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*Research Article*

# Totally Umbilical Hemi-Slant Submanifolds of Kaehler Manifolds

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We study totally umbilical hemi-slant submanifolds of a Kaehler manifold via curvature tensor. We prove some classification theorems for totally umbilical hemi-slant submanifolds of a Kaehler manifold and give an example.

## 1. Introduction

The notion of slant submanifolds of an almost Hermitian manifold was introduced by Chen [1]. These submanifolds are the generalization of both holomorphic and totally real submanifolds of an almost Hermitian manifold with an almost complex structure $J$. Recently, Sahin [2] proved that every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic. Whereas the notion of semi-slant submanifolds of Kaehler manifolds was initiated by Papaghiuc [3]. Bislant submanifolds of an almost Hermitian manifold were introduced as a natural generalization of semi-slant submanifolds by Carriazo [4]. The class of bislant submanifolds includes complex, totally real and CR submanifolds. One of the important classes of bislant submanifolds is that of antislant submanifolds which are studied by Carriazo, but the name antislant seems to refer that it has no slant factor, so Sahin named these submanifolds as hemi-slant submanifolds and studied their warped product in Kaehler setting [5]. In this paper, we study totally umbilical hemi-submanifolds of a Kaehler manifold. In fact, we obtain some classification results for totally umbilical hemi-slant submanifolds of a Kaehler manifold.

## 2. Preliminaries

Let $\overline{M}$ be a Riemannian manifold with an almost complex structure $J$ and Hermitian metric $g$ satisfying

$$\text{(a)} \quad J^2 = -I, \qquad \text{(b)} \quad g(JX, JY) = g(X, Y) \tag{2.1}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ is the tangent bundle of $\overline{M}$. If the almost complex structure $J$ satisfies

$$\left(\overline{\nabla}_X J\right) Y = 0 \tag{2.2}$$

for any $X, Y \in T_x\overline{M}$, and $x \in \overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection on $T\overline{M}$, then $\overline{M}$ is said to have a *Kaehler structure* and with the structure equation (2.2), an almost Hermitian manifold $\overline{M}$ is called a *Kaehler manifold*.

Let $\overline{M}$ be a Kaehler manifold with almost complex structure $J$ and let $M$ be a Riemannian manifold isometrically immersed in $\overline{M}$. Then $M$ is called *holomorphic* (or complex) if $J(T_xM) \subset T_xM$, for any $x \in M$ where $T_xM$ denotes the tangent space of $M$ at the point $x \in M$, and *totally realy* if $J(T_xM) \subset T_x^{\perp}M$, for every $x \in M$, where $T_x^{\perp}M$ denotes the normal space of $M$ at the point $x \in M$. There are three other important classes of submanifolds of a Kaehler manifold determined by the behavior of the tangent bundle of the submanifold under the action of the almost complex structure of the ambient manifold.

(i) A submanifold $M$ is called *CR submanifold* [6] if there exists a differentiable distribution $D : x \to D_x \subset T_xM$ such that $D$ is invariant with respect to $J$ and its orthogonal complementary distribution $D^{\perp}$ is antiinvariant with respect to $J$.

(ii) A submanifold $M$ is called *slant* [1] if for any nonzero vector $X$ tangent to $M$ the angle $\theta(X)$ between $JX$ and $T_xM$ is constant, that is, it does not depend on the choice of $x \in M$ and $X \in T_xM$.

(iii) A submanifold $M$ is called *semi-slant* [3] if it is endowed with a pair of orthogonal distribution $D$ and $D^{\theta}$ such that $D$ is invariant with respect to $J$ and $D^{\theta}$ is slant, that is, the angle between $JX$ and $D_x^{\theta}$ is constant for any $X \in D_x^{\theta}$.

It is clear that the holomorphic and totally real submanifolds are CR submanifolds (respectively, slant submanifolds) with $D^{\perp} = \{0\}$ (resp., $\theta = 0$) and $D = \{0\}$ (resp., $\theta = \pi/2$), respectively. Also, it is clear that CR submanifolds and slant submanifolds are semi slant submanifolds with $\theta = \pi/2$ and $D = \{0\}$, respectively.

For an arbitrary submanifold $M$ of a Riemannian manifold $\overline{M}$ the Gauss and Weingarten formulae are, respectively, given by

$$\begin{gathered} \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \\ \overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \end{gathered} \tag{2.3}$$

for any $X, Y \in TM$, where $\nabla$ is the induced Riemannian connection on $M$, $N$ is the vector field normal to $M$, $h$ is the second fundamental form of $M$, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$, and $A_N$ is the shape operator of the second fundamental form. Moreover, we have

$$g(A_N X, Y) = g(h(X, Y), N), \tag{2.4}$$

where $g$ denotes the Riemannian metric on $\overline{M}$ as well as the metric induced on $M$. The mean curvature vector $H$ on $M$ is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_j), \tag{2.5}$$

where $n$ is the dimension of $M$ and $\{e_1, e_2, \dots, e_n\}$ is a local orthonormal frame of vector fields on $M$.

A submanifold $M$ of a Riemannian manifold $\overline{M}$ is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H. \tag{2.6}$$

If $h(X, Y) = 0$ for any $X, Y \in TM$ then $M$ is said to be *totally geodesic submanifold*. If $H = 0$, then it is called *minimal submanifold*.

For any $X \in TM$ we write

$$JX = TX + FX, \tag{2.7}$$

where $TX$ and $FX$ are the tangential and normal components of $JX$, respectively. Similarly, for any vector field $N$ normal to $M$, we put

$$JN = tN + fN, \tag{2.8}$$

where $tN$ and $fN$ are the tangential and normal components of $JN$, respectively.

The covariant differentiation of $J$ is defined as

$$\left(\overline{\nabla}_X J\right)Y = \overline{\nabla}_X JY - J\overline{\nabla}_X Y, \tag{2.9}$$

for all $X, Y \in T\overline{M}$. Similarly, the covariant derivatives of $T, F, t$ and $f$ are

$$\left(\overline{\nabla}_X T\right)Y = \nabla_X TY - T\nabla_X Y, \tag{2.10}$$

$$\left(\overline{\nabla}_X F\right)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{2.11}$$

$$\begin{aligned} \left(\overline{\nabla}_X t\right)N &= \nabla_X tN - t\nabla^{\perp}_X N, \\ \left(\overline{\nabla}_X f\right)N &= \nabla^{\perp}_X fN - f\nabla^{\perp}_X N \end{aligned} \tag{2.12}$$

for any $X, Y \in TM$, and $N \in T^{\perp}M$.

On the other hand the covariant derivative of the second fundamental form $h$ is defined as

$$\left(\overline{\nabla}_X h\right)(Y,Z)=\nabla_X^{\perp}h(Y,Z)-h(\nabla_X Y,Z)-h(Y,\nabla_X Z) \tag{2.13}$$

for any $X,Y,Z\in TM$. Let $\overline{R}$ and $R$ be the curvature tensors of the connections $\overline{\nabla}$ and $\nabla$ on $\overline{M}$ and $M$, respectively. Then the equations of Gauss and Codazzi are given by

$$\begin{gathered}\overline{R}(X,Y,Z;W)=R(X,Y,Z;W)-g(h(X,Z),h(Y,W))+g(h(X,W),h(Y,Z)),\\ \left[\overline{R}(X,Y)Z\right]^{\perp}=\left(\overline{\nabla}_X h\right)(Y,Z)-\left(\overline{\nabla}_Y h\right)(X,Z).\end{gathered} \tag{2.14}$$

It is known that a submanifold $M$ is slant if and only if

$$T^2=\lambda I \tag{2.15}$$

for some real number $\lambda\in[-1,0]$, where $I$ is the Identity transformation of the tangent bundle $TM$ of the submanifold $M$. Moreover, if $M$ is a slant submanifold and $\theta$ is the slant angle of $M$, then $\lambda=-\cos^2\theta$ [1].

Hence, for a slant submanifold we have the following relations which are the consequences of (2.15):

$$g(TX,TX)=\cos^2\theta g(X,Y), \tag{2.16}$$

$$g(FX,FY)=\sin^2\theta g(X,Y) \tag{2.17}$$

for any $X,Y\in TM$.

Now, we define the hemi-slant submanifold of an almost Hermitian manifold as follows.

*Definition 2.1.* A submanifold $M$ of an almost Hermitian manifold $\overline{M}$ is said to be a hemi-slant submanifold if there exist two orthogonal complementary distributions $D^\theta$ and $D^\perp$ satisfying

(i) $TM=D^\theta\oplus D^\perp$,

(ii) $D^\theta$ is a slant distribution with slant angle $\theta\neq\pi/2$,

(iii) $D^\perp$ is totally real, that is, $JD^\perp\subseteq T^\perp M$.

It is clear that CR-submanifolds and slant submanifolds are hemi-slant submanifolds with $\theta=\pi/2$ and $D^\theta=\{0\}$, respectively.

If $\mu$ is the invariant subspace under the almost complex structure $J$ of the normal bundle $T^\perp M$, then in the case of pseudoslant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M=\mu\oplus FD^\theta\oplus FD^\perp. \tag{2.18}$$

On a submanifold $M$ of a Kaehler manifold $\overline{M}$, by (2.2), (2.3), and (2.7)–(2.11), we have

$$\left(\overline{\nabla}_X T\right)Y = A_{FY}X + th(X,Y), \tag{2.19}$$

$$\left(\overline{\nabla}_X F\right)Y = fh(X,Y) - h(X,TY) \tag{2.20}$$

for any $X, Y \in TM$.

## 3. Totally Umbilical Hemi-Slant Submanifolds

In this section, we study a special class of hemi-slant submanifolds which are totally umbilical. Throughout the section we consider $M$ as a totally umbilical hemi-slant submanifold of a Kaehler manifold $\overline{M}$. On a Kaehler manifold $\overline{M}$, we have the following relations [7]:

$$\text{(a)} \quad \overline{R}(JX, JY)Z = \overline{R}(X,Y)Z \qquad \text{(b)} \quad \overline{R}(X,Y)JZ = J\overline{R}(X,Y)Z \tag{3.1}$$

for any $X, Y, Z \in T\overline{M}$. The Gauss and Weigarten formulae for totally umbilical submanifold of an almost Hermitian manifold are given by

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \tag{3.2}$$

$$\overline{\nabla}_X N = -Xg(H,N) + \nabla^{\perp}_X N \tag{3.3}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $H$ is the mean curvature vector on $M$. Also, the Codazzi equation for a totally umbilical submanifold is given by

$$\overline{R}(X,Y,Z,N) = g(Y,Z)g\left(\nabla^{\perp}_X H, N\right) - g(X,Z)g\left(\nabla^{\perp}_Y H, N\right). \tag{3.4}$$

In the following thoerem we consider $M$ as a totally umbilical hemi-slant submanifold with the slant distribution $D^{\theta}$ and totally real distribution $D^{\perp}$.

**Theorem 3.1.** *Let $M$ be a totally umbilical hemi-slant submanifold of a Kaehler manifold $\overline{M}$ such that the mean curvature vector $H \in \mu$. Then one of the following statements is true:*

(i) *$M$ is totally geodesic in $\overline{M}$,*

(ii) *$M$ is a CR submanifold of $\overline{M}$,*

(iii) *$M$ is a totally real submanifold of $\overline{M}$.*

*Proof.* For any $N \in JD^{\perp}$ and $X \in D^{\theta}$, we have

$$\overline{\nabla}_X JN = J\overline{\nabla}_X N. \tag{3.5}$$

Using (3.2) and (3.3), we obtain

$$\nabla_X JN + g(X, JN)H = -JXg(H, N) + J\nabla^{\perp}_X N. \tag{3.6}$$

Then by orthogonality of two distributions and the fact that $H \in \mu$, the above equation becomes

$$\nabla_X JN = J\nabla^{\perp}_X N, \tag{3.7}$$

which implies that $\nabla^{\perp}_X N \in JD^{\perp}$, for any $N \in JD^{\perp}$. Also, we have $g(N, H) = 0$, for $N \in JD^{\perp}$, then using this fact, we derive

$$g\left(\nabla^{\perp}_X N, H\right) = -g\left(N, \nabla^{\perp}_X H\right) = 0. \tag{3.8}$$

Then (3.8), gives $\nabla^{\perp}_X H \in \mu \oplus FD^{\theta}$. Now, for any $X \in D^{\theta}$, we have

$$\overline{\nabla}_X JH = J\overline{\nabla}_X H. \tag{3.9}$$

Using (3.3), we obtain

$$-Xg(H, JH) + \nabla^{\perp}_X JH = -JXg(H, H) + J\nabla^{\perp}_X H. \tag{3.10}$$

Since $H$ and $JH$ are orthogonal, then from (2.7), the above equation takes the form

$$\nabla^{\perp}_X JH = g(H, H)TX + g(H, H)FX + J\nabla^{\perp}_X H. \tag{3.11}$$

Taking the product with $FX \in FD^{\theta}$ and using (2.17), we obtain

$$g\left(\nabla^{\perp}_X JH, FX\right) = \sin^2\theta\|H\|^2\|X\|^2 + g\left(F\nabla^{\perp}H, FX\right). \tag{3.12}$$

Then from (2.17), the last term of right hand side is identically zero using the fact that $\nabla^{\perp}_X H$ is normal vector and $X \in D^{\theta}$. Thus, the above equation becomes

$$g\left(\nabla^{\perp}_X JH, FX\right) - \sin^2\theta\|H\|^2\|X\|^2 = 0. \tag{3.13}$$

Therefore, (3.13) has a solution if either $H = 0$, that is, $M$ is totally geodesic or the angle of slant distribution $D^{\theta}$ is $\theta = 0$, that is, $M$ is CR-submanifold or if $H \neq 0$, then $D^{\theta} = \{0\}$, that is, $M$ is a totally real submanifold. □

Now, for any $X, Y \in TM$, by (2.19), we have

$$\left(\overline{\nabla}_X T\right)Y = A_{FY}X + th(X, Y). \tag{3.14}$$

In particular, if $Z \in D^{\perp}$, the above equation takes the from

$$-T\nabla_Z Z = A_{FZ}Z + th(Z,Z). \tag{3.15}$$

Then taking the product with $W \in D^{\perp}$, we get

$$-g(T\nabla_Z Z, W) = g(A_{FZ}Z, W) + g(th(Z,Z), W). \tag{3.16}$$

As $M$ is a totally umbilical hemi-slant submanifold, then the above equation takes the form

$$g(Z,W)g(H,FZ) + g(tH,W)\|Z\|^2. \tag{3.17}$$

Thus, (3.17) has a solution if either $H \in \mu$ or $\dim D^{\perp} = 1$ or $D^{\perp} = \{0\}$. If $H \notin \mu$ then $\dim D^{\perp} = 1$ or $D^{\perp} = \{0\}$.

*Remark 3.2.* For a totally umbilical hemi-slant submanifold, if we take $H \notin \mu$ and $D^{\perp} = \{0\}$, then the submanifold $M$ is proper slant. Sahin [2] proved that for a totally umbilical proper slant submanifold the mean curvature vector $H \in \mu$. Thus, in case of hemi-slant submanifold we can not take $H \notin \mu$ and $D^{\perp} = 0$, simultaneously.

**Theorem 3.3.** *Let $M$ be a totally umbilical hemi-slant submanifold of a Kaehler manifold $\overline{M}$ such that the dimension of the slant distribution* $\dim D^{\theta} \geq 4$ *and $F$ is parallel. Then*

(i) *either $M$ is an extrinsic sphere,*

(ii) *or $M$ is totally real.*

*Proof.* Since $\dim D^{\theta} \geq 4$, then we can choose a set of orthogonal vectors $X, Y \in D^{\theta}$, such that $g(X,Y) = 0$. Now from (3.1)(b), we have

$$J\overline{R}(X,Y)Z = \overline{R}(X,Y)JZ. \tag{3.18}$$

Replacing $Z$ by $TY$, we obtain

$$J\overline{R}(X,Y)TY = \overline{R}(X,Y)JTY. \tag{3.19}$$

Using (2.7) and (2.15), the above equation gives

$$J\overline{R}(X,Y)TY = -\cos^2\theta\overline{R}(X,Y)Y + \overline{R}(X,Y)FTY. \tag{3.20}$$

On the other hand, since $F$ is parallel, then we have

$$\overline{R}(X,Y)FTY = F\overline{R}(X,Y)TY. \tag{3.21}$$

Then from (3.20) and (3.21), we obtain

$$J\overline{R}(X,Y)TY = -\cos^2\theta\overline{R}(X,Y)Y + F\overline{R}(X,Y)TY. \tag{3.22}$$

Taking the product in (3.22) with $N \in T^{\perp}M$, we get

$$g\left(J\overline{R}(X,Y)TY,N\right) = -\cos^2\theta\overline{R}(X,Y,Y,N) + g\left(F\overline{R}(X,Y)TY,N\right). \tag{3.23}$$

That is,

$$\cos^2\theta\overline{R}(X,Y,Y,N) = 0. \tag{3.24}$$

Then from (3.4), we derive

$$\cos^2\theta g(Y,Y)g\left(\nabla_X^{\perp}H,N\right) - g(X,Y)g\left(\nabla_Y^{\perp}H,N\right) = 0. \tag{3.25}$$

Since $X$ and $Y$ are orthogonal vectors, then the above equation gives

$$\cos^2\theta g\left(\nabla_X^{\perp}H,N\right)\|Y\|^2 = 0. \tag{3.26}$$

Therefore, (3.26) gives either $\theta = \pi/2$ that is, $M$ is totally real or $\nabla_X^{\perp}H = 0$, for all $X \in D^{\theta}$. On the same line if we consider $X \in D^{\perp}$, then we can deduce that either $\theta = \pi/2$ or $\nabla_X^{\perp}H = 0$. This means that either $M$ is totally real or $\nabla_X^{\perp}H = 0$ for all $X \in TM$, that is, the mean curvature vector $H$ is parallel to the submanifold, thus $M$ is an extrinsic sphere. □

In our further study, we need the following theorem proved by Yamaguchi et al. [8].

**Theorem 3.4.** *A complete and simply connected extrinsic sphere $M^n$ in a Kaehler manifold $\overline{M}^{2m}$ is one of the following:*

(i) *$M^n$ is isometric to an ordinary sphere*

(ii) *$M^n$ is homothetic to a Sasakian manifold*

(iii) *$M^n$ is totally real submanifold and the f-structure is not parallel in the normal bundle.*

Now, we are in position to prove our main theorem.

**Theorem 3.5.** *Let $M$ be a complete simply connected totally umbilical hemi-slant submanifold of a Kaehler manifold $\overline{M}$. Then $M$ is one of the following:*

(i) *a totally real submanifold,*

(ii) *a totally geodesic,*

(iii) *a CR submanifold,*

(iv) $\dim D^{\perp} = 1$,

(v) $D^{\perp} \neq \{0\}$,

(vi) *$M$ is isometric to an ordinary sphere,*

(vii) *$M$ is homothetic to a Sasakian manifold.*

*The cases (vi) and (vii) hold when $F$ is parallel on $M$ and* $\dim M$ *is odd and* $\geq 5$.

*Proof.* If $H \in \mu$, then by Theorem 3.1, the parts (i), (ii), and (iii) hold. If $H \notin \mu$, then (3.17) has a solution if either $\dim D^{\perp} = 1$ or $D^{\perp} = \{0\}$ which is case (iv) and we cannot take $D^{\perp} = \{0\}$ and $H \notin \mu$, simultaneously for a totally umbilical hemi-slant submanifold due to Remark 3.2 which is case (v). Moreover, $H \notin \mu$ and $F$ is parallel on $M$, then by Theorems 3.3 and 3.4, parts (vi) and (vii) hold. This completes the proof of the theorem. □

Now, we construct an example of a hemi-slant submanifold in a Kaehler manifold.

*Example 3.6.* Consider a submanifold $M$ of $\mathbb{R}^6$ with its usual Kaehler structure as

$$\begin{gathered} x_1 = \frac{u}{k}\cos\theta_1, \qquad x_2 = \frac{u}{k}\sin\theta_1, \qquad x_3 = \frac{u}{k}, \\ x_4 = t, \qquad x_5 = t, \qquad x_6 = 0, \qquad u \neq 0, \qquad k \neq 0. \end{gathered} \tag{3.27}$$

The tangent space $TM$ is spanned by the vectors

$$\begin{gathered} e_1 = -\frac{u}{k}\sin\theta_1\frac{\partial}{\partial x_1} + \frac{u}{k}\cos\theta_1\frac{\partial}{\partial x_2}, \qquad e_2 = \frac{1}{k}\cos\theta_1\frac{\partial}{\partial x_1} + \frac{1}{k}\sin\theta_1\frac{\partial}{\partial x_2} + \frac{1}{k}\frac{\partial}{\partial x_3}, \\ e_3 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}. \end{gathered} \tag{3.28}$$

Furthermore, we see that $Je_3$ is orthogonal to $TM$. If we consider $D^{\perp}$ and $D^{\theta}$ are the totally real and slant distributions of $M$, respectively, then $D^{\perp} = \operatorname{span}\{e_3\}$ and $D^{\theta} = \operatorname{span}\{e_1, e_2\}$. Thus, $M$ is a hemi-slant submanifold of $\mathbb{R}^6$ with slant angle $\theta = \cos^{-}1(1/k)$.

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