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*Research Article*

# Application of Bifurcation Method to the Generalized Zakharov Equations

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We use the bifurcation method of dynamical systems to study the traveling wave solutions for the generalized Zakharov equations. A number of traveling wave solutions are obtained. Those solutions contain explicit periodic blow-up wave solutions and solitary wave solutions.

## 1. Introduction

The Zakharov equations

$$\begin{aligned} &iu_t + u_{xx} - uv = 0, \\ &v_{tt} - v_{xx} + \left(|u|^2\right)_{xx} = 0, \end{aligned} \tag{1.1}$$

are of the fundamental models governing dynamics of nonlinear waves in one-dimensional systems. The Zakharov equation describe the interaction between high-frequency and low-frequency waves. The physically most important example involves the interaction between the Langmuir and ion-acoustic waves in plasmas [1]. The equations can be derived from a hydrodynamic description of the plasma [2, 3]. However, some important effects such as transit-time damping and ion nonlinearities, which are also implied by the fact that the values used for the ion damping have been anomalously large from the point of view of linear ion-acoustic wave dynamics, have been ignored in (1.1). This is equivalent to saying that (1.1) is a simplified model of strong Langmuir turbulence. Thus, we have to generalize (1.1) by taking more elements into account. Starting from the dynamical plasma equations with the help of relaxed Zakharov simplification assumptions, and through making use of the time-averaged two-time-scale two-fluid plasma description, (1.1) are generalized to contain the self-generated magnetic field [4–6] and the first related study on magnetized plasmas in [7, 8]. The generalized Zakharov equations are a set of coupled equations and may be written as [9]

$$\begin{gathered} iu_t + u_{xx} - 2\lambda|u|^2 u + 2uv = 0, \\ v_{tt} - v_{xx} + \left(|u|^2\right)_{xx} = 0. \end{gathered} \tag{1.2}$$

Malomed et al. [9] analyzed internal vibrations of a solitary wave in (1.2) by means of a variational approach. Wang and Li [10] obtained a number of periodic wave solutions of (1.2) by using extended F-expansion method. Javidi and Golbabai [11] used the He's variational iteration method to obtain solitary wave solutions of (1.2). Layeni [12] obtained the exact traveling wave solutions of (1.2) by using the new rational auxiliary equation method. Zhang [13] used He's semi-inverse method to search for solitary wave solutions of (1.2). Javidi and Golbabai [14] obtained the exact and numerical solutions of (1.2) by using the variational iteration method. Li et al. [15] used the Exp-function method to seek exact solutions of (1.2). Borhanifar et al. [16] obtained the generalized solitary solutions and periodic solutions of (1.2) by using the Exp-function method. Khan et al. [17] used He's variational approach to obtain new soliton solutions of (1.2). Song and Liu [18] obtained a number of traveling wave solutions of (1.2) by using bifurcation method of dynamical systems.

In this work, we aim to study new generalized Zakharov equations.

$$\begin{gathered} iu_t + u_{xx} - 2\lambda|u|^{2n} u + 2uv = 0, \\ v_{tt} - v_{xx} + \left(|u|^{2n}\right)_{xx} = 0, \end{gathered} \tag{1.3}$$

where $n$ is positive integer. First, we aim to apply the bifurcation method of dynamical systems [18–22] to study the phase portraits for the corresponding traveling wave system of (1.3). Our second aim is to obtain exact traveling wave solutions for (1.3).

The remainder of this paper is organized as follows. In Section 2, by using the bifurcation theory of planar dynamical systems, two phase portraits for the corresponding traveling wave system of (1.3) are given under different parameter conditions. The relations between the traveling wave solutions and the Hamiltonian $h$ are presented. In Section 3, we obtain a number of traveling wave solutions of (1.3). A short conclusion will be given in Section 4.

## 2. Phase Portraits and Qualitative Analysis

We assume that the traveling wave solutions of (1.3) is of the form

$$u(x,t) = e^{i\eta}\varphi(\xi), \qquad v(x,t) = \psi(\xi), \qquad \eta = px + qt, \qquad \xi = k(x - 2pt), \tag{2.1}$$

where $\varphi(\xi)$ and $\psi(\xi)$ are real functions, $p$, $q$, and $k$ are real constants.

Substituting (2.1) into (1.3), we have

$$k^2\varphi'' + 2\varphi\psi - \left(p^2 + q\right)\varphi - 2\lambda\varphi^{2n+1} = 0,$$
$$k^2\left(4p^2 - 1\right)\psi'' + k^2\left(\varphi^{2n}\right)'' = 0. \tag{2.2}$$

Integrating the second equation of (2.2) twice and letting the first integral constant be zero, we have

$$\psi = \frac{\varphi^{2n}}{1 - 4p^2} + g, \quad p \neq \frac{1}{2}, \tag{2.3}$$

where $g$ is the second integral constant.

Substituting (2.3) into the first equation of (2.2), we have

$$k^2\varphi'' + \left(2g - p^2 - q\right)\varphi + 2\left(\frac{1}{1 - 4p^2} - \lambda\right)\varphi^{2n+1} = 0. \tag{2.4}$$

Letting $\varphi' = y$, $\alpha = (2/k^2)(\lambda - 1/(1 - 4p^2))$, and $\beta = (2g - p^2 - q)/k^2$, then we get the following planar system:

$$\frac{d\varphi}{d\xi} = y,$$
$$\frac{dy}{d\xi} = \alpha\varphi^{2n+1} - \beta\varphi. \tag{2.5}$$

Obviously, system (2.5) is a Hamiltonian system with Hamiltonian function

$$H(\varphi, y) = y^2 - \frac{\alpha}{n+1}\varphi^{2n+2} + \beta\varphi^2. \tag{2.6}$$

In order to investigate the phase portrait of (2.5), set

$$f(\varphi) = \alpha\varphi^{2n+1} - \beta\varphi. \tag{2.7}$$

Obviously, $f(\varphi)$ has three zero points, $\varphi_-$, $\varphi_0$, and $\varphi_+$, which are given as follows:

$$\varphi_- = -\left(\frac{\beta}{\alpha}\right)^{1/2n}, \qquad \varphi_0 = 0, \qquad \varphi_+ = \left(\frac{\beta}{\alpha}\right)^{1/2n}. \tag{2.8}$$

Letting $(\varphi_i, 0)$ be one of the singular points of system (2.5), then the characteristic values of the linearized system of system (2.5) at the singular points $(\varphi_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(\varphi_i)}. \tag{2.9}$$

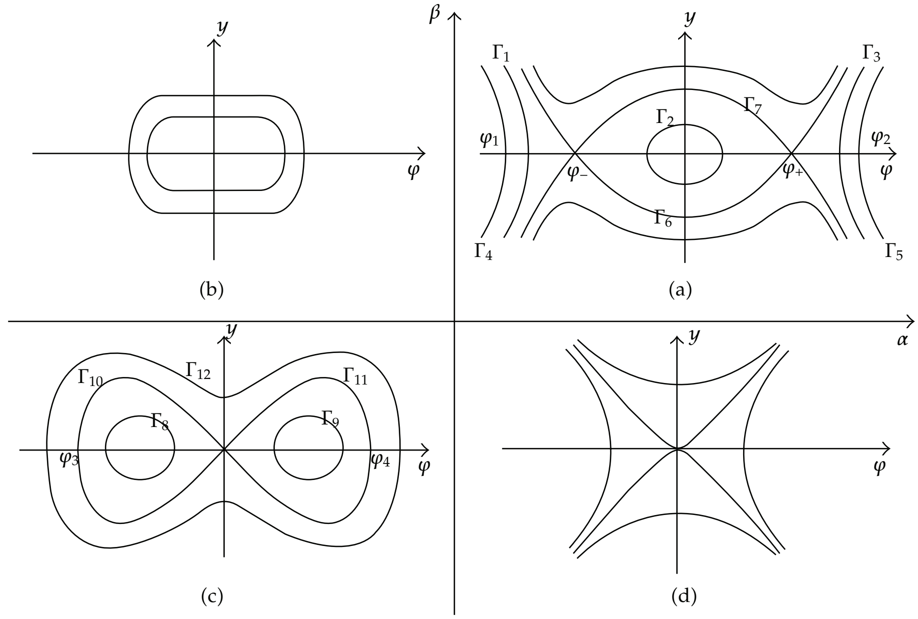


**Figure 1:** The bifurcation phase portraits of system (2.5). (a) $\alpha > 0$, $\beta > 0$, (b) $\alpha < 0$, $\beta \geqslant 0$, (c) $\alpha < 0$, $\beta < 0$, (d) $\alpha > 0$, and $\beta \leqslant 0$.

From the qualitative theory of dynamical systems, we know that

(1) if $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point;

(2) if $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point;

(3) if $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the bifurcation phase portraits of system (2.5) in Figure 1.

Let

$$H(\varphi, y) = h, \tag{2.10}$$

where $h$ is Hamiltonian.

Next, we consider the relations between the orbits of (2.5) and the Hamiltonian $h$.

Set

$$h^* = |H(\varphi_+, 0)| = |H(\varphi_-, 0)|. \tag{2.11}$$

According to Figure 1, we get the following propositions.

**Proposition 2.1.** *Suppose that $\alpha > 0$ and $\beta > 0$, one has:*

(1) *when $h < 0$ or $h > h^*$, system* (2.5) *does not have any closed orbit;*

(2) *when $0 < h < h^*$, system* (2.5) *has three periodic orbits $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$;*

(3) *when $h = 0$, system* (2.5) *has two periodic orbits $\Gamma_4$ and $\Gamma_5$;*

(4) *when $h = h^*$, system* (2.5) *has two heteroclinic orbits $\Gamma_6$ and $\Gamma_7$.*

**Proposition 2.2.** *Suppose that $\alpha < 0$ and $\beta < 0$, one has:*

(1) *when $h \leq -h^*$, system* (2.5) *does not have any closed orbit;*

(2) *when $-h^* < h < 0$, system* (2.5) *has two periodic orbits $\Gamma_8$ and $\Gamma_9$;*

(3) *when $h = 0$, system* (2.5) *has two homoclinic orbits $\Gamma_{10}$ and $\Gamma_{11}$;*

(4) *when $h > 0$, system* (2.5) *has a periodic orbit $\Gamma_{12}$.*

**Proposition 2.3.** (1) *When $\alpha < 0$, $\beta \geq 0$, and $h > 0$, system* (2.5) *has a periodic orbits.*
(2) *When $\alpha > 0$, $\beta \leqslant 0$, system* (2.5) *does not have any closed orbit.*

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink wave solution or a unbounded wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to the above analysis, we have the following propositions.

**Proposition 2.4.** *If $\alpha > 0$ and $\beta > 0$, one has the following:*

(1) *when $0 < h < h^*$,* (1.3) *has two periodic wave solutions (corresponding to the periodic orbit $\Gamma_2$ in Figure 1) and two periodic blow-up wave solutions (corresponding to the periodic orbits $\Gamma_1$ and $\Gamma_3$ in Figure 1);*

(2) *when $h = 0$,* (1.3) *has periodic blow-up wave solutions (corresponding to the periodic orbits $\Gamma_4$ and $\Gamma_5$ in Figure 1);*

(3) *when $h = h^*$,* (1.3) *has two kink profile solitary wave solutions and two unbounded wave solutions (corresponding to the heteroclinic orbits $\Gamma_6$ and $\Gamma_7$ in Figure 1).*

**Proposition 2.5.** *If $\alpha < 0$ and $\beta < 0$, one has the following:*

(1) *when $-h^* < h < 0$,* (1.3) *has two periodic wave solutions (corresponding to the periodic orbits $\Gamma_8$ and $\Gamma_9$ in Figure 1);*

(2) *when $h = 0$,* (1.3) *has two solitary wave solutions (corresponding to the homoclinic orbits $\Gamma_{10}$ and $\Gamma_{11}$ in Figure 1);*

(3) *when $h > 0$,* (1.3) *has two periodic wave solutions (corresponding to the periodic orbit $\Gamma_{12}$ in Figure 1).*

## 3. Exact Traveling Wave Solutions of (1.3)

Firstly, we will obtain the explicit expressions of traveling wave solutions for (1.3) when $\alpha > 0$ and $\beta > 0$. From the phase portrait, we note that there are two special orbits $\Gamma_4$ and $\Gamma_5$, which have the same Hamiltonian with that of the center point $(0,0)$. In $(\varphi, y)$ plane, the expressions of the orbits are given as

$$y = \pm\sqrt{\frac{\alpha}{n+1}}\varphi\sqrt{\varphi^{2n} - \frac{(n+1)\beta}{\alpha}}. \tag{3.1}$$

Substituting (3.1) into $d\varphi/d\xi = y$ and integrating them along the two orbits $\Gamma_4$ and $\Gamma_5$, it follows that

$$\begin{aligned}
&\pm\int_{\varphi}^{+\infty}\frac{1}{s\sqrt{s^{2n}-((n+1)\beta)/\alpha}}ds=\sqrt{\frac{\alpha}{n+1}}\int_0^{\xi}ds,\\
&\pm\int_{\varphi_2}^{\varphi}\frac{1}{s\sqrt{s^{2n}-((n+1)\beta)/\alpha}}ds=\sqrt{\frac{\alpha}{n+1}}\int_0^{\xi}ds,
\end{aligned}\tag{3.2}$$

where $\varphi_2 = (((n+1)\beta)/\alpha)^{1/2n}$.

Completing above integrals, we obtain

$$\begin{aligned}
\varphi&=\pm\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\csc n\sqrt{\beta}\xi\right)^{1/n},\\
\varphi&=\pm\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\sec n\sqrt{\beta}\xi\right)^{1/n}.
\end{aligned}\tag{3.3}$$

Noting (2.1) and (2.3), we get the following periodic blow-up wave solutions:

$$\begin{aligned}
u_1(x,t)&=\pm e^{i\eta}\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\csc n\sqrt{\beta}\xi\right)^{1/n},\\
v_1(x,t)&=\frac{(n+1)\beta\left(\csc n\sqrt{\beta}\xi\right)^2}{\alpha(1-4p^2)}+g,\\
u_2(x,t)&=\pm e^{i\eta}\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\sec n\sqrt{\beta}\xi\right)^{1/n},\\
v_2(x,t)&=\frac{(n+1)\beta\left(\sec n\sqrt{\beta}\xi\right)^2}{\alpha(1-4p^2)}+g,
\end{aligned}\tag{3.4}$$

where $\eta = px + qt$ and $\xi = k(x - 2pt)$.

Secondly, we will obtain the explicit expressions of traveling wave solutions for (1.3) when $\alpha < 0$ and $\beta < 0$. From the phase portrait, we see that there are two symmetric homoclinic orbits $\Gamma_{10}$ and $\Gamma_{11}$ connected at the saddle point $(0,0)$. In $(\varphi, y)$ plane, the expressions of the homoclinic orbits are given as

$$y=\pm\sqrt{-\frac{\alpha}{n+1}}\varphi\sqrt{-\varphi^{2n}+\frac{(n+1)\beta}{\alpha}}.\tag{3.5}$$

Substituting (3.5) into $d\varphi/d\xi = y$ and integrating them along the orbits $\Gamma_{10}$ and $\Gamma_{11}$, we have

$$\pm\int_{\varphi_{11}}^{\varphi}\frac{1}{s\sqrt{-s^{2n}+\left((n+1)\beta\right)/\alpha}}ds=\sqrt{-\frac{\alpha}{n+1}}\int_{0}^{\xi}ds,$$
$$\pm\int_{\varphi_{12}}^{\varphi}\frac{1}{s\sqrt{-s^{2n}+\left((n+1)\beta\right)/\alpha}}ds=\sqrt{-\frac{\alpha}{n+1}}\int_{0}^{\xi}ds. \tag{3.6}$$

Completing above integrals, we obtain

$$\varphi=\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\operatorname{sech} n\sqrt{-\beta}\xi\right)^{1/n},$$
$$\varphi=-\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\operatorname{sech} n\sqrt{-\beta}\xi\right)^{1/n}. \tag{3.7}$$

Noting (2.1) and (2.3), we get the following solitary wave solutions:

$$u_3(x,t)=e^{i\eta}\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\operatorname{sech} n\sqrt{-\beta}\xi\right)^{1/n},$$
$$v_3(x,t)=\frac{(n+1)\beta\left(\operatorname{sech} n\sqrt{-\beta}\xi\right)^{2}}{\alpha(1-4p^2)}+g,$$
$$u_4(x,t)=-e^{i\eta}\left(\sqrt{\frac{(n+1)\beta}{\alpha}}\operatorname{sech} n\sqrt{-\beta}\xi\right)^{1/n},$$
$$v_4(x,t)=\frac{(n+1)\beta\left(\operatorname{sech} n\sqrt{-\beta}\xi\right)^{2}}{\alpha(1-4p^2)}+g, \tag{3.8}$$

where $\eta = px + qt$ and $\xi = k(x - 2pt)$.

## 4. Conclusion

In this paper, we obtain phase portraits for the corresponding traveling wave system of (1.3) by using the bifurcation theory of planar dynamical systems. Furthermore, a number of exact traveling wave solutions are also obtained. The method can be applied to many other nonlinear evolution equations, and we believe that many new results wait for further discovery by this method.

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