Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 310818, 26 pages
doi:10.1155/2012/310818

Research Article

Minimax Theorems for Set-Valued Mappings under Cone-Convexities

Yen-Cherng Lin,[1] Qamrul Hasan Ansari,[2,3] and Hang-Chin Lai[4]

[1] *Department of Occupational Safety and Health, College of Public Health, China Medical University, Taichung 404, Taiwan*
[2] *Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India*
[3] *Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
[4] *Department of Mathematics, National Tsing Hua University, Hsinchu 300, Taiwan*

Correspondence should be addressed to Yen-Cherng Lin, yclin@mail.cmu.edu.tw

Received 7 September 2012; Accepted 27 October 2012

Academic Editor: Ondrej Dosly

Copyright © 2012 Yen-Cherng Lin et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The aim of this paper is to study the minimax theorems for set-valued mappings with or without linear structure. We define several kinds of cone-convexities for set-valued mappings, give some examples of such set-valued mappings, and study the relationships among these cone-convexities. By using our minimax theorems, we derive some existence results for saddle points of set-valued mappings. Some examples to illustrate our results are also given.

1. Introduction

The minimax theorems for real-valued functions were introduced by Fan [1, 2] in the early fifties. Since then, these were extended and generalized in many different directions because of their applications in variational analysis, game theory, mathematical economics, fixed-point theory, and so forth (see, for example, [3–11] and the references therein). The minimax theorems for vector-valued functions have been studied in [4, 9, 10] with applications to vector saddle point problems. However, the minimax theorems for set-valued bifunctions have been studied only in few papers, namely, [4–8] and the references therein.

In this paper, we establish some new minimax theorems for set-valued mappings. Section 2 deals with preliminaries which will be used in rest of the paper. Section 3 denotes the cone-convexities of set-valued mappings. In Section 4, we establish some minimax theorems by using separation theorems, Fan-Browder fixed-point theorem. In the last section, we discuss some existence results for different kinds of saddle points for set-valued mappings.

2. Preliminaries

Throughout the paper, unless otherwise specified, we assume that X, Y are two nonempty subsets, and $\mathcal{Z}$ is a real Hausdorff topological vector space, C is a closed convex pointed cone in $\mathcal{Z}$ with $\operatorname{int} C \neq \emptyset$. Let $\mathcal{Z}^*$ be the topological dual space of $\mathcal{Z}$, and let

$$C^* = \left\{ g \in \mathcal{Z}^* : g(c) \geq 0 \, \forall c \in C \right\}. \tag{2.1}$$

We present some fundamental concepts which will be used in the sequel.

Definition 2.1 (see [3, 4, 8]). Let A be a nonempty subset of $\mathcal{Z}$. A point $z \in A$ is called a

(a) *minimal point* of A if $A \cap (z - C) = \{z\}$; $\operatorname{Min} A$ denotes the set of all minimal points of A;

(b) *maximal point* of A if $A \cap (z + C) = \{z\}$; $\operatorname{Max} A$ denotes the set of all maximal points of A;

(c) *weakly minimal point* of A if $A \cap (z - \operatorname{int} C) = \emptyset$; $\operatorname{Min}_w A$ denotes the set of all weakly minimal points of A;

(d) *weakly maximal point* of A if $A \cap (z + \operatorname{int} C) = \emptyset$; $\operatorname{Max}_w A$ denotes the set of all weakly maximal points of A.

It can be easily seen that $\operatorname{Min} A \subset \operatorname{Min}_w A$ and $\operatorname{Max} A \subset \operatorname{Max}_w A$.

Lemma 2.2 (see [3, 4]). *Let A be a nonempty compact subset of $\mathcal{Z}$. Then,*

(a) $\operatorname{Min} A \neq \emptyset$;

(b) $A \subset \operatorname{Min} A + C$;

(c) $\operatorname{Max} A \neq \emptyset$;

(d) $A \subset \operatorname{Max} A - C$.

Following [6], we denote both Max and Max_w by max (both Min and Min_w by min) in $\mathbb{R}$ since both Max and Max_w (both Min and Min_w) are the same in $\mathbb{R}$.

Definition 2.3. Let $\mathcal{X}$, $\mathcal{Y}$ be Hausdorff topological spaces. A set-valued map $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ with nonempty values is said to be

(a) *upper semicontinuous at* $x_0 \in \mathcal{X}$ if for every $x_0 \in \mathcal{X}$ and for every open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$;

(b) *lower semi-continuous at* $x_0 \in \mathcal{X}$ if for any sequence $\{x_n\} \subset \mathcal{X}$ such that $x_n \to x_0$ and any $y_0 \in F(x_0)$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to y_0$;

(c) *continuous at* $x_0 \in \mathcal{X}$ if F is upper semi-continuous as well as lower semi-continuous at x_0.

We present the following fundamental lemmas which will be used in the sequel.

Lemma 2.4 (see [9, Lemma 3.1]). *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be three topological spaces. Let $\mathcal{Y}$ be compact, $F : \mathcal{X} \times \mathcal{Y} \rightrightarrows \mathcal{Z}$ a set-valued mapping, and the set-valued mapping $T : \mathcal{X} \rightrightarrows \mathcal{Z}$ defined by*

$$T(x) = \bigcup_{y \in \mathcal{Y}} F(x, y), \quad \forall x \in \mathcal{X}. \tag{2.2}$$

(a) *If F is upper semi-continuous on $\mathcal{X} \times \mathcal{Y}$, then T is upper semi-continuous on $\mathcal{X}$.*

(b) *If F is lower semi-continuous on $\mathcal{X}$, so is T.*

Lemma 2.5 (see [9, Lemma 3.2]). *Let $\mathcal{Z}$ be a Hausdorff topological vector space, $F : \mathcal{Z} \rightrightarrows \mathbb{R}$ a set-valued mapping with nonempty compact values, and the functions $p, q : \mathcal{Z} \to \mathbb{R}$ defined by $p(z) = \max F(z)$ and $q(z) = \min F(z)$.*

(a) *If F is upper semi-continuous, so is p.*

(b) *If F is lower semi-continuous, so is p.*

(c) *If F is continuous, so are p and q.*

We shall use the following nonlinear scalarization function to establish our results.

Definition 2.6 (see [6, 10]). Let $k \in \operatorname{int} C$ and $v \in \mathcal{Z}$. The *Gerstewitz function* $\xi_{kv} : \mathcal{Z} \to \mathbb{R}$ is defined by

$$\xi_{kv}(u) = \min\{t \in \mathbb{R} : u \in v + tk - C\}. \tag{2.3}$$

We present some fundamental properties of the scalarization function.

Proposition 2.7 (see [6, 10]). *Let $k \in \operatorname{int} C$ and $v \in \mathcal{Z}$. The Gerstewitz function $\xi_{kv} : \mathcal{Z} \to \mathbb{R}$ has the following properties:*

(a) $\xi_{kv}(u) < r \Leftrightarrow u \in v + rk - \operatorname{int} C$;

(b) $\xi_{kv}(u) \leq r \Leftrightarrow u \in v + rk - C$;

(c) $\xi_{kv}(u) = 0 \Leftrightarrow u \in v - \partial C$, *where ∂C is the topological boundary of C;*

(d) $\xi_{kv}(u) > r \Leftrightarrow u \notin v + rk - C$;

(e) $\xi_{kv}(u) \geq r \Leftrightarrow u \notin v + rk - \operatorname{int} C$;

(f) $\xi_{kv}(\cdot)$ *is a convex function;*

(g) $\xi_{kv}(\cdot)$ *is an increasing function, that is, $u_2 - u_1 \in \operatorname{int} S \Rightarrow \xi_{kv}(u_1) < \xi_{kv}(u_2)$;*

(h) $\xi_{kv}(\cdot)$ *is a continuous function.*

Theorem 2.8 (Fan-Browder fixed-point theorem (see [12])). *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and let $T : X \rightrightarrows X$ be a set-valued mapping with nonempty convex values and open fibers, that is, $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open for all $y \in X$. Then, T has a fixed point.*

3. Cone-Convexities

In this section, we present different kinds of cone-convexities for set-valued mappings and give some relations among them. Some examples of such set-valued mappings are also given.

Definition 3.1. Let X be a nonempty convex subset of a topological vector space $\mathcal{W}$. A set-valued mapping $F : X \rightrightarrows \mathfrak{Z}$ is said to be

(a) *above -C-convex* [4] (resp., *above-C-concave* [5]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C,$$
$$\left(\text{resp., } \lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) - C\right); \tag{3.1}$$

(b) *below-C-convex* [13] (resp., *below-C-concave* [9, 13]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$$
$$\left(\text{resp., } F(\lambda x_1 + (1-\lambda)x_2) \subset \lambda F(x_1) + (1-\lambda)F(x_2) + C\right); \tag{3.2}$$

(c) *above-C-quasi-convex* (resp., *below-C-quasiconcave*) [7, Definition 2.3] on X if the set

$$\mathrm{Lev}_{F\le}(z) := \{x \in X : F(x) \subset z - C\}$$
$$\left(\text{resp., } \mathrm{Lev}_{F\ge}(z) := \{x \in X : F(x) \subset z + C\}\right), \tag{3.3}$$

is convex for all $z \in \mathfrak{Z}$;

(d) *above-properly C-quasiconvex* (resp., *above-properly C-quasiconcave* [6]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, either

$$F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$$
$$\left(\text{resp., } F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) - C\right) \tag{3.4}$$

or

$$F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C$$
$$\left(\text{resp., } F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) - C\right); \tag{3.5}$$

(e) *below-properly C-quasiconvex* [7] (resp., *below-properly C-quasiconcave*) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, either

$$F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$$
$$\left(\text{resp., } F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) + C\right) \tag{3.6}$$

or

$$F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$$
$$\left(\text{resp., } F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) + C\right); \tag{3.7}$$

(f) *above-naturally C-quasiconvex* [6] on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \operatorname{co}\{F(x_1) \cup F(x_2)\} - C, \tag{3.8}$$

where co A denotes the convex hull of a set A;

(g) *above -C-convex-like* (resp., *above-C-concave-like*) on X (X is not necessarily convex) if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, there is an $x' \in X$ such that

$$F(x') \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C$$
$$\left(\text{resp., } \lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x') - C\right); \tag{3.9}$$

(h) *below -C-convex-like* [13] (resp., *below -C-concave-like*) on X (X is not necessarily convex) if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, there is an $x' \in X$ such that

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x') + C$$
$$\left(\text{resp.,} F(x') \subset \lambda F(x_1) + (1-\lambda)F(x_2) + C\right). \tag{3.10}$$

It is obvious that every above-C-convex set-valued mapping or above-properly C-quasi-convex set-valued mapping is an above-naturally C-quasi-convex set-valued mapping, and every above-C-convex (above-C-concave) set-valued mapping is an above-C-convex-like (above-C-concave-like) set-valued mapping. Similar relations hold for cases below.

Remark 3.2. The definition of above-properly C-quasi-convex (above-properly C-quasi-concave) set-valued mapping is different from the one mentioned in [7, Definition 2.3] or [5, 6]. The following Examples 3.3 and 3.4 illustrate the reason why they are different from the one mentioned in [5–7]. However, if F is a vector-valued mapping or a single-valued mapping, both mappings reduce to the ordinary definition of a properly C-quasi-convex mapping for vector-valued functions [7]. The above-C-convexity in Definition 3.1 is also different from the below-C-convexity used in [5, 9].

Example 3.3. Consider $C = \{(s,t) \in \mathbb{R}^2 : s \geq 0,\ t \geq 0\}$. Let $F : [x_1, x_2] \subset \mathbb{R} \rightrightarrows \mathbb{R}^2$ be a set-valued mapping defined by

$$F(x_1) := \left\{(s,t) \in \mathbb{R}^2 : (s-2)^2 + (t-4)^2 = 1, 2 \leq s \leq 3, 4 \leq t \leq 5\right\} \bigcup \{(s,5) : -1 \leq s \leq 2\},$$
$$F(x_2) := \left\{(s,t) \in \mathbb{R}^2 : (s-6)^2 + (t+2)^2 = 1, 6 \leq s \leq 7, -2 \leq t \leq -1\right\}, \tag{3.11}$$

and for all $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) := \left\{(s,t) \in \mathbb{R}^2 : (s-2)^2 + (t-2)^2 = 4, 0 \le s \le 2, 0 \le t \le 2\right\}. \tag{3.12}$$

Then F is an above-properly C-quasi-convex set-valued mapping, but it is not below-properly C-quasi-convex.

On the other hand, let $G : [x_1, x_2] \subset \mathbb{R} \rightrightarrows \mathbb{R}^2$ be a set-valued mapping defined by

$$\begin{aligned} G(x_1) &:= \left\{(s,t) \in \mathbb{R}^2 : (s-1)^2 + (t-4)^2 = 1, 1 \le s \le 2, 4 \le t \le 5\right\}, \\ G(x_2) &:= \left\{(s,t) \in \mathbb{R}^2 : (s-6)^2 + (t+2)^2 = 1, 6 \le s \le 7, -2 \le t \le -1\right\}, \end{aligned} \tag{3.13}$$

and for all $\lambda \in (0,1)$,

$$\begin{aligned} G(\lambda x_1 + (1-\lambda)x_2) : = \ & \left\{(s,t) \in \mathbb{R}^2 : (s-2)^2 + (t-2)^2 = 4, 0 \le s \le 2, 0 \le t \le 2\right\} \\ & \bigcup \{(s,0) : 2 \le s \le 3\}. \end{aligned} \tag{3.14}$$

Then, G is a below-properly C-quasi-convex set-valued mapping, but it is not above-properly C-quasi-convex.

Example 3.4. Let $C = \{(s,t) : s \ge 0, t \ge 0\}$. Define $F : [-1,1] \rightrightarrows \mathbb{R}^2$ by

$$F(x) = \left\{(x,t) : 1 - x^2 \le t \le 1\right\}, \quad \forall x \in [-1,1]. \tag{3.15}$$

Then F is continuous, above-C-quasi-convex, below-C-quasi-concave, above-properly C-quasi-convex, and above-properly C-quasi-concave, but it is not below-properly C-quasi-conconvex.

Proposition 3.5. *Let X be a nonempty set (not necessarily convex) and for a given set-valued mapping $F : X \rightrightarrows \mathfrak{Z}$ with nonempty compact values, define a set-valued mapping $M : X \rightrightarrows \mathfrak{Z}$ as*

$$M(x) = \mathrm{Max}_w \mathrm{F(x)}, \quad \forall \mathrm{x} \in \mathrm{X}. \tag{3.16}$$

(a) *If Fis above-C-convex-like, then M is so.*

(b) *If X is a topological space and F is a continuous mapping, then M is upper semicontinuous with nonempty compact values on X.*

Proof. (a) Let F be above-C-convex-like, and let $x_1, x_2 \in X$ be arbitrary. Since F is above-C-convex-like, for any $\alpha \in [0,1]$, there exists $x' \in X$ such that

$$F(x') \subset \alpha F(x_1) + (1-\alpha)F(x_2) - C. \tag{3.17}$$

By Lemma 2.2,

$$\mathrm{Max}_w F(x') \subset \alpha \mathrm{Max}_w F(x_1) + (1-\alpha)\mathrm{Max}_w \mathrm{F}(\mathrm{x}_2) - \mathrm{C}. \tag{3.18}$$

Therefore, $x \mapsto \mathrm{Max}_w F(x)$ is above-C-convex-like.

(b) The upper semicontinuity of M was deduced in [4, Lemma 2]. □

Proposition 3.6. *Let X be a nonempty convex set, and let $F : X \rightrightarrows \mathfrak{Z}$ be a set-valued mapping with nonempty compact values. Then, the set-valued mapping $M : X \rightrightarrows \mathfrak{Z}$ defined by*

$$M(x) = \mathrm{Max}_w \mathrm{F}(\mathrm{x}), \quad \forall \mathrm{x} \in \mathrm{X}, \tag{3.19}$$

is above-C-quasiconvex if F is so.

The following result can be easily derived, and therefore, we omit the proof.

Proposition 3.7. *Let X be a nonempty convex set and $F : X \rightrightarrows \mathbb{R}$ be above-$\mathbb{R}_+$-concave. Then the set-valued mapping $x \mapsto \max F(x)$ is above-$\mathbb{R}_+$-concave and below-$\mathbb{R}_+$-quasiconcave. Furthermore, if $F : X \rightrightarrows \mathbb{R}$ is above-properly $\mathbb{R}_+$-quasiconcave, then the set-valued mapping $x \mapsto \max F(x)$ is also above-properly $\mathbb{R}_+$-quasiconcave and below-$\mathbb{R}_+$-quasiconcave.*

Let $\xi \in C^*$ and $F : X \rightrightarrows \mathfrak{Z}$ be a set-valued mapping. Then, the composition mapping $\xi \circ F : X \rightrightarrows \mathbb{R}$ is defined by

$$(\xi \circ F)(x) = \xi(F(x)) = \bigcup_{y \in F(x)} \xi(y), \quad \forall x \in X. \tag{3.20}$$

Clearly, the composition mapping $\xi \circ F : X \rightrightarrows \mathbb{R}$ is also a set-valued mapping.

Proposition 3.8. *Let X be a nonempty set, $F : X \rightrightarrows \mathfrak{Z}$ a set-valued mapping, and $\xi \in C^*$.*

(a) *If F is above-C-convex-like, then $\xi \circ F$ is above-$\mathbb{R}_+$-convex-like.*

(b) *If F is below-C-concave-like, then $\xi \circ F$ is below-$\mathbb{R}_+$-concave-like.*

(c) *If X is a topological space and F is upper semi-continuous, then so is $\xi \circ F$.*

Proof. (a) By the definition of above-C-convex-like set-valued mapping $F : X \rightrightarrows \mathfrak{Z}$, for any $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, there exists $x' \in X$ such that $F(x') \subset \lambda F(x_1) + (1-\lambda)F(x_2) - C$. For any $y \in F(x')$, there exist $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ such that

$$\lambda y_1 + (1-\lambda)y_2 \in f - C. \tag{3.21}$$

For any $\xi \in \mathrm{C}^*$, we have $\xi(y) \leq \lambda\xi(y_1) + (1-\lambda)\xi(y_2)$. Hence, $\xi(F(x')) \subset \lambda\xi(F(x_1)) + (1-\lambda)\xi(F(x_2)) - \mathbb{R}_+$. Thus, $\xi \circ F$ is above-$\mathbb{R}_+$-convex-like.

The proof of (b) and (c) is easy, and therefore, we omit it. □

Proposition 3.9. *Let X be a nonempty convex set and $\xi \in C^*$.*

(a) *If $F : X \rightrightarrows \mathfrak{Z}$ is above-C-concave (above-properly C-quasi-concave), then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above-$\mathbb{R}_+$-concave (above-properly $\mathbb{R}_+$-quasi-concave).*

(b) *If $F : X \rightrightarrows \mathfrak{Z}$ is above-properly C-quasi-convex, then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above-$\mathbb{R}_+$-quasi-convex and above-properly $\mathbb{R}_+$-quasi-convex.*

(c) *If $F : X \rightrightarrows \mathfrak{Z}$ is above-C-convex, then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above-$\mathbb{R}_+$-convex and above-$\mathbb{R}_+$-quasi-convex.*

Lemma 3.10. *Let $\mathfrak{Z}$ be a real Hausdorff topological vector space and C a closed convex pointed cone in $\mathfrak{Z}$ with $\operatorname{int} C \neq \emptyset$. Let X be a nonempty compact subset of a topological space $\mathcal{X}$, and let $F : X \rightrightarrows \mathfrak{Z}$ be an upper semi-continuous set-valued mapping with nonempty compact values. Then, for any $\xi \in C^*$, there exists $y \in \operatorname{Max}_w F(X)$ such that $\xi(y) = \max \bigcup_{x \in X} \xi(F(x))$.*

Proof. For any given $\xi \in C^*$, the mapping $x \rightrightarrows \xi(F(x))$ is upper semi-continuous by Proposition 3.8 (c). By the compactness of X, there exist $x_0 \in X$ and $y_0 \in F(x_0)$ such that $\xi(y_0) = \max \bigcup_{x \in X} \xi(F(x))$. By Lemma 2.2, there exists $y \in \operatorname{Max}_w \bigcup_{x \in X} F(x)$ such that $y_0 - y \in -C$, and hence $\xi(y) \geq \xi(y_0)$. On the other hand, $y \in \operatorname{Max}_w \bigcup_{x \in X} F(x) \subset F(X)$, we know that $\xi(y) \in \xi(F(X))$, and then $\xi(y) \leq \max \bigcup_{x \in X} \xi(F(x)) = \xi(y_0)$. Therefore, the conclusion holds. □

Proposition 3.11. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathfrak{Z}$ is above-properly C-quasi-convex, then it is above-C-quasi-convex.*

Proof. For any $z \in \mathfrak{Z}$, let $x_1, x_2 \in \operatorname{Lev}_{F \leq}(z)$. Then, $F(x_1)$ and $F(x_2)$ are subsets of $z - C$. Since F is above-properly C-quasi-convex, for any $\lambda \in [0,1]$, $F(\lambda x_1 + (1-\lambda)x_2)$ is contained in either $F(x_1) - C$ or $F(x_2) - C$, and hence, in $z - C$. Thus, the set $\operatorname{Lev}_{F \leq}(z)$ is convex, and therefore, F is above-C-quasi-convex. □

Proposition 3.12. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathfrak{Z}$ is above-naturally C-quasi-convex, then it is above-C-quasi-convex.*

Proof. Let z, x_1, and x_2 be the same as given as in Proposition 3.11. Then, $\operatorname{co}\{F(x_1) \cup F(x_2)\} \subset z - C$ since $z - C$ is convex. By the above-naturally C-quasi-convexity, $F(\lambda x_1 + (1-\lambda)x_2)\} \subset z - C$ for all $\lambda \in [0,1]$. Thus, the set $\operatorname{Lev}_{F \leq}(z)$ is convex, and therefore, F is above-C-quasi-convex. □

Proposition 3.13. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathfrak{Z}$ is above-naturally C-quasi-convex, then $\xi \circ F$ is above-naturally $\mathbb{R}_+$-quasi-convex for any $\xi \in C^*$.*

Proof. Let $\xi \in C^*$ be given. From the above-naturally C-quasi-convexity of F, for any $x_1, x_2 \in X$ and any $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subset \operatorname{co}\{F(x_1) \cup F(x_2)\} - C. \tag{3.22}$$

For any $y \in F(\alpha x_1 + (1-\alpha)x_2)$, there is a $w \in \mathrm{co}\{F(x_1) \cup F(x_2)\}$ such that $y \in w - C$. Then there exist $y_i \in F(x_1) \cup F(x_2)$ and $\lambda_i \in [0,1]$, $1 \le i \le n$ such that $w = \sum_{i=1}^{n} \lambda_i y_i$. Hence, $\xi(w) = \sum_{i=1}^{n} \lambda_i \xi(y_i)$, and

$$\xi(y) \in \xi(w) - \mathbb{R}_+ = \sum_{i=1}^{n} \lambda_i \xi(y_i) - \mathbb{R}_+ \subset \mathrm{co}\{\xi(F(x_1)) \cup \xi(F(x_2))\} - \mathbb{R}_+. \tag{3.23}$$

Therefore, $\xi \circ F$ is a above-naturally $\mathbb{R}_+$-quasi-convex. □

Proposition 3.14. *Let $F : X \rightrightarrows \mathcal{Z}$ be a set-valued mapping with nonempty compact values. For any $\xi \in C^*$,*

(a) *if $\xi(d) = \min \bigcup_{x \in X} \xi(F(x))$ for some $d \in \mathcal{Z}$, then $d \in \mathrm{Min}_w \bigcup_{x \in X} \mathrm{F(x)}$;*

(b) *if $\xi(e) = \max \bigcup_{x \in X} \xi(F(x))$ for some $e \in \mathcal{Z}$, then $e \in \mathrm{Max}_w \bigcup_{x \in X} \mathrm{F(x)}$.*

Proof. Let $\xi(d) = \min \bigcup_{x \in X} \xi(F(x))$. Suppose that $d \notin \mathrm{Min}_w \bigcup_{x \in X} F(x)$. Then

$$\left(\bigcup_{x \in X} F(x) \right) \bigcap (d - \mathrm{int}\, C) \neq \emptyset. \tag{3.24}$$

Then, there exists $w \in \bigcup_{x \in X} F(x)$ and $w \in d - \mathrm{int}\, C$. Therefore, there exists $s \in X$ such that $w \in F(s)$ and $d - w \in \mathrm{int}\, C$. Since $\xi \in C^*$, $\xi(d) > \xi(w)$ and $\xi(w) \ge \min \bigcup_{x \in X} \xi(F(x))$. This implies that $\xi(d) > \min \bigcup_{x \in X} \xi(F(x))$, which is a contradiction. This proves (a).

Analogously, we can prove (b), so we omit it. □

Remark 3.15. Propositions 3.8 and 3.9, Lemma 3.10, and Propositions 3.13 and 3.14 are always true except Proposition 3.8 (b) if we replace ξ by any Gerstewitz function.

4. Minimax Theorems for Set-Valued Mappings

In this section, we establish some minimax theorems for set-valued mappings with or without linear structure.

Theorem 4.1. *Let X, Y be two nonempty compact subsets (not necessarily convex) of real Hausdorff topological spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ be lower semi-continuous on X and upper semi-continuous on Y such that for all $(x, y) \in X \times Y$, $F(x, y)$ is nonempty compact and satisfies the following conditions:*

(i) *for each $x \in X$, $y \mapsto F(x, y)$ is below-$\mathbb{R}_+$-concave-like on Y;*

(ii) *for each $y \in Y$, $x \mapsto F(x, y)$ is above-$\mathbb{R}_+$-convex-like on X.*

Then,

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) = \min \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y). \tag{4.1}$$

Proof. Since

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \leq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y), \tag{4.2}$$

it is sufficient to prove that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.3}$$

Choose any $\alpha \in \mathbb{R}$ such that $\alpha < \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y)$. For any $y \in Y$, let

$$\mathrm{Lev}_{F\leq}(y;\alpha) = \{x \in \mathrm{X} : F(x,y) \subset \alpha - \mathbb{R}_+\}. \tag{4.4}$$

Then, by the lower semi-continuity of the set-valued mapping $x \mapsto F(x,y)$, the set $\mathrm{Lev}_{F\leq}(y;\alpha)$ is closed, hence it is compact for all $y \in Y$. By the choice of α, we have

$$\bigcap_{y\in Y} \mathrm{Lev}_{F\leq}(y;\alpha) = \emptyset. \tag{4.5}$$

Since X is compact and the collection $\{X \setminus \mathrm{Lev}_{F\leq}(y;\alpha) : y \in Y\}$ covers X, there exist finite number of points $y_1, y_2, \ldots, y_m$ in Y such that

$$\mathrm{X} \subset \bigcup_{i=1}^{m} (\mathrm{X} \setminus \mathrm{Lev}_{F\leq}(y_i;\alpha)) \tag{4.6}$$

or

$$\bigcap_{i=1}^{m} \mathrm{Lev}_{F\leq}(y_i;\alpha) = \emptyset. \tag{4.7}$$

This implies that

$$\max \bigcup_{i=1}^{m} F(x,y_i) > \alpha, \quad \forall x \in \mathrm{X}, \tag{4.8}$$

and therefore,

$$\min \bigcup_{x\in X} \max \bigcup_{i=1}^{m} F(x,y_i) > \alpha. \tag{4.9}$$

Following the idea of Borwein and Zhuang [14], let

$$\mathfrak{E} := \left\{ (\mathbf{z}, r) \in \mathbb{R}^{m+1} : \text{there is } x \in \mathrm{X}, F(x,y_i) \subset r + z_i - \mathbb{R}_+, i = 1,2,\ldots,m \right\}, \tag{4.10}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_m)$. Then the set $\mathfrak{C}$ is convex, so is $\operatorname{int}\mathfrak{C}$. We note that the interior $\operatorname{int}\mathfrak{C}$ of $\mathfrak{C}$ is nonempty since

$$\left(\mathbf{0}, 1 + \max \bigcup_{i=1}^{m} F(x, y_i)\right) \in \operatorname{int}\mathfrak{C}, \quad \forall x \in X. \tag{4.11}$$

Since $(\mathbf{0}, \alpha) \notin \mathfrak{C}$, by separation hyperplane theorem [15, Theorem 14.2], there is a $(\Xi, \epsilon) \neq \mathbf{0} \times \{0\}$ such that

$$\langle (\Xi, \varepsilon), (\mathbf{z}, r) \rangle \geq \langle (\Xi, \varepsilon), (\mathbf{0}, \alpha) \rangle, \quad \forall (\mathbf{z}, r) \in \mathfrak{C}, \tag{4.12}$$

where $\Xi = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, that is,

$$\Xi \mathbf{z} + \varepsilon \mathrm{r} \geq \varepsilon \alpha, \quad \forall (\mathbf{z}, r) \in \mathfrak{C}. \tag{4.13}$$

By (4.11), (4.13), and the choice of α, we have that $\varepsilon > 0$. Furthermore, from the fact

$$\prod_{i=1}^{m} (F(x, y_i) + r) \times \{-r\} \subset \mathfrak{C}, \tag{4.14}$$

we have

$$(\eta_{x,1} + r, \eta_{x,2} + r, \ldots, \eta_{x,m} + r, -r) \in \mathfrak{C}, \quad \forall \eta_{x,i} \in F(x, y_i). \tag{4.15}$$

Hence, by (4.13), we have

$$\sum_{i=1}^{m} \lambda_i (\eta_{x,i} + r) + \varepsilon(-r) \geq \varepsilon \alpha \tag{4.16}$$

or

$$\sum_{i=1}^{m} \left(\frac{\lambda_i}{\varepsilon}\right) \eta_{x,i} + \left(\frac{\sum_{i=1}^{m} \lambda_i}{\varepsilon} - 1\right) r \geq \alpha, \quad \forall x \in X,\ r \in \mathbb{R}. \tag{4.17}$$

Thus, we have $\sum_{i=1}^{m} (\lambda_{\mathrm{i}} / \varepsilon) = 1$. Hence, by (4.17), we have

$$\sum_{i=1}^{m} \left(\frac{\lambda_i}{\varepsilon}\right) F(x, y_i) \subset \alpha + \mathbb{R}_+. \tag{4.18}$$

Since $F(x, y)$ is below-$\mathbb{R}_+$-concave-like in y, there is $y' \in Y$ such that

$$F(x, y') \subset \sum_{i=1}^{m} \left(\frac{\lambda_i}{\varepsilon}\right) F(x, y_i) + \mathbb{R}_+, \quad \forall x \in X. \tag{4.19}$$

Therefore,

$$\bigcup_{x\in X} F(x,y') \subset \alpha + \mathbb{R}_+, \tag{4.20}$$

and hence,

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \alpha. \tag{4.21}$$

This completes the proof. □

Remark 4.2. Theorem 4.1 is a modification of [14, Theorem A]. If F is a real-valued function, then Theorem 4.1 reduces to the well-known minimax theorem due to Fan [2].

We next establish a minimax theorem for set-valued mappings defined on the sets with linear structure.

Theorem 4.3. *Let X, Y be two nonempty compact convex subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ be lower semi-continuous on X and upper semi-continuous on Y such that for all $(x,y) \in X \times Y$, $F(x,y)$ is nonempty compact, and satisfies the following conditions:*

(i) *for each $y \in Y$, $x \mapsto F(x,y)$ is above-$\mathbb{R}_+$-quasi-convex on X;*

(ii) *for each $x \in X$, $y \mapsto F(x,y)$ is above-$\mathbb{R}_+$-concave, or above-properly $\mathbb{R}_+$-quasi-concave on Y;*

(iii) *for each $y \in Y$, there is a $x_y \in Y$ such that*

$$\max F(x_y, y) \leq \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y). \tag{4.22}$$

Then,

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) = \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y). \tag{4.23}$$

Proof. We only need to prove that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) < \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) \tag{4.24}$$

is impossible, since it is always true that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \leq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.25}$$

Suppose that there is an $\alpha \in \mathbb{R}$ such that

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) < \alpha < \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.26}$$

Define $G : X \times Y \rightrightarrows X \times Y$ by

$$G(x,y) = \{s \in X : \max F(s,y) < \alpha\} \times \{t \in Y : \max F(x,t) > \alpha\}. \tag{4.27}$$

For each $x \in X$, $\max \bigcup_{y\in Y} F(x,y) \geq \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) > \alpha$. Since Y is compact and the set-valued mapping $y \mapsto \max F(x,y)$ is upper semi-continuous, there is a $t \in Y$ such that $\max F(x,t) = \max \bigcup_{y\in Y} F(x,y) > \alpha$.

On the other hand, from the condition (iii), for each $y \in Y$, there is a $x_y \in Y$ such that $\max F(x_y,y) < \alpha$. Hence, for each $(x,y) \in X \times Y$, $G(x,y) \neq \emptyset$. By (i) and Proposition 3.6, the mapping $x \to \max F(x,y)$ is above-$\mathbb{R}_+$-quasi-convex on X. By (ii) and Proposition 3.7, the mapping $y \to \max F(x,y)$ is below-$\mathbb{R}_+$-quasi-concave on y. Hence, for each $(x,y) \in X \times Y$, the set $G(x,y)$ is convex. From the lower semi-continuities on X and upper semi-continuity on Y of F, the set

$$G^{-1}(s,t) = \{x \in X : \max F(x,t) > \alpha\} \times \{y \in Y : \max F(s,y) < \alpha\} \tag{4.28}$$

is open in $X \times Y$. By Fan-Browder fixed-point Theorem 2.8, there exists $(\overline{x},\overline{y}) \in X \times Y$ such that

$$(\overline{x},\overline{y}) \in G(\overline{x},\overline{y}), \tag{4.29}$$

that is,

$$\max F(\overline{x},\overline{y}) > \alpha > \max F(\overline{x},\overline{y}), \tag{4.30}$$

which is a contradiction. This completes the proof. □

Remark 4.4. [5, Propositions 2.7 and 2.1] can be deduced from Theorem 4.3. Indeed, in [5, Proposition 2.1], the above-naturally C-quasi-convexity is used. By Proposition 3.12, the condition (i) of Theorem 4.3 holds. Hence the conclusion of Proposition 2.1 in [5] holds. We also note that, in Theorem 4.3, the mapping F need not be continuous on $X \times Y$. Hence Theorem 4.3 is a slight generalization of [7, Theorem 3.1].

Theorem 4.5. *Let X and Y be nonempty compact (not necessarily convex) subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the mapping $F : X \times Y \rightrightarrows \mathcal{Z}$ be upper semi-continuous with nonempty compact values and lower semi-continuous on X such that*

(i) *for each $x \in X$, $y \to F(x,y)$ is below-C-concave-like on Y;*

(ii) *for each $y \in Y$, $x \to F(x,y)$ is above-C-convex-like on X;*

(iii) *for every* $y \in Y$,

$$\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} F(x, y) \subset \operatorname{Min}_w \bigcup_{x \in X} F(x, y) + C. \tag{4.31}$$

Then for any

$$z_1 \in \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} F(x, y), \tag{4.32}$$

there is a

$$z_2 \in \operatorname{Min}\left(\operatorname{co}\left\{ \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) \tag{4.33}$$

such that

$$z_1 \in z_2 + C, \tag{4.34}$$

that is,

$$\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w \bigcup_{x \in X} F(x, y) \subset \operatorname{Min}\left(\operatorname{co}\left\{ \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) + C. \tag{4.35}$$

Proof. Let $\Gamma(x) := \operatorname{Max}_w \bigcup_{y \in Y} F(x, y)$ for all $x \in X$. From Lemma 2.4 and Proposition 3.5, the set-valued mapping $x \mapsto \Gamma(x)$ is upper semi-continuous with nonempty compact values on X. Hence the set $\Gamma(X)$ is compact, and so is $\operatorname{co}\{\Gamma(X)\}$. Then $\operatorname{co}\{\Gamma(X)\}+C$ is a closed convex set with nonempty interior. Suppose that $v \notin \operatorname{co}\{\Gamma(X)\} + C$. By separation hyperplane theorem [15, Theorem 14.2], there exist $k \in \mathbb{R}$, $\varepsilon > 0$ and a nonzero continuous linear functional $\xi : Z \to \mathbb{R}$ such that

$$\xi(v) \le k - \varepsilon < k \le \xi(u + c), \quad \text{for every } u \in \operatorname{co}\{\Gamma(X)\},\ c \in C. \tag{4.36}$$

Therefore,

$$\xi(c) > \xi(v - u), \quad \text{for every } u \in \operatorname{co}\{\Gamma(X)\},\ c \in C. \tag{4.37}$$

This implies that $\xi \in C^*$ and $\xi(v) < \xi(u)$ for all $u \in \operatorname{co}\{\Gamma(X)\}$.

Let $g := \xi F : X \times Y \rightrightarrows \mathbb{R}$. From Lemma 3.10, for each fixed $x \in X$, there exist $y_x^* \in Y$ and $f(x, y_x^*) \in F(x, y_x^*)$ with $f(x, y_x^*) \in \Gamma(x)$ such that $\xi(f(x, y_x^*)) = \max \bigcup_{y \in Y} \xi(F(x, y))$. Choosing $c = 0$ and $u = f(x, y_x^*)$ in (4.36), we have

$$\max \bigcup_{y \in X} \xi(F(x, y)) = \xi f(x, y_x^*) \ge k > k - \varepsilon \ge \xi(v), \quad \forall x \in X. \tag{4.38}$$

Therefore,

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} \xi(F(x,y)) > \xi(v). \tag{4.39}$$

By the conditions (i), (ii) and Proposition 3.8, the set-valued mapping $y \mapsto \xi(F(x,y))$ is below-$\mathbb{R}_+$-concave-like on Y for all $x \in X$, and the set-valued mapping $x \mapsto \xi(F(x,y))$ is above-$\mathbb{R}_+$-convex-like on X for all $y \in Y$. From Theorem 4.1, we have

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} \xi(F(x,y)) > \xi(v). \tag{4.40}$$

Since Y is compact, there is an $y' \in Y$ such that $\min \bigcup_{x\in X} \xi(F(x,y')) > \xi(v)$. For any $x \in X$ and all $g(x,y') \in F(x,y')$, we have

$$\xi(g(x,y')) > \xi(v), \tag{4.41}$$

that is,

$$\xi(g(x,y') - v) > 0, \quad \forall x \in X, \; g(x,y') \in F(x,y'). \tag{4.42}$$

Thus, $v \notin \bigcup_{x\in X} F(x,y') + C$, and hence,

$$v \notin \mathrm{Min}_w \bigcup_{x\in X} F(x,y') + C. \tag{4.43}$$

If $v \in \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y)$, by the condition (iii), $v \in \mathrm{Min}_w \bigcup_{x\in X} F(x,y') + C$ which contradicts (4.43). Hence, for every $v \in \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y)$,

$$v \in \mathrm{co}\left\{ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) \right\} + C, \tag{4.44}$$

that is,

$$\mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) \subset \mathrm{co}\left\{ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) \right\} + C \tag{4.45}$$

or

$$\mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) \subset \mathrm{Min}\left(\mathrm{co}\left\{ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) \right\} \right) + C. \tag{4.46}$$

□

The following examples illustrate Theorem 4.5.

Example 4.6. Let $X = Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$, $C = \mathbb{R}^2_+$ and

$$F(x,y) = \left\{(s,t) \in \mathbb{R}^2 : s = x^2, t = 1 - y^2\right\}, \quad \forall (x,y) \in X \times Y. \tag{4.47}$$

It is obviously that F is below-$\mathbb{R}^2_+$-concave-like on Y and above-$\mathbb{R}^2_+$-convex-like on X. We now verify the condition (iii) of Theorem 4.5. Indeed, for any $y \in Y$,

$$\begin{gathered} \bigcup_{x \in X} F(x,y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left\{1 - y^2\right\}, \\ \mathrm{Min}_w \bigcup_{x \in X} F(x,y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left\{1 - y^2\right\}. \end{gathered} \tag{4.48}$$

Then,

$$\begin{gathered} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F(x,y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right), \\ \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F(x,y) = \{(1,1)\}. \end{gathered} \tag{4.49}$$

Thus, for every $y \in Y$,

$$\begin{aligned} \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F(x,y) &\subset \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left\{1 - y^2\right\} + C \\ &= \mathrm{Min}_w \bigcup_{x \in X} F(x,y) + C, \end{aligned} \tag{4.50}$$

and the condition (iii) of Theorem 4.5 holds.

Furthermore, for any $x \in X$,

$$\begin{gathered} \bigcup_{y \in Y} F(x,y) = \left\{x^2\right\} \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right), \\ \mathrm{Max}_w \bigcup_{y \in Y} F(x,y) = \left\{x^2\right\} \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right). \end{gathered} \tag{4.51}$$

Then,

$$\begin{gathered} \bigcup_{x \in X} \mathrm{Max}_w \bigcup_{y \in Y} F(x,y) = \left(\{0\} \cup \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}\right) \times \left(\{1\} \cup \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}\right), \\ \mathrm{co}\left\{\bigcup_{x \in X} \mathrm{Max}_w \bigcup_{y \in Y} F(x,y)\right\} = [0,1] \times [0,1]. \end{gathered} \tag{4.52}$$

Thus,

$$\begin{gathered}
\operatorname{Min}\left(\operatorname{co}\left\{\bigcup_{x\in X}\operatorname{Max}_w\bigcup_{y\in Y}F(x,y)\right\}\right)=\{(0,0)\},\\
\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\{(1,1)\}\subset\operatorname{Min}\left(\operatorname{co}\left\{\bigcup_{x\in X}\operatorname{Max}_w\bigcup_{y\in Y}F(x,y)\right\}\right)+C.
\end{gathered}\tag{4.53}$$

Hence, the conclusion of Theorem 4.5 holds.

Example 4.7. Let $X=[0,1]$, $Y=[-1,0]$, $C=\mathbb{R}^2_+$, and $G:Y\rightrightarrows Y$ be defined by

$$G(y)=\begin{cases}[-1,0], & y=0,\\ \{0\}, & y\neq 0.\end{cases}\tag{4.54}$$

Let $F(x,y)=\{x^2\}\times G(y)$ for all $(x,y)\in X\times Y$. Then G is upper semi-continuous, but not lower semi-continuous on $\mathbb{R}$, and F is not continuous but is upper semi-continuous on $X\times Y$. Moreover, F has nonempty compact values and is lower semi-continuous on X. It is easy to see that F is below-C-concave-like on Y and is above-C-convex-like on X. We verify the condition (iii) of Theorem 4.5. Indeed, for all $y\in Y$, $\bigcup_{x\in X}F(x,y)=[0,1]\times G(y)$.

$$\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\begin{cases}[0,1]\times\{0\}, & y\neq 0,\\ (\{0\}\times[-1,0])\cup([0,1]\times\{-1\}), & y=0.\end{cases}\tag{4.55}$$

Then,

$$\begin{gathered}
\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=(\{0\}\times[-1,0])\cup([0,1]\times\{-1\})\cup([0,1]\times\{0\}),\\
\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\{(1,0)\}\subset\operatorname{Min}_w\bigcup_{x\in X}F(x,y)+C.
\end{gathered}\tag{4.56}$$

Therefore, the condition (iii) of Theorem 4.5 holds.

Since

$$F(x,y)=\begin{cases}\{x^2\}\times[-1,0], & y=0,\\ \{x^2\}\times\{0\}, & y\neq 0,\end{cases}\tag{4.57}$$

for all $(x,y)\in X\times Y$, and $\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)=\{(1,0)\}$, for each $y\in Y$, we can choose $x_y=0\in X$ such that

$$\operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}_w\bigcup_{x\in X}F(x,y)\subset F(x_y,y)+C.\tag{4.58}$$

Furthermore,

$$\begin{aligned}\bigcup_{y\in Y} F(x,y) &= \left\{x^2\right\} \times \left(\bigcup_{y\in Y} G(y)\right)\\ &= \left\{x^2\right\} \times ([-1,0] \cup \{0\})\\ &= \left\{x^2\right\} \times [-1,0],\\ \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) &= [0,1] \times [-1,0].\end{aligned} \tag{4.59}$$

Therefore,

$$\begin{aligned}\mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) &= \{(1,0)\} \subset \{(0,-1)\} + C\\ &= \mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) + C.\end{aligned} \tag{4.60}$$

Hence, the conclusion of Theorem 4.5 holds.

Remark 4.8. Theorem 3.1 in [5] Theorem 3.1 in [6], or Theorem 4.2 in [7] cannot be applied to Examples 4.6 and 4.7 because of the following reasons:

(i) the two sets X and Y are not convex in Example 4.6;

(ii) F is not continuous on $X \times Y$ in Examples 4.6 and 4.7.

Theorem 4.9. *Let X, Y be two nonempty compact convex subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathfrak{Z}$ has nonempty compact values, and it is continuous on Y and lower semi-continuous on X such that*

(i) *for each $y \in Y$, $x \mapsto F(x,y)$ is above-naturally C-quasi-convex on X;*

(ii) *for each $x \in X$, $y \mapsto F(x,y)$ is above-C-concave or above-properly C-quasi-concave on Y;*

(iii) *for every $y \in Y$,*

$$\mathrm{Max} \bigcup_{y\in Y} Min_w \bigcup_{x\in X} F(x,y) \subset Min_w \bigcup_{x\in X} F(x,y) + C; \tag{4.61}$$

(iv) for any continuous increasing function h and for each $y \in Y$, there exists $x_y \in X$ such that

$$\max h(F(x_y,y)) \leq \max \bigcup_{y\in Y} \min \bigcup_{x\in X} h(F(x,y)). \tag{4.62}$$

Then, for any $z_1 \in \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y)$, *there is a*

$$z_2 \in \mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) \tag{4.63}$$

such that $z_1 \in z_2 + C$, *that is,*

$$\mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) \subset \mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) + C. \tag{4.64}$$

Proof. Let $\Gamma(x)$ be defined as the same as in the proof of Theorem 4.5. Following the same perspective as in the proof of Theorem 4.5, suppose that $v \notin \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) + C$. For any $k \in \mathrm{int}\, C$ and Gerstewitz function $\xi_{kv} : \mathcal{Z} \rightrightarrows \mathbb{R}$. By Proposition 2.7(d), we have

$$\xi_{kv}(u) > 0, \quad \text{for every } u \in \Gamma(X). \tag{4.65}$$

Let $g := \xi_{kv} \circ F : X \times Y \rightrightarrows \mathbb{R}$. From Lemma 3.10, for the mapping ξ_{kv} and Remark 3.15, for each $x \in X$, there exist $y_x^* \in Y$and $f(x, y_x^*) \in F(x, y_x^*)$ with $f(x, y_x^*) \in \mathrm{Max}_w \bigcup_{y\in Y} F(x,y)$ such that $\xi_{kv} f(x, y_x^*) = \max \bigcup_{y\in Y} \xi_{kv}(F(x,y))$. Choosing $u = f(x, y_x^*)$ in (4.65), we have

$$\max \bigcup_{y\in Y} \xi_{kv}(F(x,y)) > 0, \quad \forall x \in X. \tag{4.66}$$

Therefore,

$$\min \bigcup_{x\in X} \max \bigcup_{y\in Y} \xi_{kv}(F(x,y)) > 0. \tag{4.67}$$

By conditions (i), (ii) and Remark 3.15, the set-valued mapping $y \mapsto \xi_{kv}(F(x,y))$ is upper semi-continuous, and either above-$\mathbb{R}_+$-concave or above-properly $\mathbb{R}_+$-quasi-concave on Y, and the set-valued mapping $x \mapsto \xi_{kv}(F(x,y))$ is lower semi-continuous and above-$\mathbb{R}_+$-quasi-convex on X. From Theorem 4.3, we have

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} \xi_{kv}(F(x,y)) > 0. \tag{4.68}$$

Since the set-valued mapping $y \mapsto F(x,y)$ is lower semi-continuous on Y, by Lemma 2.4 (b) and Lemma 2.5 (b), the set-valued mapping $y \mapsto \min \bigcup_{x\in X} \xi_{kv}(F(x,y))$ is upper semi-continuous on Y. By the compactness of Y, there exists $y' \in Y$ such that $\min \bigcup_{x\in X} \xi_{kv}(F(x,y')) > 0$. For all $x \in X$ and all $g(x,y') \in F(x,y')$, we have $\xi_{kv}(g(x,y')) > 0$. Thus, $v \notin \bigcup_{x\in X} F(x,y') + C$, and hence,

$$v \notin \mathrm{Min}_w \bigcup_{x\in X} F(x,y') + C. \tag{4.69}$$

If $v \in \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F(x,y)$, by the condition (iii), $v \in \operatorname{Min}_w \bigcup_{x\in X} F(x,y') + C$ which contradicts (4.69). Hence, for every $v \in \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F(x,y)$,

$$v \in \operatorname{Min} \bigcup_{x\in X} \operatorname{Max}_w \bigcup_{y\in Y} F(x,y) + C, \tag{4.70}$$

that is,

$$\operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F(x,y) \subset \operatorname{Min} \bigcup_{x\in X} \operatorname{Max}_w \bigcup_{y\in Y} F(x,y) + C. \tag{4.71}$$

This completes the proof. □

The following example illustrates Theorem 4.9.

Example 4.10. Let $X = Y = [0,1]$, $C = \mathbb{R}^2_+$ and $G : X \rightrightarrows Y$ be a set-valued mapping defined as

$$G(x) = \begin{cases} [0,1], & x \neq 0, \\ \{0\}, & x = 0. \end{cases} \tag{4.72}$$

Let $F(x,y) = G(x) \times \{-y^2\}$ for all $(x,y) \in X \times Y$. Then G is lower semi-continuous, but not upper semi-continuous on $\mathbb{R}$, and F is continuous on Y, and F has nonempty compact values and is lower semi-continuous on X. It is easy to see that F is above-C-concave or above-properly C-quasi-concave on Y and is above-naturally C-quasi-convex on X.

We verify the condition (iii) of Theorem 4.9. Indeed, for all $y \in Y$, $\bigcup_{x\in X} F(x,y) = [0,1] \times \{-y^2\}$ and $\operatorname{Min}_w \bigcup_{x\in X} F(x,y) = [0,1] \times \{-y^2\}$. Hence,

$$\begin{gathered} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F(x,y) = [0,1] \times [-1,0], \\ \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F(x,y) = \{(1,0)\} \subset \operatorname{Min}_w \bigcup_{x\in X} F(x,y) + C. \end{gathered} \tag{4.73}$$

Therefore, the condition (iii) of Theorem 4.9 holds.

Since $\operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F(x,y) = \{(1,0)\}$ for any $y \in Y$, we can choose $x_y = 0 \in X$ such that

$$F(x_y, y) = \left\{ \left(0, -y^2\right) \right\} \subset \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F(x,y) - C. \tag{4.74}$$

For any continuous increasing function h, the condition (iv) of Theorem 4.9 holds.

Furthermore, since for each $x \in X$,

$$\bigcup_{y\in Y} F(x,y) = G(x) \times [-1,0],$$
$$\mathrm{Max}_w \bigcup_{y\in Y} F(x,y) = \begin{cases} \{0\} \times [-1,0], & x = 0, \\ (\{1\} \times [-1,0]) \bigcup ([0,1] \times \{0\}), & x \neq 0, \end{cases} \tag{4.75}$$

we have

$$\bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) = (\{0\} \times [-1,0]) \bigcup ([0,1] \times \{0\}) \bigcup (\{1\} \times [-1,0]),$$
$$\mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) = \{(0,-1)\}. \tag{4.76}$$

Thus,

$$\begin{aligned} \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w \bigcup_{x\in X} F(x,y) &= \{(1,0)\} \subset \{(0,-1)\} + C \\ &= \mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w \bigcup_{y\in Y} F(x,y) + C. \end{aligned} \tag{4.77}$$

Therefore, the conclusion of Theorem 4.9 holds.

Remark 4.11. Theorem 3.1 in [5], Theorem 3.1 in [6], or Theorem 4.2 in [7] cannot be applied to Example 4.10 as F is not continuous on $X \times Y$.

If we choose $Z = \mathbb{R}$ and $C = \mathbb{R}_+$ in Theorems 4.5 and 4.9, we always have $C^* = \mathbb{R}_+$ and for every $y \in Y$,

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \min \bigcup_{x\in X} F(x,y). \tag{4.78}$$

Hence, the condition (iii) of Theorem 4.5 holds. Thus, we have the following corollaries.

Corollary 4.12. *Let X, Y be nonempty compact (not necessarily convex) subsets of real Hausdorff topological vector space $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ has nonempty compact values such that it is lower semi-continuous on X and is upper semi-continuous on $X \times Y$. Assume that the following conditions hold:*

(i) *for each $x \in X$, $y \mapsto F(x,y)$ is below-$\mathbb{R}_+$-concave-like on Y;*

(ii) *for each $y \in Y$, $x \mapsto F(x,y)$ is above-$\mathbb{R}_+$-convex-like on X;*

(iii) *for every $y \in Y$,*

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) \geq \min \bigcup_{x\in X} F(x,y). \tag{4.79}$$

Then, for any

$$z_1 \in \max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y), \tag{4.80}$$

there is a

$$z_2 \in \min\left(\operatorname{co}\left\{ \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right\}\right) \tag{4.81}$$

such that

$$z_1 \geq z_2, \tag{4.82}$$

that is,

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min\left(\operatorname{co}\left\{ \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right\}\right). \tag{4.83}$$

Corollary 4.13. *Under the framework of Corollary 4.12, in addition, let X, Y be two convex subsets, and let F be upper semi-continuous on $X \times Y$. Then,*

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) = \min \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y). \tag{4.84}$$

Proof. By Corollary 4.12, we have

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min\left(\operatorname{co}\left\{ \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right\}\right). \tag{4.85}$$

Since the set-valued mapping F is upper semi-continuous on $X \times Y$ and Y is compact, by Lemmas 2.4 and 2.5, the set-valued mapping $x \mapsto \max \bigcup_{y \in Y} F(x, y)$ is upper semi-continuous on X. Since X is convex, it is connected. By [16, Theorem 3.1],

$$\bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \tag{4.86}$$

is connected in $\mathbb{R}$, and hence, it is convex. From (4.85),

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min\left(\bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) \right). \tag{4.87}$$

This completes the proof. □

When $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, from Theorem 4.9, we deduce the following corollary.

Corollary 4.14. *Let X, Y be two nonempty compact convex subsets in real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ has nonempty compact values such that it is continuous on Y and is lower semi-continuous on X. Assume that the following conditions hold:*

(i) *for each $y \in Y$, $x \to F(x,y)$ is above-naturally $\mathbb{R}_+$-quasi-convex on X;*

(ii) *for each $x \in X$, $y \to F(x,y)$ is above-$\mathbb{R}_+$-concave or above-properly $\mathbb{R}_+$-quasi-concave on Y;*

(iii) *for each $y \in Y$, there exists $x_y \in X$ such that*

$$\max F(x_y, y) \leq \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y). \tag{4.88}$$

Then,

$$\max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) = \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y). \tag{4.89}$$

Remark 4.15. Corollary 4.14 includes Proposition 2.1 in [5].

5. Saddle Points for Set-Valued Mappings

In this section, we discuss the existence of several kinds of saddle points for set-valued mappings including the C-loose saddle points, weak C-saddle points, $\mathbb{R}_+$-loose saddle points, and R_+-saddle points of F on $X \times Y$.

Definition 5.1. Let $\mathrm{F} : \mathrm{X} \times \mathrm{Y} \rightrightarrows \mathcal{Z}$ be a set-valued mapping. A point $(\overline{x}, \overline{y}) \in X \times Y$ is said to be a

(a) C-loose saddle point [7] of F on $X \times Y$ if

$$\begin{gathered} F(\overline{x},\overline{y}) \bigcap \left(\operatorname{Max} \bigcup_{y\in Y} F(\overline{x},y) \right) \neq \emptyset, \\ F(\overline{x},\overline{y}) \bigcap \left(\operatorname{Min} \bigcup_{x\in X} F(x,\overline{y}) \right) \neq \emptyset; \end{gathered} \tag{5.1}$$

(b) weak C-saddle point [7] of F on $X \times Y$ if

$$F(\overline{x},\overline{y}) \bigcap \left(\operatorname{Max}_w \bigcup_{y\in Y} F(\overline{x},y) \right) \bigcap \left(\operatorname{Min}_w \bigcup_{x\in X} F(x,\overline{y}) \right) \neq \emptyset; \tag{5.2}$$

(c) $\mathbb{R}_+$-*loose saddle point* of F on $X \times Y$ if $Z = \mathbb{R}$ and

$$F(\overline{x},\overline{y}) = \left[\min \bigcup_{x\in X} F(x,\overline{y}), \max \bigcup_{y\in Y} F(\overline{x},y)\right]; \tag{5.3}$$

(d) $\mathbb{R}_+$-*saddle point* of F on $X \times Y$ if $Z = \mathbb{R}$ and

$$\max \bigcup_{y\in Y} F(\overline{x},y) = \min \bigcup_{x\in X} F(x,\overline{y}) = F(\overline{x},\overline{y}). \tag{5.4}$$

It is obvious that every weak C-saddle point is a C-loose saddle point and every $\mathbb{R}_+$-saddle point is a $\mathbb{R}_+$-loose saddle point.

Theorem 5.2. *Under the framework of Theorem 4.1, F has $\mathbb{R}_+$-saddle point if the set-valued mapping $y \mapsto F(x,y)$ is continuous.*

Proof. By Lemmas 2.4 and 2.5, we attained the max and min in Theorem 4.1. By the compactness of X and Y and the lower semi-continuity of F on X and Y, respectively, there exists $(\overline{x},\overline{y}) \in X \times Y$ such that

$$\begin{aligned} \max \bigcup_{y\in Y} \min \bigcup_{x\in X} F(x,y) &= \min \bigcup_{x\in X} F(x,\overline{y}), \\ \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x,y) &= \max \bigcup_{y\in Y} F(\overline{x},y). \end{aligned} \tag{5.5}$$

Combining this with Theorem 4.1, we have

$$\max \bigcup_{y\in Y} F(\overline{x},y) = \min \bigcup_{x\in X} F(x,\overline{y}) = F(\overline{x},\overline{y}), \tag{5.6}$$

and hence, F has $\mathbb{R}_+$-saddle point. □

Theorem 5.3. *Under the framework of Theorem 4.3, F has $\mathbb{R}_+$-saddle point if the set-valued mapping $y \mapsto F(x,y)$ is continuous.*

Theorem 5.4. *Under the framework of Theorem 4.5 or Theorem 4.9, F has weak C-saddle point if the set-valued mapping $y \mapsto F(x,y)$ is continuous.*

Proof. For any $\xi \in C^*$, the set-valued mapping $\xi \circ F$ satisfies all the conditions of Theorem 5.2 or Theorem 5.3. Hence, $\xi \circ F$ has $\mathbb{R}_+$-saddle point, that is, there exists $(\overline{x},\overline{y}) \in X \times Y$ such that

$$\max \bigcup_{y\in Y} \xi(F(\overline{x},y)) = \min \bigcup_{x\in X} \xi(F(x,\overline{y})) = \xi(F(\overline{x},\overline{y})). \tag{5.7}$$

Then, for any $z \in F(\overline{x}, \overline{y})$,

$$\xi(z) \in \min \bigcup_{x \in X} \xi(F(x, \overline{y})),$$
$$\xi(z) \in \max \bigcup_{y \in Y} \xi(F(\overline{x}, y)). \tag{5.8}$$

Thus, by Proposition 3.14,

$$z \in \mathrm{Min}_w \bigcup_{x \in X} F(x, \overline{y}) \bigcap \mathrm{Max}_w \bigcup_{y \in Y} F(\overline{x}, y), \tag{5.9}$$

and $(\overline{x}, \overline{y})$ is a weak C-saddle point of F. □

Acknowledgments

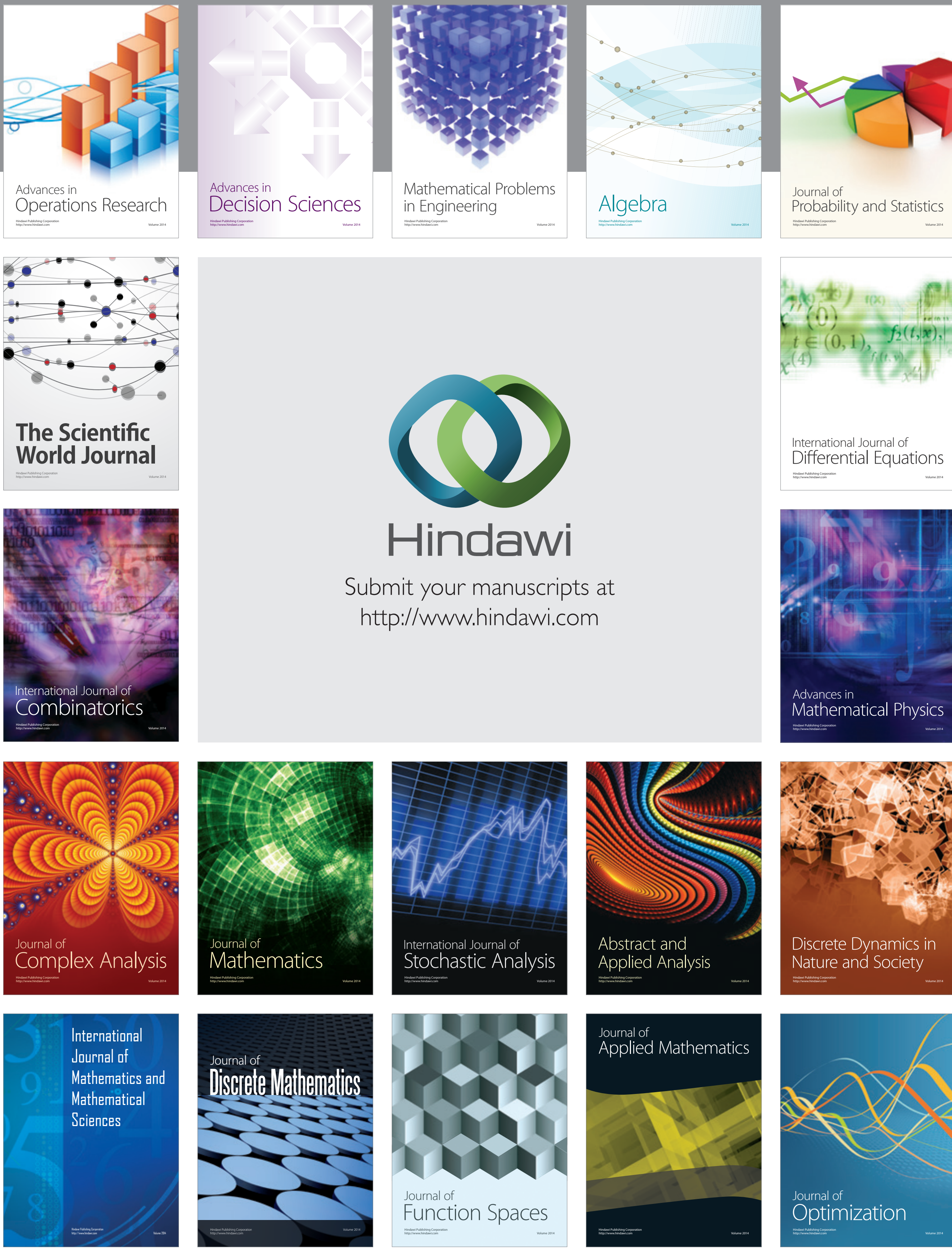
In this paper, the first author was partially supported by Grant NSC101-2115-M-039-001- from the National Science Council of Taiwan. The research part of the second author was done during his visit to King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

References

[1] Ky. Fan, "Fixed-point and minimax theorems in locally convex topological linear spaces," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 38, pp. 121–126, 1952.
[2] K. Fan, "Minimax theorems," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 39, pp. 42–47, 1953.
[3] F. Ferro, "A minimax theorem for vector-valued functions," *Journal of Optimization Theory and Applications*, vol. 60, no. 1, pp. 19–31, 1989.
[4] G. Y. Chen, "Improvement for Ferro's minimax theorem," *Systems Science and Mathematical Sciences*, vol. 7, no. 1, pp. 1–4, 1994.
[5] S. J. Li, G. Y. Chen, and G. M. Lee, "Minimax theorems for set-valued mappings," *Journal of Optimization Theory and Applications*, vol. 106, no. 1, pp. 183–199, 2000.
[6] S. J. Li, G. Y. Chen, K. L. Teo, and X. Q. Yang, "Generalized minimax inequalities for set-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 2, pp. 707–723, 2003.
[7] Y. Zhang, S. J. Li, and S. K. Zhu, "Minimax problems for set-valued mappings," *Numerical Functional Analysis and Optimization*, vol. 33, no. 2, pp. 239–253, 2012.
[8] Y.-C. Lin, "On generalized vector equilibrium problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 2, pp. 1040–1048, 2009.
[9] K. K. Tan, J. Yu, and X. Z. Yuan, "Existence theorems for saddle points of vector-valued maps," *Journal of Optimization Theory and Applications*, vol. 89, no. 3, pp. 731–747, 1996.
[10] C. Gerth and P. Weidner, "Nonconvex separation theorems and some applications in vector optimization," *Journal of Optimization Theory and Applications*, vol. 67, no. 2, pp. 297–320, 1990.
[11] G. X.-Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker Inc., New York, NY, USA, 1999.
[12] F. E. Browder, "The fixed point theory of multi-valued mappings in topological vector spaces," *Mathematische Annalen*, vol. 177, pp. 283–301, 1968.
[13] D. Kuroiwa, "Convexity for set-valued maps," *Applied Mathematics Letters*, vol. 9, no. 2, pp. 97–101, 1996.
[14] J. M. Borwein and D. Zhuang, "On Fan's minimax theorem," *Mathematical Programming*, vol. 34, no. 2, pp. 232–234, 1986.

[15] J. L. Kelley and I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Co., Inc., Princeton, NJ, USA, 1963.

[16] A. R. Warburton, "Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives," *Journal of Optimization Theory and Applications*, vol. 40, no. 4, pp. 537–557, 1983.


Advances in
Operations Research
Advances in
Decision Sciences
Mathematical Problems
in Engineering
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
International Journal of
Stochastic Analysis
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
Journal of
Applied Mathematics
Journal of
Optimization