_v^0}. \quad (2.29)$$

Fix $0 < t < 1$ and f in the unit ball of E . Then

$$|[T \circ (I - L_m)](f)(z)| = |\langle \delta_z, [T \circ (I - L_m)](f) \rangle| = |\langle (I - L_m^*) \circ T^*(\delta_z), f \rangle|. \quad (2.30)$$

The sequence $(I - L_m^*)_{m \in \mathbb{N}} \subset \mathcal{L}(E^*)$ is bounded because of (2.2) in Proposition 2.1. Since for each fixed $z \in \mathbb{B}$, and using (2.1) in Proposition 2.1

$$\lim_{m \rightarrow \infty} \|(I - L_m^*)(\delta_z)\|_{E^*} = \lim_{m \rightarrow \infty} \sup_{\|f\|_E \leq 1} |(I - L_m)(f)(z)| = 0, \quad (2.31)$$

the sequence $(I - L_m^*)_{m \in \mathbb{N}}$ converges to zero on every point in the total set $\{\delta_z : z \in \mathbb{B}\} \subset E^*$ as $m \rightarrow \infty$. So we appeal to the Banach-Steinhaus type theorem [15, III 4.5] to conclude that $(I - L_m^*)_{m \in \mathbb{N}}$ converges to zero uniformly on compact subsets of E^* . In particular on the image $\{T^*(\delta_z) : |z| \leq t\} \subset E^*$ of the compact set $\{\delta_z : |z| \leq t\} \subset G_v^\infty$, that is,

$$0 = \lim_{m \rightarrow \infty} \sup_{\{|z| \leq t\}} \|(I - L_m^*)(T^*(\delta_z))\|_{E^*} = \lim_{m \rightarrow \infty} \sup_{\{\|f\|_E \leq 1\}} \sup_{\{|z| \leq t\}} |\langle (I - L_m^*) \circ T^*(\delta_z), f \rangle|. \quad (2.32)$$

Therefore since the weight \tilde{v} is bounded,

$$\lim_{m \rightarrow \infty} \sup_{\{\|f\|_E \leq 1\}} \sup_{\{|z| \leq t\}} |[T \circ (I - L_m)](f)(z)| \tilde{v}(z) = 0. \quad (2.33)$$

On the other hand, for $|z| > t$, and as in the proof of [3, Theorem 3.1]

$$\sup_{\{\|f\|_E \leq 1\}} |[T \circ (I - L_m)](f)(z)| \tilde{v}(z) \leq \|I - L_m\|_{E \rightarrow H_v^0} \cdot \sup_{|z| > t} \frac{\|T^*(\delta_z)\|_{E^*}}{\|\delta_z\|_v}. \quad (2.34)$$

Therefore,

$$\begin{aligned}
\limsup_{m \rightarrow \infty} \|T - T \circ L_m\|_{E \rightarrow H_v^0} &= \limsup_{m \rightarrow \infty} \sup_{\{\|f\|_E \leq 1\}} \sup_{\{|z| < 1\}} |[T \circ (I - L_m)](f)(z)| \tilde{v}(z) \\
&= \limsup_{m \rightarrow \infty} \sup_{\{\|f\|_E \leq 1\}} \sup_{\{|z| > t\}} |[T \circ (I - L_m)](f)(z)| \tilde{v}(z) \\
&\leq \limsup_{m \rightarrow \infty} \|I - L_m\|_{E \rightarrow H_v^0} \cdot \sup_{|z| > t} \frac{\|T^*(\delta_z)\|_{E^*}}{\|\delta_z\|_v}.
\end{aligned} \tag{2.35}$$

Applying Proposition 2.1 in (2.35) and letting $t \rightarrow 1$ we obtain $\|T\|_e \leq \ell$. \square

Corollary 2.6. *Assume that E satisfies conditions (C1)–(C6). If $uC_\varphi : E \rightarrow H_v^0$ is a bounded weighted composition operator, then*

$$\|uC_\varphi\|_e = \limsup_{|z| \rightarrow 1^-} |u(z)| \tilde{v}(z) \|\delta_{\varphi(z)}\|_{E^*}. \tag{2.36}$$

Proof. We apply Theorem 2.5 and just use that $(uC_\varphi)^*(\delta_z) = u(z)\delta_{\varphi(z)}$ and that $\|\delta_z\|_v = 1/\tilde{v}(z)$. \square

Example 2.7. If $E = H_w^0$, then from Corollary 2.6 we get the following extension of the one-dimensional result in [5, Theorem 2.2]

$$\|uC_\varphi\|_e = \limsup_{|z| \rightarrow 1^-} \frac{\tilde{v}(z)|u(z)|}{\tilde{w}(\varphi(z))}. \tag{2.37}$$

Corollary 2.8. *Let F be a Banach space of analytic functions in \mathbb{B} that is isometrically isomorphic to the bidual of a space E satisfying conditions (C1)–(C6). Assume that E has the λ -metric approximation property or is a dual space. Let $T : F \rightarrow H_v^\infty$ be a bounded operator such that $T(E) \subseteq H_v^0$ and whose restrictions to bounded subsets of F are τ - τ continuous. Then*

$$\|T\|_e \simeq \limsup_{|z| \rightarrow 1^-} \frac{\|T^*(\delta_z)\|_{F^*}}{\|\delta_z\|_v}. \tag{2.38}$$

Proof. To see that $T = (T|_E)^{**}$, it suffices to remark that T is weak*-weak* continuous, that is, to check that $l \circ T \in E^*$ for all $l \in G_v^\infty$. And for this, it is enough to check that $l \circ T$ is weak* continuous on the closed unit ball $B_{E^{**}}$ of $F = E^{**}$, a fact that follows from observing that on (B_F, w^*) the weak*-topology coincides with the Hausdorff coarser one τ , for which both T and l are also continuous on bounded sets.

Since E has the λ -metric approximation property or is a dual space, we may use Axler et al. [16] to obtain that $\|T\|_e = \|(T|_E)^*\|_e \simeq \|T|_E\|_e$. From this and Theorem 2.5 the corollary follows. \square

Notice that spaces like H_v^∞ , the Bloch space and $BMOA$ satisfy the assumptions on F in the above corollary.

Corollary 2.9. *Let $\beta > -1, \gamma > 0$ and assume that $T : A_\beta^1 \rightarrow A_\beta^1$ is a bounded operator. If $T'(H_\gamma^0) \subseteq H_\gamma^0$, where $T' : H_\gamma^\infty \rightarrow H_\gamma^\infty$ is the dual operator with respect to the duality $\langle \cdot, \cdot \rangle_{\beta, \gamma}$, then*

$$\|T\|_e \simeq \limsup_{|z| \rightarrow 1^-} \frac{\|T(K_z^{\beta+\gamma})\|_{A_\beta^1}}{\|K_z^{\beta+\gamma}\|_{A_\beta^1}}. \tag{2.39}$$

Proof. Note that $\|K_z^{\beta+\gamma}\|_{A_\beta^1} \simeq (1 - |z|^2)^{-\gamma}$. Let $T'|_{H_\gamma^0}$ be the restriction to H_γ^0 of T' . Then we can apply Theorem 2.5 so that

$$\|T'|_{H_\gamma^0}\|_e = \limsup_{|z| \rightarrow 1^-} \frac{\|(T'|_{H_\gamma^0})^*(\delta_z)\|_{(H_\gamma^0)^*}}{\|\delta_z\|_{\mathfrak{v}_\gamma}}. \tag{2.40}$$

For $\|f\|_{H_\gamma^\infty} \leq 1, f \in H_\gamma^0$ and $z \in \mathbb{D}$,

$$\left| (T'|_{H_\gamma^0})^* \delta_z(f) \right| = |T'f(z)| = \left| \langle K_z^{\beta+\gamma}, T'f \rangle_{\beta, \gamma} \right| = \left| \langle T(K_z^{\beta+\gamma}), f \rangle_{\beta, \gamma} \right|. \tag{2.41}$$

Since by [12] H_γ^0 is isomorphic to c_0 , we obtain that

$$\|T\|_e \simeq \|(T'|_{H_\gamma^0})^*\|_e \simeq \|T'|_{H_\gamma^0}\|_e, \tag{2.42}$$

as wanted. □

Now we apply Theorem 2.5 to bounded operators on the subspaces of p -Bloch and little p -Bloch spaces consisting of all f such that $f(0) = 0$. We denote the spaces by \mathcal{B}^p/\mathbb{C} and $\mathcal{B}_0^p/\mathbb{C}$ correspondingly. Similar subspaces of H_μ^∞ and H_μ^0 are denoted by H_μ^∞/\mathbb{C} and H_μ^0/\mathbb{C} correspondingly.

Observe that for any $z \in \mathbb{B}$, $\|\delta_z\|_{(\mathcal{B}_0^p)^*} \leq 1 + \|\delta_z\|_{(\mathcal{B}_0^p/\mathbb{C})^*}$. Hence $\mathcal{B}_0^p/\mathbb{C}$ being a subspace of \mathcal{B}_0^p fulfills conditions (C1)–(C6).

Set $\widehat{\delta}_z(f) := (1 - |z|^2)^p \mathfrak{A}f(z)$ for $z \in \mathbb{B}$ and $f \in \mathcal{B}_p/\mathbb{C}$.

Corollary 2.10. (a) *Let $T : \mathcal{B}_0^p/\mathbb{C} \rightarrow \mathcal{B}_0^p/\mathbb{C}$ be a bounded operator. Then*

$$\|T\|_e = \limsup_{|z| \rightarrow 1^-} \|T^*(\widehat{\delta}_z)\|_{(\mathcal{B}_0^p/\mathbb{C})^*}. \tag{2.43}$$

(b) *Let $T : \mathcal{B}^p/\mathbb{C} \rightarrow \mathcal{B}^p/\mathbb{C}$ be a bounded operator such that $\mathcal{B}_0^p/\mathbb{C}$ is an invariant subspace for T and whose restrictions to the bounded subsets are τ - τ continuous. Then*

$$\|T\|_e \simeq \limsup_{|z| \rightarrow 1^-} \|T^*(\widehat{\delta}_z)\|_{(\mathcal{B}^p/\mathbb{C})^*}. \tag{2.44}$$

Proof. Let $\mathfrak{R} : \mathcal{B}^p/\mathbb{C} \rightarrow H_p^\infty/\mathbb{C}$ be defined by $\mathfrak{R}(f)(z) = \mathfrak{R}f(z)$. Then both \mathfrak{R} and $\mathfrak{R} : \mathcal{B}_0^p/\mathbb{C} \rightarrow H_p^0/\mathbb{C}$ are onto isometries (see, e.g., [17]). Therefore, if we put $L = \mathfrak{R} \circ T$, we have $\|L\|_e = \|T\|_e$. Since $\|\delta_z\|_{v_p} = (1 - |z|^2)^{-p}$, and $\mathfrak{R}(f)(z) = \mathfrak{R}^*(\delta_z)(f)$ we also have that

$$L^* \left(\frac{\delta_z}{\|\delta_z\|_{v_p}} \right) = T^* \circ \mathfrak{R}^* \left(\frac{\delta_z}{\|\delta_z\|_{v_p}} \right) = T^* \left((1 - |z|^2)^p \mathfrak{R}^*(\delta_z) \right) = T^* \left(\widehat{\delta}_z \right). \quad (2.45)$$

Hence, $\|T^*(\widehat{\delta}_z)\|_{(\mathcal{B}_0^p/\mathbb{C})^*} = \|L^*(\delta_z/\|\delta_z\|_{v_p})\|_{(\mathcal{B}_0^p/\mathbb{C})^*}$.

Now for (a), apply Theorem 2.5 to L . For (b), observe that $L(\mathcal{B}_0^p/\mathbb{C}) \subseteq H_p^0$ and also that the quotient norm on $\mathcal{B}_0^p/\mathbb{C}$ coincides with b_p , and therefore using the quotient and the subspace duality, $(\mathcal{B}_0^p/\mathbb{C})^{**} = \mathcal{B}^p/\mathbb{C}$. Moreover, L is τ - τ continuous on bounded sets, because \mathfrak{R} preserves the τ - τ continuity. Then use Corollary 2.8. \square

For a given $g \in H(\mathbb{B})$, $g(0) = 0$ and a holomorphic self-map φ of \mathbb{B} , the following integral-type operator on $H(\mathbb{B})$ is introduced in [17]

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}. \quad (2.46)$$

This operator has also been studied later, for example, in [6, 8, 18, 19]. For a closely related operator see also [20]. In [6] Stević calculated the essential norm of the operator $P_\varphi^g : A_\beta^2 \rightarrow \mathcal{B}_\mu$. Motivated by this result we calculate here the essential norm of the operator $P_\varphi^g : A_\beta^p \rightarrow \mathcal{B}_{\mu,0}/\mathbb{C}$, but in terms of the associated weight of μ .

Corollary 2.11. *Assume that $\beta > -1$, $p > 1$, $g \in H(\mathbb{B})$, $g(0) = 0$, φ is a holomorphic self-map of \mathbb{B} and $P_\varphi^g : A_\beta^p \rightarrow \mathcal{B}_{\mu,0}/\mathbb{C}$ is bounded. Then*

$$\|P_\varphi^g\|_e = \limsup_{|z| \rightarrow 1^-} \frac{\tilde{\mu}(z)|g(z)|}{(1 - |\varphi(z)|^2)^{(n+1+\beta)/p}}. \quad (2.47)$$

Proof. Since $\mathfrak{R}P_\varphi^g(f)(z) = f(\varphi(z))g(z)$, we have $\mathfrak{R}P_\varphi^g = gC_\varphi : A_\beta^p \rightarrow H_\mu^0/\mathbb{C}$. From this and the fact that the operator \mathfrak{R} is an onto isometry between $\mathcal{B}_{\mu,0}/\mathbb{C}$ and H_μ^0/\mathbb{C} it follows that $\|P_\varphi^g\|_e = \|gC_\varphi\|_e$. By Corollary 2.6, we have

$$\|gC_\varphi\|_e = \limsup_{|z| \rightarrow 1^-} |g(z)|\tilde{\mu}(z)\|\delta_{\varphi(z)}\|_{A_\beta^q}, \quad (2.48)$$

where $1/p + 1/q = 1$. Therefore equality (2.47) follows simply bearing in mind (2.24). \square

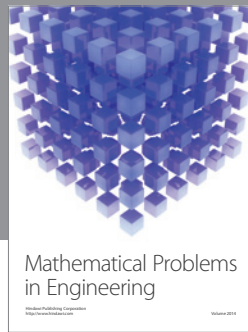
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