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Research Article

The Cesáro Core of Double Sequences

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We have characterized a new type of core for double sequences, P_C-core, and determined the necessary and sufficient conditions on a four-dimensional matrix A to yield P_C-core$\{Ax\} \subseteq \alpha(P$-core$\{x\})$ for all ℓ_2^∞.

1. Introduction

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ is said to be convergent in the Pringsheim sense or P-convergent if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{jk} - \ell| < \varepsilon$ whenever $j, k > N$, [1]. In this case, we write $P-\lim x = \ell$. By c_2, we mean the space of all P-convergent sequences.

A double sequence x is bounded if

$$\|x\| = \sup_{j,k\geq 0}\left|x_{jk}\right| < \infty. \tag{1.1}$$

By ℓ_∞^2, we denote the space of all bounded double sequences.

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. So, we denote by c_2^∞ the space of double sequences which are bounded and convergent.

A double sequence $x = [x_{jk}]$ is said to converge regularly if it converges in Pringsheim's sense and, in addition, the following finite limits exist:

$$\begin{aligned} &\lim_{k\to\infty} x_{jk} = \ell_j, \quad (j = 1, 2, 3, \ldots),\\ &\lim_{j\to\infty} x_{jk} = t_j, \quad (k = 1, 2, 3, \ldots). \end{aligned} \tag{1.2}$$

Let $A = [a_{jk}^{mn}]_{j,k=0}^{\infty}$ be a four-dimensional infinite matrix of real numbers for all $m, n = 0, 1, \ldots$. The sums

$$y_{mn} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk} \tag{1.3}$$

are called the A-transforms of the double sequence $x = [x_{jk}]$. We say that a sequence $x = [x_{jk}]$ is A-summable to the limit ℓ if the A-transform of $x = [x_{jk}]$ exists for all $m, n = 0, 1, \ldots$ and is convergent to ℓ in the Pringsheim sense, that is,

$$\begin{gathered} \lim_{p,q\to\infty}\sum_{j=0}^{p}\sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = y_{mn}, \\ \lim_{m,n\to\infty} y_{mn} = \ell. \end{gathered} \tag{1.4}$$

We say that a matrix A is bounded-regular if every bounded-convergent sequence x is A-summable to the same limit and the A-transforms are also bounded. The necessary and sufficient conditions for A to be bounded-regular or RH-regular (cf., Robison [2]) are

$$\begin{gathered} \lim_{m,n\to\infty} a_{jk}^{mn} = 0, \quad (j, k = 0, 1, \ldots), \\ \lim_{m,n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}^{mn} = 1, \\ \lim_{m,n\to\infty}\sum_{j=0}^{\infty}\left|a_{jk}^{mn}\right| = 0, \quad (k = 0, 1, \ldots), \\ \lim_{m,n\to\infty}\sum_{k=0}^{\infty}\left|a_{jk}^{mn}\right| = 0, \quad (j = 0, 1, \ldots), \\ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|a_{jk}^{mn}\right| \le C < \infty \quad (m, n = 0, 1, \ldots). \end{gathered} \tag{1.5}$$

A double sequence $x = [x_{jk}]$ is said to be almost convergent (see [3]) to a number L if

$$\lim_{p,q\to\infty}\sup_{s,t\ge0}\frac{1}{pq}\sum_{j=0}^{p}\sum_{k=0}^{q} x_{s+j,t+k} = L. \tag{1.6}$$

Let σ be a one-to-one mapping from $\mathbb{N}$ into itself. The almost convergence of double sequences has been generalized to the σ-convergence in [4] as follows:

$$\lim_{p,q\to\infty}\sup_{s,t\ge0}\frac{1}{pq}\sum_{j=0}^{p}\sum_{k=0}^{q} x_{\sigma^{j}(s),\sigma^{k}(t)} = \ell, \tag{1.7}$$

where $\sigma^j(s) = \sigma(\sigma^{j-1}(s))$. In this case, we write $\sigma - \lim x = \ell$. By V_σ^2, we denote the set of all σ-convergent and bounded double sequences. One can see that in contrast to the case for single sequences, a convergent double sequence need not be σ-convergent. But every bounded convergent double sequence is σ-convergent. So, $c_2^\infty \subset V_\sigma^2 \subset \ell_2^\infty$. In the case $\sigma(i) = i+1$, σ-convergence of double sequences reduces to the almost convergence. A matrix $A = [a_{jk}^{mn}]_{j,k=0}^\infty$ is said to be σ-regular if $Ax \in V_2^\sigma$ for $x = [x_{jk}] \in c_2^\infty$ with $\sigma - \lim Ax = \lim x$, and we denote this by $A \in (c_2^\infty, V_2^\sigma)_{\text{reg}}$, (see [5, 6]). Mursaleen and Mohiuddine defined and characterized σ-conservative and σ-coercive matrices for double sequences [6].

A double sequence $x = [x_{jk}]$ of real numbers is said to be Cesáro convergent (or C_1-convergent) to a number L if and only if $x \in C_1$, where

$$C_1 = \left\{ x \in \ell_2^\infty : \lim_{p,q\to\infty} T_{pq}(x) = L; L = C_1 - \lim x \right\},$$
$$T_{pq}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=1}^{p}\sum_{k=1}^{q} x_{jk}^{mn}. \tag{1.8}$$

We shall denote by C_1 the space of Cesáro convergent (C_1-convergent) double sequences.

A matrix $A = (a_{jk}^{mn})$ is said to be C_1-multiplicative if $Ax \in C_1$ for $x = [x_{jk}] \in c_2^\infty$ with $C_1 - \lim Ax = \alpha \lim x$, and in this case we write $A \in (c_2^\infty, C_1)_\alpha$. Note that if $\alpha = 1$, then C_1-multiplicative matrices are said to be C_1-regular matrices.

Recall that the Knopp core (or K-core) of a real number single sequence $x = (x_k)$ is defined by the closed interval $[\ell(x), L(x)]$, where $\ell(x) = \liminf x$ and $L(x) = \limsup x$. The well-known Knopp core theorem states (cf., Maddox [7] and Knopp [8]) that in order that $L(Ax) \le L(x)$ for every bounded real sequence x, it is necessary and sufficient that $A = (a_{nk})$ should be regular and $\lim_{n\to\infty}\sum_{k=0}^\infty |a_{nk}| = 1$. Patterson [9] extended this idea for double sequences by defining the Pringsheim core (or P-core) of a real bounded double sequence $x = [x_{jk}]$ as the closed interval $[P - \liminf x, P - \limsup x]$. Some inequalities related to the these concepts have been studied in [5, 9, 10]. Let

$$L^*(x) = \limsup_{p,q\to\infty} \sup_{s,t} \frac{1}{pq} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{j+s,k+t},$$
$$C_\sigma(x) = \limsup_{p,q\to\infty} \sup_{s,t} \frac{1}{pq} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}. \tag{1.9}$$

Then, MR- (Moricz-Rhoades) and σ-core of a double sequence have been introduced by the closed intervals $[-L^*(-x), L^*(-x)]$ and $[-C_\sigma(-x), C_\sigma(x)]$, and also the inequalities

$$L(Ax) \le L^*(x), L^*(Ax) \le L(x), L^*(Ax) \le L^*(x), L(Ax) \le C_\sigma(x), C_\sigma(Ax) \le L(x) \tag{1.10}$$

have been studies in [3–5, 11].

In this paper, we introduce the concept of C_1-multiplicative matrices and determine the necessary and sufficient conditions for a matrix $A = (a_{jk}^{mn})$ to belong to the class $(c_2^{\infty}, C_1)_{\alpha}$. Further we investigate the necessary and sufficient conditions for the inequality

$$C_1^*(Ax) \leq \alpha L(x) \tag{1.11}$$

for all $x \in \ell_{\infty}^2$.

2. Main Results

Let us write

$$C_1^*(x) = \limsup_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{jk}. \tag{2.1}$$

Then, we will define the P_C-core of a realvalued bounded double sequence $x = [x_{jk}]$ by the closed interval $[-C_1^*(-x), C_1^*(x)]$. Since every bounded convergent double sequence is Cesáro convergent, we have $C_1^*(x) \leq P - \limsup x$, and hence it follows that P_C-core$(x) \subseteq P$-core(x) for a bounded double sequence $x = [x_{jk}]$.

Lemma 2.1. *A matrix $A = (a_{jk}^{mn})$ is C_1-multiplicative if and only if*

$$\lim_{p,q\to\infty} \beta(j,k,p,q) = 0 \quad (j,k = 0,1,\ldots), \tag{2.2}$$

$$\lim_{p,q\to\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \beta(j,k,p,q) = \alpha, \tag{2.3}$$

$$\lim_{p,q\to\infty} \sum_{j=0}^{\infty} |\beta(j,k,p,q)| = 0 \quad (k = 0,1,\ldots), \tag{2.4}$$

$$\lim_{p,q\to\infty} \sum_{k=0}^{\infty} |\beta(j,k,p,q)| = 0 \quad (j = 0,1,\ldots), \tag{2.5}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left|a_{jk}^{mn}\right| \leq C < \infty, \quad (m,n = 0,1,\ldots), \tag{2.6}$$

where the lim *means* $P - \lim$ *and*

$$\beta(j,k,p,q) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} a_{jk}^{mn}. \tag{2.7}$$

Proof. Sufficiency. Suppose that the conditions (2.2)-(2.6) hold and $x = [x_{jk}] \in c_2^{\infty}$ with $P - \lim_{j,k} x_{jk} = L$, say. So that for every $\epsilon > 0$ there exists $N > 0$ such that $|x_{jk}| < |\ell| + \epsilon$ whenever $j,k > N$.

Then, we can write

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta(j,k,p,q)x_{jk} = \sum_{j=0}^{N}\sum_{k=0}^{N}\beta(j,k,p,q)x_{jk} + \sum_{j=N}^{\infty}\sum_{k=0}^{N-1}\beta(j,k,p,q)x_{jk} + \sum_{j=0}^{N-1}\sum_{k=N}^{\infty}\beta(j,k,p,q)x_{jk} + \sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\beta(j,k,p,q)x_{jk}. \tag{2.8}$$

Therefore,

$$\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta(j,k,p,q)x_{jk}\right| \le \|x\|\sum_{j=0}^{N}\sum_{k=0}^{N}|\beta(j,k,p,q)| + \|x\|\sum_{j=N}^{\infty}\sum_{k=0}^{N-1}|\beta(j,k,p,q)x_{jk}| + \|x\|\sum_{j=0}^{N-1}\sum_{k=N}^{\infty}|\beta(j,k,p,q)| + (|L|+\epsilon)\left|\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\beta(j,k,p,q)\right|. \tag{2.9}$$

Letting $p,q \to \infty$ and using the conditions (2.2)–(2.6), we get

$$\left|\lim_{p,q\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta(j,k,p,q)x_{jk}\right| \le (|L|+\epsilon)\alpha. \tag{2.10}$$

Since ϵ is arbitrary, $C_1 - \lim Ax = \alpha L$. Hence $A \in (c_2^{\infty}, C_1)_{\alpha}$, that is, A is C_1-multiplicative. □

Necessity 1. Suppose that A is C_1-multiplicative. Then, by the definition, the A-transform of x exists and $Ax \in C_1$ for each $x \in c_2^{\infty}$. Therefore, Ax is also bounded. Then, we can write

$$\sup_{m,n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|a_{jk}^{mn}x_{jk}\right| < M < \infty, \tag{2.11}$$

for each $x \in c_2^{\infty}$. Now, let us define a sequence $y = [y_{jk}]$ by

$$y_{jk} = \begin{cases} \operatorname{sgn} a_{jk}^{mn}, & 0 \le j \le r,\ 0 \le k \le r, \\ 0, & \text{otherwise}, \end{cases} \tag{2.12}$$

$m,n = 0,1,2,\ldots$. Then, the necessity of (10) follows by considering the sequence $y = [y_{jk}]$ in (2.11).

Also, by the assumption, we have

$$\lim_{p,q\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta(j,k,p,q)x_{jk} = \alpha \lim_{j,k\to\infty} x_{jk}. \tag{2.13}$$

Now let us define the sequence e^{il} as follows:

$$e^{il} = \begin{cases} 1, & (j,k) = (i,l), \\ 0, & \text{otherwise}, \end{cases} \tag{2.14}$$

and write $s^l = \sum_i e^{il} (l \in \mathbb{N})$, $r^i = \sum_l e^{il} (i \in \mathbb{N})$. Then, the necessity of (2.2), (2.4), and (2.5) follows from $C_1 - \lim Ae^{il}$, $C_1 - \lim Ar^j$ and $C_1 - \lim As^k$, respectively.

Note that when $\alpha = 1$, the above theorem gives the characterization of $A \in (c_2^{\infty}, C_1)_{\text{reg}}$. Now, we are ready to construct our main theorem.

Theorem 2.2. *For every bounded double sequence x,*

$$C_1^*(Ax) \leq \alpha L(x), \tag{2.15}$$

or ($P_C - core\{Ax\} \subseteq \alpha(P - core\{x\})$) if and only if A is C_1-multiplicative and

$$\limsup_{p,q\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|\beta(j,k,p,q)\right| = \alpha. \tag{2.16}$$

Proof. Necessity. Let (2.15) hold and for all $x \in \ell_{\infty}^2$. So, since $c_2^{\infty} \subset \ell_{\infty}^2$, then, we get

$$\alpha(-L(-x)) \leq -C_1^*(-Ax) \leq C_1^*(Ax) \leq \alpha L(x). \tag{2.17}$$

That is,

$$\alpha \liminf x \leq -C_1^*(-Ax) \leq C_1^*(Ax) \leq \alpha \limsup x, \tag{2.18}$$

where

$$-C_1^*(-Ax) = \liminf_{p,q\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta(j,k,p,q)x_{jk}. \tag{2.19}$$

By choosing $x = [x_{jk}] \in c_{\infty}^2$, we get from (2.17) that

$$-C_1^*(-Ax) = C_1^*(Ax) = C_1 - \lim Ax = \alpha \lim x. \tag{2.20}$$

This means that A is C_1-multiplicative.

By Lemma 3.1 of Patterson [9], there exists a $y \in \ell_\infty^2$ with $\|y\| \le 1$ such that

$$C_1^*(Ay) = \limsup_{p,q\to\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \beta(j,k,p,q). \tag{2.21}$$

If we choose $y = v = [v_{jk}]$, it follows

$$v_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0, & \text{elsewhere}. \end{cases} \tag{2.22}$$

Since $\|v_{jk}\| \le 1$, we have from (2.15) that

$$\alpha = C_1^*(Av) = \limsup_{p,q\to\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\beta(j,k,p,q)| \le \alpha L(v_{jk}) \le \alpha\|v\| \le \alpha. \tag{2.23}$$

This gives the necessity of (2.16). □

Sufficiency 1. Suppose that A is C_1-regular and (2.16) holds. Let $x = [x_{jk}]$ be an arbitrary bounded sequence. Then, there exist $M, N > 0$ such that $x_{jk} \le K$ for all $j, k \ge 0$. Now, we can write the following inequality:

$$\begin{aligned}
\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \beta(j,k,p,q) x_{jk}\right| &= \left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left(\frac{|\beta(j,k,p,q)| + \beta(j,k,p,q)}{2} \right.\right. \\
&\qquad \left.\left. - \frac{|\beta(j,k,p,q)| - \beta(j,k,p,q)}{2} \right) x_{jk} \right| \\
&\le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\beta(j,k,p,q)||x_{jk}| \\
&\quad + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |(|\beta(j,k,p,q)| - \beta(j,k,p,q)) x_{jk}| \\
&\le \|x\| \sum_{j=0}^{M}\sum_{k=0}^{N} |\beta(j,k,p,q)| \\
&\quad + \|x\| \sum_{j=M+1}^{\infty}\sum_{k=0}^{N} |\beta(j,k,p,q)| \\
&\quad + \|x\| \sum_{j=0}^{M}\sum_{k=N+1}^{\infty} |\beta(j,k,p,q)|
\end{aligned}$$

$$+\sup_{j,k\geq M,N}|x_{jk}|\sum_{j=M+1}^{\infty}\sum_{k=N+1}^{\infty}|\beta(j,k,p,q)|$$
$$+\|x\|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(|\beta(j,k,p,q)|-\beta(j,k,p,q)). \tag{2.24}$$

Using the condition of C_1-multiplicative and condition (2.16), we get

$$C_1^*(Ax)\leq \alpha L(x). \tag{2.25}$$

This completes the proof of the theorem.

Theorem 2.3. *For $x,y\in \ell_2^{\infty}$, if $C_1-\lim|x-y|=0$, then $C_1-\text{core}\{x\}=C_1-\text{core}\{y\}$.*

Proof. Since $C_2-\lim|x-y|=0$, we have $C_1-\lim(x-y)=0$ and $C_1-\lim(-(x-y))=0$. Using definition of C_1- core, we take $C_1^*(x-y)=-C_1^*(-(x-y))=0$. Since C_1^* is sublinear,

$$0=-C_1^*(-(x-y))\leq -C_1^*(-x)-C_1^*(y). \tag{2.26}$$

Therefore, $C_1^*(y)\leq -C_1^*(-x)$. Since $-C_1^*(-x)\leq C_1^*(x)$, this implies that $C_1^*(y)\leq C_1^*(x)$. By an argument similar as above, we can show that $C_1^*(x)\leq C_1^*(y)$. This completes the proof. □

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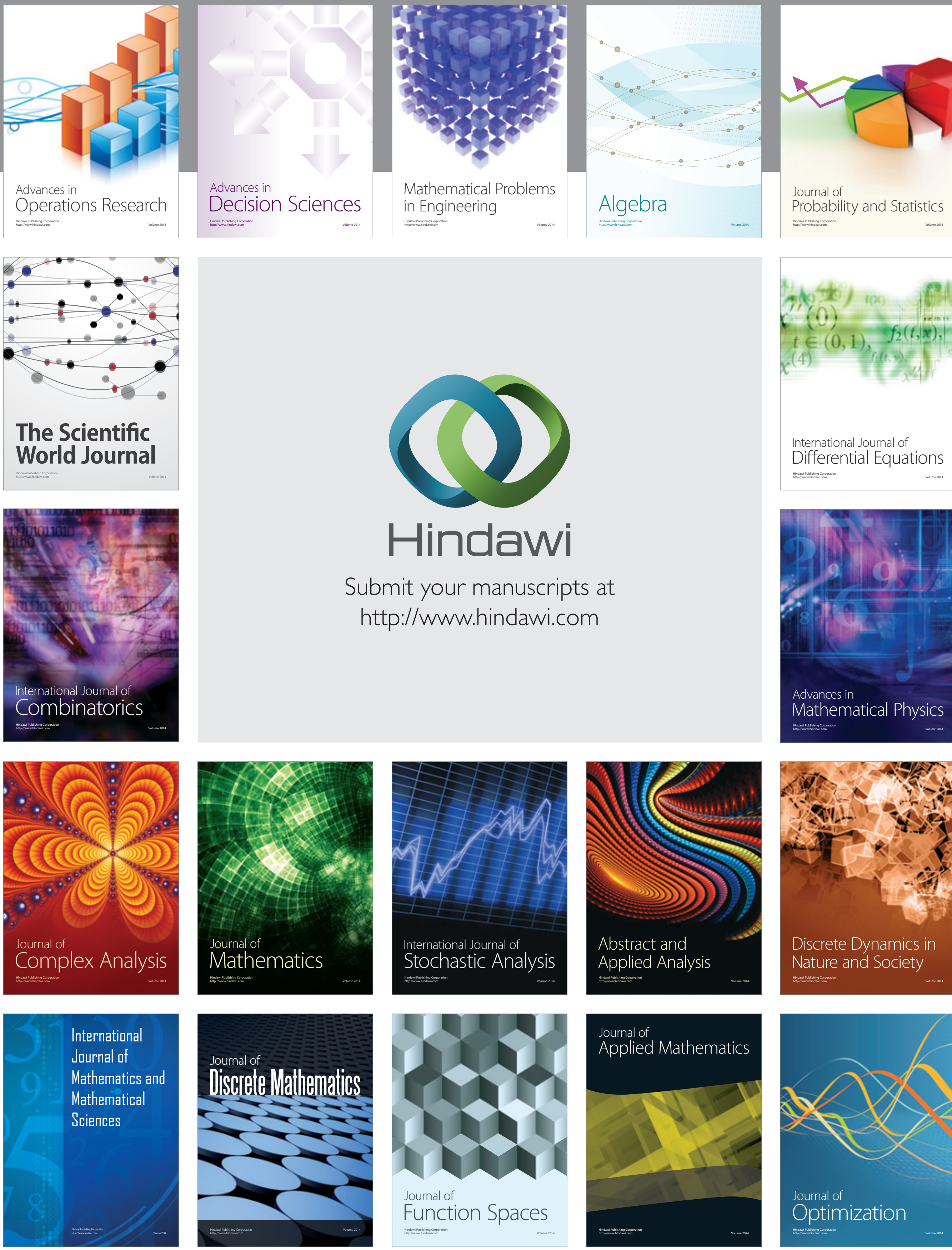
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