

## Research Article

# Strong Convergence of Non-Implicit Iteration Process with Errors in Banach Spaces

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The purpose of this paper is to study the strong convergence of a non-implicit iteration process with errors for asymptotically  $I$ -nonexpansive mappings in the intermediate sense in the framework of Banach spaces. The results presented in this paper extend and improve the corresponding results recently announced.

## 1. Introduction and Preliminaries

Let  $K$  be a nonempty, closed, and convex subset of a real Banach space  $X$  and let  $T : K \rightarrow K$  be a mapping. In this paper, we use  $F(T)$  to stand for the set of fixed points of  $T$ , that is  $F(T) = \{x \in K : Tx = x\}$ .

Recall that  $T$  is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K. \quad (1.1)$$

$T$  is said to be asymptotically nonexpansive if there exists a sequence  $\{h_n\}$  with  $h_n \in [1, +\infty)$  with  $\lim_{n \rightarrow \infty} h_n = 1$  such that

$$\|T^n x - T^n y\| \leq h_n \|x - y\|, \quad \forall x, y \in K, n \geq 1. \quad (1.2)$$

$T$  is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \quad (1.3)$$

Observe that if we define  $a_n = \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|)$ ,  $\sigma_n = \max\{0, a_n\}$ , then  $\sigma_n \rightarrow 0$  as  $n \rightarrow \infty$  and (1.3) reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + \sigma_n, \quad \forall x, y \in K, n \geq 1. \quad (1.4)$$

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive. And every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediated sense. In [1], Goebel and Kirk proved that, if  $K$  is a nonempty closed convex bounded subset of a real uniformly convex Banach space  $X$ , and  $T$  is an asymptotically nonexpansive self-mapping on  $K$ , then  $T$  has a fixed point in  $K$ . The class of mappings which are asymptotically nonexpansive in the intermediat sense was investigated by Bruck et al. [2] and Kirk [3]. Since then, many authors have investigated the fixed point problem of these mappings based on implicit iterative methods or non-implicit iterative methods; see, for example, [4–21].

Let  $I : K \rightarrow K$  be a mapping. Recall that  $T$  is said to be asymptotically  $I$ -nonexpansive if there exists a sequence  $\{h_n\}$  with  $\{h_n\} \subset [1, +\infty)$  with  $\lim_{n \rightarrow \infty} h_n = 1$  such that

$$\|T^n x - T^n y\| \leq h_n \|I^n x - I^n y\|, \quad \forall x, y \in K, n \geq 1. \quad (1.5)$$

Recently, weak and strong convergence theorems for fixed points of  $I$ -nonexpansive mappings, and asymptotically  $I$ -nonexpansive mappings have been established by many scholar, see, for example, [22–25].

In this paper, we consider a new mapping based on asymptotically nonexpansive mappings in the intermediate sense and asymptotically  $I$ -nonexpansive mappings.

Let  $T : K \rightarrow K, I : K \rightarrow K$  be two mappings.  $T$  is said to be asymptotically  $I$ -nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in K} (\|T^n x - T^n y\| - \|I^n x - I^n y\|) \leq 0. \quad (1.6)$$

Observe that if we define  $a_n = \sup_{x, y \in K} (\|T^n x - T^n y\| - \|I^n x - I^n y\|)$ ,  $\sigma_n = \max\{0, a_n\}$ , then  $\sigma_n \rightarrow 0$  as  $n \rightarrow \infty$  and (1.6) reduces to

$$\|T^n x - T^n y\| \leq \|I^n x - I^n y\| + \sigma_n, \quad \forall x, y \in K, n \geq 1. \quad (1.7)$$

Note that if  $I = \text{Id}$ , where  $\text{Id}$  is the identity mapping, then (1.7) reduces to (1.4).

In this paper, we investigate asymptotically  $I$ -nonexpansive mappings in the intermediate sense based on a non-implicit iterative algorithm. Strong convergence of the implicit iterative algorithm is obtained in the framework of Banach spaces.

In order to prove our main results, we need the following lemmas.

**Lemma 1.1** (see [21]). *Let  $X$  be a uniformly convex Banach space. Let  $b$  and  $c$  be two constants with  $0 < b < c < 1$ . Suppose that  $\{t_n\}$  is a sequence in  $[b, c]$ . Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $X$  such that*

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x_n\| &\leq d, & \limsup_{n \rightarrow \infty} \|y_n\| &\leq d, \\ \lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| &= d \end{aligned} \quad (1.8)$$

hold for some  $d \geq 0$ , then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

**Lemma 1.2** (see [26]). *Let  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0, \quad (1.9)$$

where  $n_0$  is some nonnegative integer,  $\sum_{n=1}^{\infty} b_n < \infty$  and  $\sum_{n=1}^{\infty} c_n < \infty$ . Then the limit  $\lim_{n \rightarrow \infty} a_n$  exists.

## 2. Main Results

**Lemma 2.1.** *Let  $X$  be a real Banach space and  $K$  a nonempty closed and convex subset of  $X$ . Let  $T : K \rightarrow K$  be a asymptotically  $I$ -nonexpansive in the intermediate sense and  $I : K \rightarrow K$  a asymptotically nonexpansive in the intermediate sense. Assume that  $F := F(T) \cap F(I) \neq \emptyset$ . Let  $\sigma_n = \max\{0, \sup_{x, y \in K} (\|T^n x - T^n y\| - \|I^n x - I^n y\|)\}$  and  $\bar{\sigma}_n = \max\{0, \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|)\}$ . Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\hat{\alpha}_n\}$ ,  $\{\hat{\beta}_n\}$ ,  $\{\hat{\gamma}_n\}$  be six real number sequences in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:*

$$\begin{aligned} x_1 &\in C, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n I^n x_n + \hat{\gamma}_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1, \end{aligned} \quad (2.1)$$

where  $\{u_n\}$  and  $\{v_n\}$  be two bounded sequences in  $K$ . Assume that the following restrictions are satisfied:

- (a)  $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$ ;
- (b)  $\sum_{n=1}^{\infty} \sigma_n < \infty$ ,  $\sum_{n=1}^{\infty} \bar{\sigma}_n < \infty$ ;
- (c)  $\sum_{n=1}^{\infty} \gamma_n < \infty$ ,  $\sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$ .

Then  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F$ .

*Proof.* Letting  $p \in F$ , we see that

$$\begin{aligned}
\|y_n - p\| &= \left\| (1 - \hat{\beta}_n - \hat{\gamma}_n)x_n + \hat{\beta}_n I^n x_n + \hat{\gamma}_n v_n - p \right\| \\
&\leq (1 - \hat{\beta}_n - \hat{\gamma}_n) \|x_n - p\| + \hat{\beta}_n \|I^n x_n - p\| + \hat{\gamma}_n \|v_n - p\| \\
&\leq (1 - \hat{\beta}_n - \hat{\gamma}_n) \|x_n - p\| + \hat{\beta}_n (\|x_n - p\| + \bar{\sigma}_n) + \hat{\gamma}_n \|v_n - p\| \\
&= (1 - \hat{\gamma}_n) \|x_n - p\| + \hat{\gamma}_n \|v_n - p\| + \hat{\beta}_n \bar{\sigma}_n \\
&\leq \|x_n - p\| + \hat{\gamma}_n \|v_n - p\| + \hat{\beta}_n \bar{\sigma}_n,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| (1 - \beta_n - \gamma_n)x_n + \beta_n T^n y_n + \gamma_n u_n - p \right\| \\
&\leq (1 - \beta_n - \gamma_n) \|x_n - p\| + \beta_n \|T^n y_n - p\| + \gamma_n \|u_n - p\| \\
&= (1 - \beta_n - \gamma_n) \|x_n - p\| + \beta_n \|T^n y_n - T^n p\| + \gamma_n \|u_n - p\| \\
&\leq (1 - \beta_n - \gamma_n) \|x_n - p\| + \beta_n (\|I^n y_n - I^n p\| + \sigma_n) + \gamma_n \|u_n - p\| \\
&= (1 - \beta_n - \gamma_n) \|x_n - p\| + \beta_n \|I^n y_n - I^n p\| + \beta_n \sigma_n + \gamma_n \|u_n - p\| \\
&\leq (1 - \beta_n - \gamma_n) \|x_n - p\| + \beta_n (\|y_n - p\| + \bar{\sigma}_n) + \beta_n \sigma_n + \gamma_n \|u_n - p\| \\
&= (1 - \beta_n - \gamma_n) \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|u_n - p\| + \beta_n (\sigma_n + \bar{\sigma}_n) \\
&\leq (1 - \beta_n) \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|u_n - p\| + \beta_n (\sigma_n + \bar{\sigma}_n).
\end{aligned} \tag{2.3}$$

Substituting (2.2) into (2.3), we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \beta_n) \|x_n - p\| + \beta_n (\|x_n - p\| + \hat{\gamma}_n \|v_n - p\| + \hat{\beta}_n \bar{\sigma}_n) \\
&\quad + \gamma_n \|u_n - p\| + \beta_n (\sigma_n + \bar{\sigma}_n) \\
&= \|x_n - p\| + \left[ \beta_n \hat{\gamma}_n \|v_n - p\| + \gamma_n \|u_n - p\| + \beta_n \sigma_n + \beta_n \bar{\sigma}_n (1 + \hat{\beta}_n) \right].
\end{aligned} \tag{2.4}$$

Let  $a_n = \|x_n - p\|$ ,  $b_n = 0$ , and

$$c_n = \beta_n \hat{\gamma}_n \|v_n - p\| + \gamma_n \|u_n - p\| + \beta_n \sigma_n + \beta_n \bar{\sigma}_n (1 + \hat{\beta}_n). \tag{2.5}$$

It follows from (2.4) that

$$a_{n+1} \leq a_n + c_n. \tag{2.6}$$

In view of the restrictions (b) and (c), we see that  $\sum_{n=1}^{\infty} c_n < \infty$ . We can easily conclude the desired conclusion with the aid of Lemma 1.2. This completes the proof of Lemma 2.1.  $\square$

**Theorem 2.2.** Let  $X$  be a real Banach space and  $K$  a nonempty closed and convex subset of  $X$ . Let  $T : K \rightarrow K$  be a asymptotically  $I$ -nonexpansive in the intermediate sense and  $I : K \rightarrow K$  a asymptotically nonexpansive in the intermediate sense. Assume that  $F := F(T) \cap F(I) \neq \emptyset$ . Let  $\sigma_n = \max\{0, \sup_{x,y \in K} (\|T^n x - T^n y\| - \|I^n x - I^n y\|)\}$  and  $\bar{\sigma}_n = \max\{0, \sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|)\}$ . Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}, \{\hat{\gamma}_n\}$  be six real number sequences in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:

$$\begin{aligned} x_1 &\in C, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n I^n x_n + \hat{\gamma}_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1, \end{aligned} \tag{2.7}$$

where  $\{u_n\}$  and  $\{v_n\}$  be two bounded sequences in  $K$ . Assume that the following restrictions are satisfied:

- (a)  $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$ ;
- (b)  $\sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \bar{\sigma}_n < \infty$ ;
- (c)  $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$ .

If both  $T$  and  $I$  are continuous, then the sequence  $\{x_n\}$  strongly converges to a common fixed point of  $T$  and  $I$  if and only if

$$\liminf_{n \rightarrow \infty} d(x_n, F) = 0. \tag{2.8}$$

*Proof.* The necessity is obvious. Next, we prove the sufficiency part of the theorem. Note that continuity of  $T$  and  $I$  implies that the set  $F(T)$  and  $F(I)$  are closed. It follows from (2.6) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + c_n. \tag{2.9}$$

This implies in turn that

$$d(x_{n+1}, F) \leq d(x_n, F) + c_n. \tag{2.10}$$

Now applying Lemma 1.2 to (2.10), we obtain the existence of the limit  $\lim_{n \rightarrow \infty} d(x_n, F)$ . By condition (2.8), we have

$$\lim_{n \rightarrow \infty} d(x_n, F) = \liminf_{n \rightarrow \infty} d(x_n, F) = 0. \tag{2.11}$$

Next we prove that the sequence  $\{x_n\}$  is a Cauchy sequence in  $K$ . For any positive integers  $n, m$ , from (2.9) it follows that

$$\begin{aligned}
 \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + c_{n+m-1} \\
 &\leq (\|x_{n+m-2} - p\| + c_{n+m-2}) + c_{n+m-1} \\
 &\leq \dots \\
 &\leq \|x_n - p\| + \sum_{i=n}^{n+m-1} c_i \\
 &\leq \|x_n - p\| + \sum_{i=n}^{\infty} c_i.
 \end{aligned} \tag{2.12}$$

Since  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ , and  $\sum_{n=1}^{\infty} c_n < \infty$ , for any given  $\epsilon > 0$ , there exists a positive integer  $n_0$  such that

$$d(x_n, F) < \frac{\epsilon}{8}, \quad \sum_{i=n}^{\infty} c_i < \frac{\epsilon}{2}, \quad \forall n \geq n_0. \tag{2.13}$$

Therefore there exists  $p_1 \in F$  such that  $d(x_n, p_1) < (\epsilon/4)$ ,  $\forall n \geq n_0$ . Consequently, for any  $n \geq n_0$  and for all  $m \geq 1$ , we have

$$\begin{aligned}
 \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
 &\leq 2\|x_n - p_1\| + \sum_{i=n}^{\infty} c_i \\
 &\leq \frac{\epsilon}{4} \cdot 2 + \frac{\epsilon}{2} = \epsilon.
 \end{aligned} \tag{2.14}$$

This implies that  $\{x_n\}$  is a Cauchy sequence in  $K$ . Let  $x_n \rightarrow x^* \in K$ . Since  $F$  is closed, this implies that  $x^* \in F$ . This shows that  $\{x_n\}$  strongly converges to a common fixed of  $T$  and  $I$ . This completes the proof of Theorem 2.2.  $\square$

**Lemma 2.3.** *Let  $X$  be a real Banach space and  $K$  a nonempty closed and convex subset of  $X$ . Let  $T : K \rightarrow K$  be a asymptotically  $I$ -nonexpansive in the intermediate sense and  $I : K \rightarrow K$  a asymptotically nonexpansive in the intermediate sense. Assume that  $F := F(T) \cap F(I) \neq \emptyset$ . Let  $\sigma_n = \max\{0, \sup_{x,y \in K} (\|T^n x - T^n y\| - \|I^n x - I^n y\|)\}$  and  $\bar{\sigma}_n = \max\{0, \sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|)\}$ . Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}, \{\hat{\gamma}_n\}$  be six real number sequences in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:*

$$\begin{aligned}
 x_1 &\in C, \\
 y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n I^n x_n + \hat{\gamma}_n v_n, \\
 x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1,
 \end{aligned} \tag{2.15}$$

where  $\{u_n\}$  and  $\{v_n\}$  be two bounded sequences in  $K$ . Assume that the following restrictions are satisfied:

$$(a) \alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \forall n \geq 1;$$

$$(b) \sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \overline{\sigma}_n < \infty;$$

$$(c) \text{ there exist constants } \tau_1, \tau_2 \in (0, 1) \text{ such that } \tau_1 \leq \beta_n, \widehat{\beta}_n \leq \tau_2, \forall n \geq 1;$$

$$(d) \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty.$$

Then

$$\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0, \quad (2.16)$$

$$\lim_{n \rightarrow \infty} \|x_n - I^n x_n\| = 0. \quad (2.17)$$

*Proof.* According to Lemma 2.1, for any  $p \in F$ , we have  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists. Without loss of generality, we may assume that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = d, \quad (2.18)$$

where  $d > 0$  is some constant. It follows that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \lim_{n \rightarrow \infty} \left\| (1 - \beta_n) [x_n - p + \gamma_n(u_n - x_n)] + \beta_n [T^n y_n - p + \gamma_n(u_n - x_n)] \right\| = d. \quad (2.19)$$

Notice that

$$\|x_n - p + \gamma_n(u_n - x_n)\| \leq \|x_n - p\| + \gamma_n \|u_n - x_n\|. \quad (2.20)$$

It follows from the restriction (d) and (2.18) that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n(u_n - x_n)\| \leq d. \quad (2.21)$$

Notice that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \|T^n y_n - p + \gamma_n(u_n - x_n)\| &\leq \limsup_{n \rightarrow \infty} \|T^n y_n - p\| + \limsup_{n \rightarrow \infty} \gamma_n \|u_n - x_n\| \\
&= \limsup_{n \rightarrow \infty} \|T^n y_n - p\| \\
&\leq \limsup_{n \rightarrow \infty} (\|I^n y_n - I^n p\| + \sigma_n) \\
&\leq \limsup_{n \rightarrow \infty} (\|y_n - p\| + \bar{\sigma}_n + \sigma_n) \\
&= \limsup_{n \rightarrow \infty} \|y_n - p\| \\
&\leq \limsup_{n \rightarrow \infty} (\|x_n - p\| + \hat{\gamma}_n \|v_n - p\| + \hat{\beta}_n \bar{\sigma}_n) \\
&= d.
\end{aligned} \tag{2.22}$$

In view of (2.19), (2.21) and (2.22), we obtain from Lemma 1.1 that

$$\lim_{n \rightarrow \infty} \|x_n - T^n y_n\| = 0. \tag{2.23}$$

Notice that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n(T^n y_n - x_n) + \gamma_n(u_n - x_n)\| \\
&\leq \beta_n \|T^n y_n - x_n\| + \gamma_n \|u_n - x_n\|.
\end{aligned} \tag{2.24}$$

It follows from (2.23) and the restriction (d) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{2.25}$$

Notice that

$$\|I^n x_n - p\| \leq \|x_n - p\| + \bar{\sigma}_n. \tag{2.26}$$

It follows that

$$\limsup_{n \rightarrow \infty} \|I^n x_n - p\| \leq d. \tag{2.27}$$

On the other hand, we have

$$\lim_{n \rightarrow \infty} \|y_n - p\| = \lim_{n \rightarrow \infty} \left\| (1 - \hat{\beta}_n)[x_n - p + \hat{\gamma}_n(v_n - x_n)] + \hat{\beta}_n[I^n x_n - p + \hat{\gamma}_n(v_n - x_n)] \right\| = d. \tag{2.28}$$



Notice that

$$\|x_n - p + \hat{\gamma}_n(v_n - x_n)\| \leq \|x_n - p\| + \hat{\gamma}_n\|v_n - x_n\|. \quad (2.29)$$

It follows that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \hat{\gamma}_n(v_n - x_n)\| \leq d. \quad (2.30)$$

Notice that

$$\|I^n x_n - p + \hat{\gamma}_n(v_n - x_n)\| \leq \|I^n x_n - p\| + \hat{\gamma}_n\|v_n - x_n\|. \quad (2.31)$$

It follows from (2.27) that

$$\limsup_{n \rightarrow \infty} \|I^n x_n - p + \hat{\gamma}_n(v_n - x_n)\| \leq d. \quad (2.32)$$

In view of (2.28), (2.30), and (2.32), we obtain from Lemma 1.1 that

$$\lim_{n \rightarrow \infty} \|x_n - I^n x_n\| = 0. \quad (2.33)$$

On the other hand, we have

$$\begin{aligned} \|x_n - T^n x_n\| &\leq \|x_n - T^n y_n\| + \|T^n y_n - T^n x_n\| \\ &\leq \|x_n - T^n y_n\| + \|y_n - x_n\| + \sigma_n + \bar{\sigma}_n \\ &\leq \|x_n - T^n y_n\| + \hat{\beta}_n \|I^n x_n - x_n\| + \hat{\gamma}_n \|v_n - x_n\| + \sigma_n + \bar{\sigma}_n. \end{aligned} \quad (2.34)$$

In view of (2.23) and (2.33), we have  $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$ . This completes the proof of Lemma 2.3.  $\square$

**Theorem 2.4.** *Let  $X$  be a real Banach space and  $K$  a nonempty closed and convex subset of  $X$ . Let  $T : K \rightarrow K$  be a asymptotically  $I$ -nonexpansive in the intermediate sense and  $I : K \rightarrow K$  a asymptotically nonexpansive in the intermediate sense. Assume that  $F := F(T) \cap F(I) \neq \emptyset$ . Let  $\sigma_n = \max\{0, \sup_{x,y \in K} (\|T^n x - T^n y\| - \|I^n x - I^n y\|)\}$  and  $\bar{\sigma}_n = \max\{0, \sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|)\}$ . Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}, \{\hat{\gamma}_n\}$  be six real number sequences in  $(0, 1)$ . Assume that both  $T$  and  $I$  are Lipschitz continuous. Let  $\{x_n\}$  are a sequence generated in the following iterative process:*

$$\begin{aligned} x_1 &\in C, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n I^n x_n + \hat{\gamma}_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1, \end{aligned} \quad (2.35)$$

where  $\{u_n\}$  and  $\{v_n\}$  be two bounded sequences in  $K$ . Assume that the following restrictions are satisfied:

- (a)  $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \forall n \geq 1$ ;
- (b)  $\sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \overline{\sigma}_n < \infty$ ;
- (c) there exist constants  $\tau_1, \tau_2 \in (0, 1)$  such that  $\tau_1 \leq \beta_n, \widehat{\beta}_n \leq \tau_2, \forall n \geq 1$ ;
- (d)  $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty$ .

If at least one of the mappings  $T$  and  $I$  is compact, then the sequence convergence strongly to a common fixed point of  $T$  and  $I$ .

*Proof.* Without loss of generality, we may assume that  $T$  is compact; this means that there exists a subsequence  $\{T^{n_k} x_{n_k}\}$  of  $\{T^n x_n\}$  such that  $\{T^{n_k} x_{n_k}\}$  converges strongly to  $x^* \in K$ , then (2.16) implies that  $\{x_{n_k}\}$  converges strongly to  $x^*$ . Since  $T$  is continuous, then  $\{T^{n_k+1} x_{n_k}\}$  converges strongly to  $Tx^*$ . On the other hand, according to (2.17) and the continuity of  $I$ , we obtain that  $\{I^{n_k} x_{n_k}\}, \{I^{n_k+1} x_{n_k}\}$  converge strongly to  $x^*, Ix^*$ , respectively. Since  $\lim_{k \rightarrow \infty} \|x_{n_k+1} - x_{n_k}\| = 0$ , then

$$\begin{aligned} \|I^{n_k+1} x_{n_k+1} - I^{n_k+1} x_{n_k}\| &\leq \|x_{n_k+1} - x_{n_k}\| + \overline{\sigma}_{n_k} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \\ \|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\| &\leq \|I^{n_k+1} x_{n_k+1} - I^{n_k+1} x_{n_k}\| + \sigma_{n_k} \\ &\leq \|x_{n_k+1} - x_{n_k}\| + \overline{\sigma}_{n_k} + \sigma_{n_k} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \end{aligned} \quad (2.36)$$

Observe that

$$\begin{aligned} \|x^* - Tx^*\| &\leq \|x^* - x_{n_k+1}\| + \|x_{n_k+1} - T^{n_k+1} x_{n_k+1}\| \\ &\quad + \|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\| + \|T^{n_k+1} x_{n_k} - Tx^*\|, \\ \|x^* - Ix^*\| &\leq \|x^* - x_{n_k+1}\| + \|x_{n_k+1} - I^{n_k+1} x_{n_k+1}\| \\ &\quad + \|I^{n_k+1} x_{n_k+1} - I^{n_k+1} x_{n_k}\| + \|I^{n_k+1} x_{n_k} - Ix^*\|. \end{aligned} \quad (2.37)$$

Taking limit as  $k \rightarrow \infty$  in the above inequality, we find  $x^* = Tx^*, x^* = Ix^*$ , which means  $x^* \in F$ . However, due to Lemma 2.1, the limit  $\lim_{n \rightarrow \infty} \|x_n - x^*\|$  exists, therefore

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| = \lim_{k \rightarrow \infty} \|x_{n_k} - x^*\| = 0, \quad (2.38)$$

which means that  $\{x_n\}$  converges strongly to  $x^* \in F$ . This completes the proof of Theorem 2.4.  $\square$

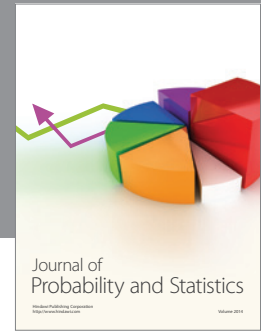
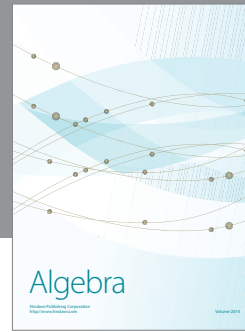
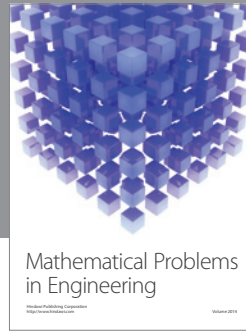
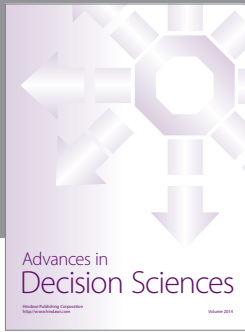
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