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*Research Article*

# Robust Exponential Stability Criteria of LPD Systems with Mixed Time-Varying Delays and Nonlinear Perturbations

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This paper investigates the problem of robust exponential stability for linear parameter-dependent (LPD) systems with discrete and distributed time-varying delays and nonlinear perturbations. Parameter dependent Lyapunov-Krasovskii functional, Leibniz-Newton formula, and linear matrix inequality are proposed to analyze the stability. On the basis of the estimation and by utilizing free-weighting matrices, new delay-dependent exponential stability criteria are established in terms of linear matrix inequalities (LMIs). Numerical examples are given to demonstrate the effectiveness and less conservativeness of the proposed methods.

## 1. Introduction

Over the past decades, dynamical systems with state delays have attracted much interest in the literature over the half century, especially in the last decade. Since time delay is frequently a source of instability or poor performances in various systems such as electric, chemical processes, and long transmission line in pneumatic systems [1]. The problems of stability and stabilization for dynamical systems with or without state delays have been intensively studied in the past years by many researchers in mathematics and control communities [2, 3]. Stability criteria for dynamical systems with time delay is generally divided into two classes: delay-independent one and delay-dependent one. Delay-independent stability criteria tends to be more conservative, especially for small size delay, such criteria do not give any information on the size of the delay. On the other hand, delay-dependent

stability criteria concerned with the size of the delay and usually provide a maximal delay size. Various stability of linear continuous-time and discrete-time systems subject to time-invariant parametric uncertainty have received considerable attention. An important class of linear time-invariant parametric uncertain system is linear parameter-dependent (LPD) system in which the uncertain state matrices are in the polytope consisting of all convex combination of known matrices. Most of sufficient (or necessary and sufficient) conditions have been obtained via Lyapunov-Krasovskii theory approaches in which parameter-dependent Lyapunov-Krasovskii functional has been employed. These conditions are always expressed in terms of linear matrix inequalities (LMIs). The results have been obtained for robust stability for LPD systems in which time delay occurs in state variable such as [4–6] which present sufficient conditions for robust stability of LPD continuous-time system with delays.

Recently, many researchers have studied the problem of stability for time-delay systems with nonlinear perturbations such as [7] which considers the robust stability for a class of linear systems with interval time-varying delay and nonlinear perturbations. In [8], exponential stability of time-delay systems with nonlinear uncertainties is studied. Based on the Lyapunov theory approach and the approaches of decomposing the matrix, a new exponential stability criterion is derived in terms of LMI. In [9], they propose a new delay-dependent stability criterion in terms of linear matrix inequality for dynamic systems with time-varying delays and nonlinear perturbations by using Lyapunov theory. However, many researchers have studied the problem of stability for systems with discrete and distributed delays such as [10] which presented some stability conditions for uncertain neutral systems with discrete and distributed delays. The robust stability of uncertain linear neutral systems with discrete and distributed delays has been studied in [11]. In [12, 13], they studied the problem of stability for linear switching system with discrete and distributed delays. Moreover, a descriptor model transformation and a corresponding Lyapunov-Krasovskii functionals have been introduced for stability analysis of systems with delays in [14, 15].

In this paper, we will investigate the problems of robust exponential stability for LPD system with mixed time-varying delays and nonlinear perturbations. Based on the combination of Leibniz-Newton formula and linear matrix inequality, the use of suitable Lyapunov-Krasovskii functional, new delay-dependent exponential stability criteria will be obtained in terms of LMIs. Finally, numerical examples will be given to show the effectiveness of the obtained results.

## 2. Problem Formulation and Preliminaries

We introduce some notations, definition, and lemmas that will be used throughout the paper. $R^+$ denotes the set of all real nonnegative numbers; $R^n$ denotes the $n$-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in R^n$; $R^{n\times r}$ denotes the set $n\times r$ real matrices; $A^T$ denotes the transpose of the matrix $A$; $A$ is symmetric if $A = A^T$; $I$ denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of $A$; $\lambda_{\max}(A) = \max\{\mathrm{Re}\ \lambda : \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{\mathrm{Re}\ \lambda : \lambda \in \lambda(A)\}$; $\lambda_{\max}(A(\alpha)) = \max\{\lambda_{\max}(A_i) : i = 1,2,\ldots,N\}$; $\lambda_{\min}(A(\alpha)) = \min\{\lambda_{\min}(A_i) : i = 1,2,\ldots,N\}$; matrix $A$ is called a semipositive definite ($A \geq 0$) if $x^TAx \geq 0$, for all $x \in R^n$; $A$ is a positive definite ($A > 0$) if $x^TAx > 0$ for all $x \neq 0$; matrix $B$ is called a seminegative definite ($B \leq 0$) if $x^TBx \leq 0$, for all $x \in R^n$; $B$ is a negative definite ($B < 0$) if $x^TBx < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; $A \geq B$ means $A - B \geq 0$; $C([-\overline{h}, 0], R^n)$ denotes the space of all continuous vector functions mapping $[-\overline{h}, 0]$ into $R^n$

where $\overline{h} = \max\{h, g\}$; $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the system described by the following state equation of the form

$$\begin{aligned}\dot{x}(t) &= A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t)))\\ &\quad + C(\alpha)\int_{t-g(t)}^{t} x(s)ds, \quad t > 0;\\ x(t) &= \phi(t), \quad \dot{x}(t) = \psi(t), \quad t \in \left[-\overline{h}, 0\right],\end{aligned} \tag{2.1}$$

where $x(t) \in R^n$ is the state variable, $A(\alpha), B(\alpha), C(\alpha) \in R^{n\times n}$ are uncertain matrices belonging to the polytope

$$\begin{aligned}&A(\alpha) = \sum_{i=1}^{N}\alpha_i A_i, \quad B(\alpha) = \sum_{i=1}^{N}\alpha_i B_i, \quad C(\alpha) = \sum_{i=1}^{N}\alpha_i C_i,\\ &\sum_{i=1}^{N}\alpha_i = 1, \quad \alpha_i \geq 0, A_i,\ B_i, C_i \in R^{n\times n},\ i = 1,\ldots,N.\end{aligned} \tag{2.2}$$

$h(t)$ and $g(t)$ are discrete and distributed time-varying delays, respectively, satisfying

$$0 \leq h(t) \leq h, \qquad \dot{h}(t) \leq h_d, \qquad 0 \leq g(t) \leq g, \tag{2.3}$$

where $h, h_d, g$ are given positive real constants. Consider the initial functions $\phi(t)$, $\psi(t) \in C([-\overline{h}, 0], R^n)$ with the norm $\|\phi\| = \sup_{t\in[-h,0]}\|\phi(t)\|$ and $\|\psi\| = \sup_{t\in[-h,0]}\|\psi(t)\|$. The uncertainties $f(\cdot), g(\cdot)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h(t))$, respectively, and are bounded in magnitude of the form

$$\begin{aligned}f^T(t,x(t))f(t,x(t)) &\leq \eta^2 x^T(t)x(t),\\ g^T(t,x(t-h(t)))g(t,x(t-h(t))) &\leq \rho^2 x^T(t-h(t))x(t-h(t)),\end{aligned} \tag{2.4}$$

where $\eta, \rho$ are given real constants.

*Definition 2.1.* The system (2.1) is robustly exponentially stable, if there exist positive real numbers $\beta$ and $M$ such that for each $\phi(t), \psi(t) \in C([-\overline{h}, 0], R^n)$, the solution $x(t, \phi, \psi)$ of the system (2.1) satisfies

$$\left\|x(t,\phi,\psi)\right\| \leq M\max\{\left\|\phi\right\|, \left\|\psi\right\|\}e^{-\beta t}, \quad \forall t \in R^+. \tag{2.5}$$

**Lemma 2.2** (Schur complement lemma, see [9]). *Given constant symmetric matrices $X, Y, Z$ where $Y > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0. \tag{2.6}$$

**Lemma 2.3** (Jensen's inequality, see [1]). *For any constant matrix $Q \in R^{n\times n}$, $Q = Q^T > 0$, scalar $h > 0$, vector function $\dot{x} : [0, h] \to R^n$ such that the integrations concerned are well defined, then*

$$-h \int_{-h}^{0} \dot{x}^T(s+t) Q \dot{x}(s+t) ds \le -\left( \int_{-h}^{0} \dot{x}(s+t) ds \right)^T Q \left( \int_{-h}^{0} \dot{x}(s+t) ds \right). \tag{2.7}$$

*Rearranging the term $\int_{-h}^{0} \dot{x}(s+t)ds$ with $x(t) - x(t-h)$, one can yield the following inequality:*

$$-h \int_{-h}^{0} \dot{x}^T(s+t) Q \dot{x}(s+t) ds \le \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -Q & Q \\ Q & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}. \tag{2.8}$$

**Lemma 2.4** (see [16]). *Let $x(t) \in R^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $X, M_i \in R^{n\times n}$, $i = 1, 2, \ldots, 5$ and a scalar function $h := h(t) \ge 0$:*

$$\begin{aligned} -\int_{t-h}^{t} \dot{x}^T(s) X \dot{x}(s) ds \le & \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ -M_1 + M_2^T & -M_2^T - M_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ & + h \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 \\ M_4^T & M_5 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}, \end{aligned} \tag{2.9}$$

*where*

$$\begin{bmatrix} X & M_1 & M_2 \\ M_1^T & M_3 & M_4 \\ M_2^T & M_4^T & M_5 \end{bmatrix} \ge 0. \tag{2.10}$$

## 3. Main Results

In this section, we first study the robust exponential stability criteria for the system (2.1) by using the combination of linear matrix inequality (LMI) technique and Lyapunov theory method. We introduce the following notations for later use:

$$P_j(\alpha)=\sum_{i=1}^{N}\alpha_i P_i^j,\quad W_j(\alpha)=\sum_{i=1}^{N}\alpha_i W_i^j,\quad N_j(\alpha)=\sum_{i=1}^{N}\alpha_i N_i^j,\quad Q_j(\alpha)=\sum_{i=1}^{N}\alpha_i Q_i^j,$$
$$M_l(\alpha)=\sum_{i=1}^{N}\alpha_i M_i^l,\quad R_s(\alpha)=\sum_{i=1}^{N}\alpha_i R_i^s,\quad \sum_{i=1}^{N}\alpha_i=1,\quad \alpha_i\geq 0, \tag{3.1}$$

$$P_i^j, W_i^j, N_i^j, Q_i^j, M_i^l, R_i^s\in R^{n\times n},\quad j=1,2,\ldots,6,\ l=1,2,\ldots,5,\ s=1,2,3,\ i=1,2,\ldots,N;$$

$$\prod_{i,j,k}=\begin{bmatrix}
\Sigma_{i,j,k}^{11} & \Sigma_{i,j,k}^{12} & \Sigma_{i,j}^{13} & \Sigma_{i,j}^{14} & \Sigma_{i,j}^{15} & \Sigma_{i}^{16} & \Sigma_{i,j,k}^{17}\\
* & \Sigma_{i,j,k}^{22} & \Sigma_{i,j}^{23} & \Sigma_{i,j}^{24} & \Sigma_{i,j}^{25} & \Sigma_{i}^{26} & \Sigma_{i,j,k}^{27}\\
* & * & \Sigma_{i}^{33} & \Sigma_{i}^{34} & \Sigma_{i}^{35} & -N_i^{3T} & \Sigma_{i,j}^{37}\\
* & * & * & \Sigma_{i}^{44} & \Sigma_{i}^{45} & -N_i^{4T} & \Sigma_{i,j}^{47}\\
* & * & * & * & \Sigma_{i}^{55} & \Sigma_{i}^{56} & \Sigma_{i,j}^{57}\\
* & * & * & * & * & \Sigma_{i}^{66} & 0\\
* & * & * & * & * & * & \Sigma_{i,j,k}^{77}
\end{bmatrix}, \tag{3.2}$$

where

$$\Sigma_{i,j,k}^{11}=2\beta P_i^1+P_i^1A_j+A_i^TP_j^1+P_i^2+h^2A_i^TP_j^5A_k-e^{-2\beta h}P_i^5+Q_i^1+Q_i^{1T}+Q_i^{4T}A_j+A_i^TQ_j^4$$
$$+N_i^{1T}+N_i^1+W_i^{1T}A_j+A_i^TW_j^1+hM_i^{1T}+hM_i^1+h^2M_i^3+\epsilon_1\eta^2I+g^2P_i^6,$$
$$\Sigma_{i,j,k}^{12}=P_i^1B_j+Q_i^2-Q_i^{1T}+A_i^TQ_j^5+Q_i^{4T}B_j+h^2A_i^TP_j^5B_k+e^{-2\beta h}P_i^5+hR_i^{2T}-hM_i^{1T}$$
$$-N_i^{1T}+N_i^2+W_i^{1T}B_j+W_i^{2T}A_j+hM_i^2+h^2M_i^4,$$
$$\Sigma_{i,j}^{13}=N_i^3+W_i^{1T}+A_i^TW_j^3+P_i^1+Q_i^{4T}+h^2A_i^TP_j^5,$$
$$\Sigma_{i,j}^{14}=N_i^4+W_i^{1T}+A_i^TW_j^4+P_i^1+Q_i^{4T}+h^2A_i^TP_j^5,$$
$$\Sigma_{i,j}^{15}=N_i^5-W_i^{1T}+A_i^TW_j^5+Q_i^3-Q_i^{4T}+A_i^TQ_j^6,$$

$$
\begin{aligned}
\Sigma_i^{16} &= -{Q_i^1}^T - {N_i^1}^T + N_i^6,\\
\Sigma_{i,j,k}^{17} &= h^2 A_i^T P_j^5 C_k + {W_i^1}^T C_j + A_i^T W_j^6 + P_i^1 C_j + {Q_i^4}^T C_j,\\
\Sigma_{i,j,k}^{22} &= -{Q_i^2}^T - Q_i^2 + {Q_i^5}^T B_j + B_i^T Q_j^5 - e^{-2\beta h} P_i^2 + h_d P_i^2 + h^2 B_i^T P_j^5 B_k - e^{-2\beta h} P_i^5 + h^2 R_i^1\\
&\quad - {N_i^2}^T - N_i^2 + {W_i^2}^T B_j + B_i^T W_j^2 - h{R_i^2}^T - hR_i^2 - h{M_i^2}^T - hM_i^2 + h^2 M_i^5 + \epsilon_2 \rho^2 I,\\
\Sigma_{i,j}^{23} &= {W_i^2}^T - N_i^3 + B_i^T W_j^3 + {Q_i^5}^T + h^2 B_i^T P_j^5,\\
\Sigma_{i,j}^{24} &= {W_i^2}^T - N_i^4 + B_i^T W_j^4 + {Q_i^5}^T + h^2 B_i^T P_j^5,\\
\Sigma_{i,j}^{25} &= -{W_i^2}^T - N_i^5 + B_i^T W_j^5 + B_i^T Q_j^6 - Q_i^3 - {Q_i^5}^T,\\
\Sigma_i^{26} &= -{Q_i^2}^T - {N_i^2}^T - N_i^6,\\
\Sigma_{i,j,k}^{27} &= h^2 B_i^T P_j^5 C_k + {W_i^2}^T C_j + B_i^T W_j^6 + {Q_i^5}^T C_j,\\
\Sigma_i^{33} &= {W_i^3}^T + W_i^3 + h^2 P_i^5 - \epsilon_1 I,\\
\Sigma_i^{34} &= {W_i^3}^T + W_i^4 + h^2 P_i^5,\\
\Sigma_i^{35} &= -{W_i^3}^T + W_i^5 + Q_i^6,\\
\Sigma_{i,j}^{37} &= h^2 P_i^5 C_j + {W_i^3}^T C_j + W_i^6,\\
\Sigma_i^{44} &= {W_i^4}^T + W_i^4 + h^2 P_i^5 - \epsilon_2 I,\\
\Sigma_i^{45} &= -{W_i^4}^T + W_i^5 + Q_i^6,\\
\Sigma_{i,j}^{47} &= h^2 P_i^5 C_j + {W_i^4}^T C_j + W_i^6,\\
\Sigma_i^{55} &= -{W_i^5}^T - W_i^5 - {Q_i^6}^T - Q_i^6 + h^2 P_i^3 + h^2 P_i^4,\\
\Sigma_i^{56} &= -{Q_i^3}^T - {N_i^5}^T,\\
\Sigma_{i,j}^{57} &= {W_i^5}^T C_j - W_i^6 + {Q_i^6}^T C_j,\\
\Sigma_i^{66} &= -{N_i^6}^T - N_i^6 - e^{-2\beta h} P_i^4,\\
\Sigma_{i,j,k}^{77} &= -e^{2\beta g} P_i^6 + h^2 C_i^T P_j^5 C_k + C_i^T W_j^6 + {W_i^6}^T C_j.
\end{aligned}
\tag{3.3}
$$

**Theorem 3.1.** *For given positive real constants $h, h_d, g, \eta$ and $\rho$, system* (2.1) *is robustly exponentially stable with a decay rate $\beta$, if there exist positive definite symmetric matrices $P_i^s$, any*

*appropriate dimensional matrices* $W_i^s$, $Q_i^s$, $N_i^s$, $M_i^r$, $R_i^t$, $s = 1,2,\ldots,6$, $r = 1,2,\ldots,5$, $t = 1, 2, 3$, $i = 1, 2,\ldots, N$ *and positive real constants* $\epsilon_1$ *and* $\epsilon_2$ *satisfying the following LMIs:*

$$\begin{bmatrix} R_i^1 & R_i^2 \\ * & R_i^3 \end{bmatrix} > 0, \quad i = 1,2,\ldots,N, \tag{3.4}$$

$$\begin{bmatrix} e^{-2\beta h}P_i^3 - R_i^3 & M_i^1 & M_i^2 \\ * & M_i^3 & M_i^4 \\ * & * & M_i^5 \end{bmatrix} \geq 0, \quad i = 1,2,\ldots,N, \tag{3.5}$$

$$\prod_{i,i,i} < -I, \quad i = 1,2,\ldots,N, \tag{3.6}$$

$$\prod_{i,i,j} + \prod_{i,j,i} + \prod_{j,i,i} < \frac{1}{(N-1)^2} I, \quad i = 1,2,\ldots,N,\ i \neq j,\ j = 1,2,\ldots,N, \tag{3.7}$$

$$\begin{aligned} &\prod_{i,j,k} + \prod_{i,k,j} + \prod_{j,i,k} + \prod_{j,k,i} + \prod_{k,i,j} + \prod_{k,j,i} < \frac{6}{(N-1)^2} I, \\ &i = 1,2,\ldots,\ N-2, \quad j = i+1,\ldots,\ N-1, k = j+1,\ldots,N. \end{aligned} \tag{3.8}$$

*Moreover, the solution* $x(t,\phi,\psi)$ *satisfies the inequality*

$$\|x(t,\phi,\psi)\| \leq \sqrt{\frac{N}{\lambda_{\min}(P_1(\alpha))}} \max\left[\|\phi\|, \|\psi\|\right] e^{-\beta t}, \quad \forall t \in R^+, \tag{3.9}$$

*where* $N = \lambda_{\max}(P_1(\alpha)) + h\lambda_{\max}(P_2(\alpha)) + h^3\lambda_{\max}(P_3(\alpha)) + h^3\lambda_{\max}(P_4(\alpha)) + h^3\lambda_{\max}(P_5(\alpha)) + h^3\lambda_{\max}(\left[\begin{smallmatrix} R_1(\alpha) & R_2(\alpha) \\ R_2^T(\alpha) & R_3(\alpha) \end{smallmatrix}\right])$.

*Proof.* Choose a parameter-dependent Lyapunov-Krasovskii functional candidate for the system (2.1) of the form

$$V(t) = \sum_{i=1}^{7} V_i(t), \tag{3.10}$$

where

$$\begin{aligned} V_1(t) &= x^T(t)P_1(\alpha)x(t) \\ &= \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1(\alpha) & 0 & 0 \\ Q_1(\alpha) & Q_2(\alpha) & Q_3(\alpha) \\ Q_4(\alpha) & Q_5(\alpha) & Q_6(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
V_2(t) &= \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) P_2(\alpha) x(s) ds,\\
V_3(t) &= h \int_{-h}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s) P_3(\alpha) \dot{x}(s) ds\, d\theta,\\
V_4(t) &= h \int_{-h}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s) P_4(\alpha) \dot{x}(s) ds\, d\theta,\\
V_5(t) &= h \int_{-h}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s) P_5(\alpha) \dot{x}(s) ds\, d\theta,\\
V_6(t) &= h \int_{-h}^{t} \int_{\theta-h(\theta)}^{\theta} e^{2\beta(\theta-t)} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} R_1(\alpha) & R_2(\alpha) \\ R_2^T(\alpha) & R_3(\alpha) \end{bmatrix} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix} ds\, d\theta,\\
V_7(t) &= g \int_{-g}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} x^T(s) P_6(\alpha) x(s) ds d\theta.
\end{aligned}
\tag{3.11}
$$

Calculating the time derivatives of $V_i(t)$, $i = 1, 2, 3, \ldots, 6$, along the trajectory of (2.1) yields

$$
\begin{aligned}
\dot{V}_1(t) &= 2 \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} P_1(\alpha) & Q_1^T(\alpha) & Q_4^T(\alpha) \\ 0 & Q_2^T(\alpha) & Q_5^T(\alpha) \\ 0 & Q_3^T(\alpha) & Q_6^T(\alpha) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ 0 \\ 0 \end{bmatrix}\\
&= 2 \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} P_1(\alpha) & Q_1^T(\alpha) & Q_4^T(\alpha) \\ 0 & Q_2^T(\alpha) & Q_5^T(\alpha) \\ 0 & Q_3^T(\alpha) & Q_6^T(\alpha) \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{21} \\ \omega_{31} \end{bmatrix},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
\omega_{11} &= A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) + C(\alpha)\int_{t-g(t)}^{t} x(s)ds,\\
\omega_{21} &= x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} \dot{x}(s)ds,\\
\omega_{31} &= A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) + C(\alpha)\int_{t-g(t)}^{t} x(s)ds - \dot{x}(t).
\end{aligned}
\tag{3.13}
$$

Taking the time-derivative of $V_2(t)$ leads to

$$\begin{aligned}\dot{V}_2(t) &= x^T(t)P_2(\alpha)x(t) - \left(1-\dot{h}(t)\right)e^{-2\beta h(t)}x^T(t-h(t))P_2(\alpha)x(t-h(t)) - 2\beta V_2(t)\\ &\le x^T(t)P_2(\alpha)x(t) - e^{-2\beta h}x^T(t-h(t))P_2(\alpha)x(t-h(t)) + h_d x^T(t-h(t))P_2(\alpha)x(t-h(t))\\ &\quad - 2\beta V_2(t).\end{aligned} \tag{3.14}$$

Obviously, for any scalar $s \in [t-h, t]$, we get $e^{-2\beta h} \le e^{-2\beta(s-t)} \le 1$. Together with Lemma 2.3 (Jensen's inequality), we obtain

$$\begin{aligned}\dot{V}_3(t) &= h^2\dot{x}^T(t)P_3(\alpha)\dot{x}(t) - h\int_{-h}^{0} e^{2\beta s}\dot{x}^T(t+s)P_3(\alpha)\dot{x}(t+s)ds - 2\beta V_3(t)\\ &\le h^2\dot{x}^T(t)P_3(\alpha)\dot{x}(t) - h\int_{t-h}^{t} e^{2\beta(s-t)}\dot{x}^T(s)P_3(\alpha)\dot{x}(s)ds - 2\beta V_3(t)\\ &\le h^2\dot{x}^T(t)P_3(\alpha)\dot{x}(t) - he^{-2\beta h}\int_{t-h}^{t}\dot{x}^T(s)P_3(\alpha)\dot{x}(s)ds - 2\beta V_3(t).\end{aligned} \tag{3.15}$$

Following the estimation of $\dot{V}_3(t)$, we have

$$\begin{aligned}\dot{V}_4(t) &\le h^2\dot{x}^T(t)P_4(\alpha)\dot{x}(t) - he^{-2\beta h}\int_{t-h}^{t}\dot{x}^T(s)P_4(\alpha)\dot{x}(s)ds - 2\beta V_4(t)\\ &\le h^2\dot{x}^T(t)P_4(\alpha)\dot{x}(t) - e^{-2\beta h}\int_{t-h}^{t}\dot{x}^T(s)dsP_4(\alpha)\int_{t-h}^{t}\dot{x}(s)ds - 2\beta V_4(t)\\ &\le h^2\dot{x}^T(t)P_4(\alpha)\dot{x}(t) - e^{-2\beta h}\int_{t-h(t)}^{t}\dot{x}^T(s)dsP_4(\alpha)\int_{t-h(t)}^{t}\dot{x}(s)ds - 2\beta V_4(t).\end{aligned} \tag{3.16}$$

From (3.16), it follows that

$$\begin{aligned}\dot{V}_5(t) &\le h^2\dot{x}^T(t)P_5(\alpha)\dot{x}(t) - e^{-2\beta h}\int_{t-h(t)}^{t}\dot{x}^T(s)dsP_5(\alpha)\int_{t-h(t)}^{t}\dot{x}(s)ds - 2\beta V_5(t)\\ &= h^2\dot{x}^T(t)P_5(\alpha)\dot{x}(t) - e^{-2\beta h}\left[x^T(t) - x^T(t-h(t))\right]P_5(\alpha)[x(t) - x(t-h(t))] - 2\beta V_5(t)\\ &= h^2\left[A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) + C(\alpha)\int_{t-g(t)}^{t}x(s)ds\right]^T\end{aligned}$$

$$\times P_5(\alpha)\Bigg[A(\alpha)x(t)+B(\alpha)x(t-h(t))+f(t,x(t))+g(t,x(t-h(t)))$$
$$+C(\alpha)\int_{t-g(t)}^{t}x(s)ds\Bigg]-e^{-2\beta h}\Big[x^T(t)-x^T(t-h(t))\Big]P_5(\alpha)[x(t)-x(t-h(t))]$$
$$-2\beta V_5(t).$$
(3.17)

Taking the time derivative of $V_6(t)$ and $V_7(t)$, we obtain

$$\begin{aligned}\dot{V}_6(t)&=hh(t)x^T(t-h(t))R_1(\alpha)x(t-h(t))+2hx^T(t-h(t))R_2(\alpha)x(t)\\&\quad-2hx^T(t-h(t))R_2(\alpha)x(t-h(t))+h\int_{t-h}^{t}\dot{x}^T(s)R_3(\alpha)\dot{x}(s)ds-2\beta V_6(t)\\&\le h^2x^T(t-h(t))R_1(\alpha)x(t-h(t))+2hx^T(t-h(t))R_2(\alpha)x(t)\\&\quad-2hx^T(t-h(t))R_2(\alpha)x(t-h(t))+h\int_{t-h}^{t}\dot{x}^T(s)R_3(\alpha)\dot{x}(s)ds-2\beta V_6(t);\\\dot{V}_7(t)&\le g^2x^T(t)P_6(\alpha)x(t)-e^{-2\beta g}\int_{t-g(t)}^{t}x^T(s)dsP_6(\alpha)\int_{t-g(t)}^{t}x(s)ds-2\beta V_7(t).\end{aligned}$$
(3.18)

From the Leibinz-Newton formula, the following equation is true for any real matrices $N_i(\alpha)$, $i=1,2,\ldots,6$ with appropriate dimensions

$$2\Bigg[x^T(t)N_1^T(\alpha)+x^T(t-h(t))N_2^T(\alpha)+f^T(t,x(t))N_3^T(\alpha)+g^T(t,x(t-h(t)))N_4^T(\alpha)$$
$$+\dot{x}^T(t)N_5^T(\alpha)+\int_{t-h(t)}^{t}\dot{x}^T(s)dsN_6^T(\alpha)\Bigg]\times\Bigg[x(t)-x(t-h(t))-\int_{t-h(t)}^{t}\dot{x}(s)ds\Bigg]=0.$$
(3.19)

From the utilization of zero equation, the following equation is true for any real matrices $W_i$, $i=1,2,\ldots,5$ with appropriate dimensions

$$2\Bigg[x^T(t)W_1^T(\alpha)+x^T(t-h(t))W_2^T(\alpha)+f^T(t,x(t))W_3^T(\alpha)+g^T(t,x(t-h(t)))W_4^T(\alpha)$$
$$+\dot{x}^T(t)W_5^T(\alpha)+\int_{t-g(t)}^{t}x^T(s)dsW_6^T(\alpha)\Bigg]$$
$$\times\Bigg[A(\alpha)x(t)+B(\alpha)x(t-h(t))+f(t,x(t))+g(t,x(t-h(t)))+C(\alpha)\int_{t-g(t)}^{t}x(s)ds-\dot{x}(t)\Bigg]=0.$$
(3.20)

From (2.4), we obtain for any positive real constants $\epsilon_1$ and $\epsilon_2$,

$$\begin{aligned} 0 &\le \epsilon_1\eta^2 x^T(t)x(t) - \epsilon_1 f^T(t,x(t))f(t,x(t)), \\ 0 &\le \epsilon_2\rho^2 x^T(t-h(t))x(t-h(t)) - \epsilon_2 g^T(t,x(t-h(t)))g(t,x(t-h(t))). \end{aligned} \tag{3.21}$$

By (3.5), Lemma 2.4 and the integral term of the right-hand side of $\dot{V}_3(t)$ and $\dot{V}_6(t)$, we obtain

$$\begin{aligned} &-h\int_{t-h}^{t} \dot{x}^T(s)\left[e^{-2\beta h}P_3(\alpha) - R_3(\alpha)\right]\dot{x}(s)ds \\ &\quad\le h\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} M_1^T(\alpha)+M_1(\alpha) & -M_1^T(\alpha)+M_2(\alpha) \\ -M_1(\alpha)+M_2^T(\alpha) & -M_2^T(\alpha)-M_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\ &\qquad + h^2\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} M_3(\alpha) & M_4(\alpha) \\ M_4^T(\alpha) & M_5(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}. \end{aligned} \tag{3.22}$$

According to (3.12)–(3.22), it is straightforward to see that

$$\dot{V}(t) \le \zeta^T(t)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_i\alpha_j\alpha_k\prod_{i,j,k}\zeta(t) - 2\beta V(t), \tag{3.23}$$

where $\zeta^T(t) = [x^T(t),\ x^T(t-h(t)),\ f^T(t,x(t)),\ g^T(t,\ x(t-h(t))),\ \dot{x}^T(t),\ \int_{t-h(t)}^{t}\dot{x}^T(s)ds,\ \int_{t-g(t)}^{t}x^T(s)ds]$ and $\prod_{i,j,k}$ is defined in (3.2). The facts that $\sum_{i=1}^{N}\alpha_i = 1$, we obtain the following identities:

$$\begin{aligned} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_i\alpha_j\alpha_k\prod_{i,j,k} &= \sum_{i=1}^{N}\alpha_i^3\prod_{i,i,i} + \sum_{i=1}^{N}\sum_{i\ne j,j=1}^{N}\alpha_i^2\alpha_j\left[\prod_{i,i,j}+\prod_{i,j,i}+\prod_{j,i,i}\right] \\ &\quad + \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N}\alpha_i\alpha_j\alpha_l\left[\prod_{i,j,k}+\prod_{i,k,j}+\prod_{j,i,k}+\prod_{j,k,i}+\prod_{k,i,j}+\prod_{k,j,i}\right]. \end{aligned} \tag{3.24}$$

We define $\Phi$ and $\Lambda$ as

$$\begin{aligned} \Phi &\equiv \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\left(\alpha_i-\alpha_j\right)^2 = (N-1)\sum_{i=1}^{N}\alpha_i^3 - \sum_{i=1}^{N}\sum_{j\ne i;j=1}^{N}\alpha_i^2\alpha_j \ge 0, \\ \Lambda &\equiv \sum_{i=1}^{N}\sum_{j\ne i;j=1}^{N-1}\sum_{k\ne i;k=2}^{N}\alpha_i\left[\alpha_j-\alpha_k\right]^2 = (N-2)\sum_{i=1}^{N}\sum_{j\ne i;j=1}^{N-1}\alpha_i^2\alpha_j - 6\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N}\alpha_i\alpha_j\alpha_l \ge 0. \end{aligned} \tag{3.25}$$

From $(N-1)\Phi + \Lambda \geq 0$, we obtain

$$\sum_{i=1}^{N}\alpha_i^3 - \frac{1}{(N-1)^2}\sum_{i=1}^{N}\sum_{i\neq j,j=1}^{N}\alpha_i^2\alpha_j - \frac{6}{(N-1)^2}\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N}\alpha_i\alpha_j\alpha_l \geq 0. \tag{3.26}$$

By (3.23)–(3.26), if the conditions (3.6)–(3.8) are true, then

$$\dot{V}(t) + 2\beta V(t) \leq 0, \quad \forall t \in R^+, \tag{3.27}$$

which gives

$$V(t) \leq V(0)e^{-2\beta t}, \quad \forall t \in R^+. \tag{3.28}$$

From (3.28), it is easy to see that

$$\begin{gathered}\lambda_{\min}(P_1(\alpha))\|x(t)\|^2 \leq V(t) \leq V(0)e^{-2\beta t},\\ V(0) = \sum_{i=1}^{6} V_i(0),\end{gathered} \tag{3.29}$$

where

$$\begin{aligned}
V_1(0) &= x^T(0)P_1(\alpha)x(0),\\
V_2(0) &= \int_{-h(0)}^{t} e^{2\beta s}x^T(s)P_2(\alpha)x(s)ds,\\
V_3(0) &= h\int_{-h}^{0}\int_{\theta}^{0} e^{2\beta s}\dot{x}^T(s)P_3(\alpha)\dot{x}(s)ds\,d\theta,\\
V_4(0) &= h\int_{-h}^{0}\int_{\theta}^{0} e^{2\beta s}\dot{x}^T(s)P_4(\alpha)\dot{x}(s)ds\,d\theta,\\
V_5(0) &= h\int_{-h}^{0}\int_{\theta}^{0} e^{2\beta s}\dot{x}^T(s)P_5(\alpha)\dot{x}(s)ds\,d\theta,\\
V_6(0) &= h\int_{-h}^{0}\int_{\theta-h(\theta)}^{0} e^{2\beta\theta}\begin{bmatrix}x(\theta-h(\theta))\\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix}R_1(\alpha) & R_2(\alpha)\\ R_2^T(\alpha) & R_3(\alpha)\end{bmatrix}\begin{bmatrix}x(\theta-h(\theta))\\ \dot{x}(s)\end{bmatrix}ds\,d\theta,\\
V_7(0) &= g\int_{-g}^{0}\int_{\theta}^{0} e^{2\beta s}x^T(s)P_6(\alpha)x(s)ds\,d\theta.
\end{aligned} \tag{3.30}$$

Therefore, we get

$$\lambda_{\min}(P_1(\alpha))\|x(t)\|^2 \leq V(0)e^{-2\beta t} \leq N\max\left[\|\phi\|, \|\varphi\|\right]^2 e^{-2\beta t}, \tag{3.31}$$

where $N = \lambda_{\max}(P_1(\alpha)) + h\lambda_{\max}(P_2(\alpha)) + h^3\lambda_{\max}(P_3(\alpha)) + h^3\lambda_{\max}(P_4(\alpha)) + h^3\lambda_{\max}(P_5(\alpha)) + h^3\lambda_{\max}(\begin{bmatrix} R_1(\alpha) & R_2(\alpha) \\ R_2^T(\alpha) & R_3(\alpha) \end{bmatrix})$. From (3.31), we get

$$\|x(t,\phi,\varphi)\| \le \sqrt{\frac{N}{\lambda_{\min}(P_1(\alpha))}} \max[\|\phi\|, \|\varphi\|] e^{-\beta t}, \quad \forall t \in R^+. \tag{3.32}$$

This means that system (2.1) is robustly exponentially stable. The proof of the theorem is complete. □

If $A(\alpha) = A$, $B(\alpha) = B$ and $C(\alpha) = 0$ when $A$ and $B$ are appropriate dimensional constant matrices, then system (2.1) reduces to the following system:

$$\begin{gathered} \dot{x}(t) = Ax(t) + Bx(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))), \quad t > 0; \\ x(t) = \phi(t), \quad \dot{x}(t) = \varphi(t), \quad t \in [-h, 0]. \end{gathered} \tag{3.33}$$

Take the Lyapunov-Krasovskii functional as

$$V(t) = \sum_{i=1}^{6} V_i(t), \tag{3.34}$$

where

$$\begin{aligned} V_1(t) &= x^T(t)P_1x(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 & 0 \\ Q_1 & Q_2 & Q_3 \\ Q_4 & Q_5 & Q_6 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}, \\ V_2(t) &= \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s)P_2x(s)ds, \\ V_3(t) &= h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s)P_3\dot{x}(s)ds\,d\theta, \\ V_4(t) &= h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s)P_4\dot{x}(s)ds\,d\theta, \\ V_5(t) &= h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s)P_5\dot{x}(s)ds\,d\theta, \\ V_6(t) &= h\int_{-h}^{t}\int_{\theta-h(\theta)}^{\theta} e^{2\beta(\theta-t)} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix} ds\,d\theta. \end{aligned} \tag{3.35}$$

According to Theorem 3.1, we have the following Corollary 3.2 for the delay-dependent exponential stability criteria of system (3.33).

**Corollary 3.2.** *For given positive real constants $h, h_d, \eta$ and $\rho$, system* (3.33) *is exponentially stable with a decay rate $\beta$, if there exist positive definite symmetric matrices $P_i, i = 1, 2, \ldots, 5$, any appropriate*

*dimensional matrices* $Q_i, N_i, i = 1,2,\ldots,6$, $W_i, M_i, i = 1,2,\ldots,5$, $R_i, i = 1,2,3$ *and positive real constants* $\epsilon_1$ *and* $\epsilon_2$ *satisfying the following LMIs:*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} > 0,$$

$$\begin{bmatrix} e^{-2\beta h}P_3 - R_3 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} \\ * & * & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & -N_3^T \\ * & * & * & \Sigma_{44} & \Sigma_{45} & -N_4^T \\ * & * & * & * & \Sigma_{55} & \Sigma_{56} \\ * & * & * & * & * & \Sigma_{66} \end{bmatrix} < 0, \tag{3.36}$$

*where*

$$\begin{aligned}
\Sigma_{11} &= 2\beta P_1 + P_1 A + A^T P_1 + P_2 + h^2 A^T P_5 A - e^{-2\beta h}P_5 + Q_1 + Q_1^T + Q_4^T A + A^T Q_4 \\
&\quad + N_1^T + N_1 + W_1^T A + A^T W_1 + hM_1^T + hM_1 + h^2 M_3 + \epsilon_1 \eta^2 I, \\
\Sigma_{12} &= P_1 B + Q_2 - Q_1^T + A^T Q_5 + Q_4^T B + h^2 A^T P_5 B + e^{-2\beta h}P_5 + hR_2^T - hM_1^T \\
&\quad - N_1^T + N_2 + W_1^T B + W_2^T A + hM_2 + h^2 M_4, \\
\Sigma_{13} &= N_3 + W_1^T + A^T W_3 + P_1 + Q_4^T + h^2 A^T P_5, \\
\Sigma_{14} &= N_4 + W_1^T + A^T W_4 + P_1 + Q_4^T + h^2 A^T P_5, \\
\Sigma_{15} &= N_5 - W_1^T + A^T W_5 + Q_3 - Q_4^T + A^T Q_6, \\
\Sigma_{16} &= -Q_1^T - N_1^T + N_6, \\
\Sigma_{22} &= Q_2^T - Q_2 + Q_5^T B + B^T Q_5 - e^{-2\beta h}P_2 + h_d P_2 + h^2 B^T P_5 B - e^{-2\beta h}P_5 + h^2 R_1 \\
&\quad - N_2^T - N_2 + W_2^T B + B^T W_2 - hR_2^T - hR_2 - hM_2^T - hM_2 + h^2 M_5 + \epsilon_2 \rho^2 I, \\
\Sigma_{23} &= W_2^T - N_3 + B^T W_3 + Q_5^T + h^2 B^T P_5, \\
\Sigma_{24} &= W_2^T - N_4 + B^T W_4 + Q_5^T + h^2 B^T P_5, \\
\Sigma_{25} &= -W_2^T - N_5 + B^T W_5 + B^T Q_6 - Q_3 - Q_5^T, \\
\Sigma_{26} &= -Q_2^T - N_2^T - N_6, \\
\Sigma_{33} &= W_3^T + W_3 + h^2 P_5 - \epsilon_1 I,
\end{aligned}$$

$$\begin{aligned}
\Sigma_{34} &= W_3^T + W_4 + h^2 P_5,\\
\Sigma_{35} &= -W_3^T + W_5 + Q_6,\\
\Sigma_{44} &= W_4^T + W_4 + h^2 P_5 - \epsilon_2 I,\\
\Sigma_{45} &= -W_4^T + W_5 + Q_6,\\
\Sigma_{55} &= -W_5^T - W_5 - Q_6^T - Q_6 + h^2 P_3 + h^2 P_4,\\
\Sigma_{56} &= -Q_3^T - N_5^T,\\
\Sigma_{66} &= -N_6^T - N_6 - e^{-2\beta h} P_4.
\end{aligned} \tag{3.37}$$

*Moreover, the solution $x(t,\phi,\psi)$ satisfies the inequality*

$$\|x(t,\phi,\psi)\| \le \sqrt{\frac{N}{\lambda_{\min}(P_1)}} \max\left[\|\phi\|, \|\psi\|\right] e^{-\beta t}, \quad \forall t \in R^+, \tag{3.38}$$

*where* $N = \lambda_{\max}(P_1) + h\lambda_{\max}(P_2) + h^3\lambda_{\max}(P_3) + h^3\lambda_{\max}(P_4) + h^3\lambda_{\max}(P_5) + h^3\lambda_{\max}\left(\begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}\right)$.

## 4. Numerical Examples

In order to show the effectiveness of the approaches presented in Section 3, four numerical examples are provided.

*Example 4.1.* Consider the LPD time-delay system (2.1) with the following parameters ($N = 3$):

$$\begin{gathered}
A_1 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 1 \\ 0 & -4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix},\\
B_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\\
C_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & -0.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & 0.5 \end{bmatrix},\\
f(t,x(t)) = \begin{bmatrix} 0.2\sin t x_1(t) \\ 0.2\cos t x_2(t) \end{bmatrix}, \quad g(t,x(t-h(t))) = \begin{bmatrix} 0.3\sin t x_1(t-h(t)) \\ 0.3\cos t x_2(t-h(t)) \end{bmatrix},\\
h(t) = 0.2134\sin^2\left(\frac{0.3t}{0.4268}\right), \quad g(t) = 0.4\cos^2(t), \quad \phi(t) = \begin{bmatrix} -7 \\ 5 \end{bmatrix}, \quad t \in [-0.4, 0].
\end{gathered} \tag{4.1}$$

It is easy to see that $h_d = 0.3$, $\eta = 0.2$, $\rho = 0.3$, and $g = 0.4$. Find the discrete delay time $h$ to guarantee system (2.1) with the above parameters to be robustly exponentially stable with a decay rate $\beta = 0.15$.

*Solution 1.* By using the LMI Toolbox in Matlab (with accuracy 0.01) and conditions (3.4)–(3.8) of Theorem 3.1, this system is robustly exponentially stable for discrete delay satisfying $h = 0.2134$ and

$$P_1^1 = \begin{bmatrix} 225.4987 & -143.6565 \\ -143.6565 & 300.6876 \end{bmatrix}, \quad P_2^1 = \begin{bmatrix} 316.6633 & 0.3122 \\ 0.3122 & 426.8816 \end{bmatrix}, \quad P_3^1 = \begin{bmatrix} 361.5534 & -0.4974 \\ -0.4974 & 306.5360 \end{bmatrix},$$

$$P_1^2 = \begin{bmatrix} 153.3987 & -105.3621 \\ -105.3621 & 242.2669 \end{bmatrix}, \quad P_2^2 = \begin{bmatrix} 284.5598 & -17.5849 \\ -17.5849 & 306.8985 \end{bmatrix}, \quad P_3^2 = \begin{bmatrix} 283.8792 & -0.3896 \\ -0.3896 & 254.6003 \end{bmatrix},$$

$$P_1^3 = \begin{bmatrix} 484.7945 & 31.1598 \\ 31.1598 & 398.0539 \end{bmatrix}, \quad P_2^3 = \begin{bmatrix} 380.0951 & 9.9856 \\ 9.9856 & 387.3373 \end{bmatrix}, \quad P_3^3 = \begin{bmatrix} 391.5353 & -0.0741 \\ -0.0741 & 392.5712 \end{bmatrix},$$

$$P_1^4 = \begin{bmatrix} 285.1892 & -5.6549 \\ -5.6549 & 230.3520 \end{bmatrix}, \quad P_2^4 = \begin{bmatrix} 229.7296 & -0.3010 \\ -0.3010 & 239.7532 \end{bmatrix}, \quad P_3^4 = \begin{bmatrix} 239.9236 & 0.0430 \\ 0.0430 & 251.9889 \end{bmatrix},$$

$$P_1^5 = \begin{bmatrix} 285.3879 & -8.7124 \\ -8.7124 & 227.7653 \end{bmatrix}, \quad P_2^5 = \begin{bmatrix} 231.9963 & -0.7154 \\ -0.7154 & 239.6643 \end{bmatrix}, \quad P_3^5 = \begin{bmatrix} 239.9146 & -0.0007 \\ -0.0007 & 251.9987 \end{bmatrix},$$

$$P_1^6 = \begin{bmatrix} 195.2662 & -48.4444 \\ -48.4444 & 261.3741 \end{bmatrix}, \quad P_2^6 = \begin{bmatrix} 285.1103 & -2.8941 \\ -2.8941 & 286.4945 \end{bmatrix}, \quad P_3^6 = \begin{bmatrix} 291.3897 & 2.4418 \\ 2.4418 & 303.1851 \end{bmatrix},$$

$$Q_1^1 = \begin{bmatrix} 153.8322 & -6.2058 \\ -6.2058 & 511.1849 \end{bmatrix}, \quad Q_2^1 = \begin{bmatrix} 1,057.7 & 119.5 \\ 119.5 & 1,113.8 \end{bmatrix}, \quad Q_3^1 = \begin{bmatrix} -442.4 & 2.6 \\ 2.6 & -1,942.6 \end{bmatrix},$$

$$Q_1^2 = \begin{bmatrix} -1,438.7 & 472.6 \\ 472.6 & 503.4 \end{bmatrix}, \quad Q_2^2 = \begin{bmatrix} -3,162.4 & -435.4 \\ -435.4 & -107.8 \end{bmatrix}, \quad Q_3^2 = \begin{bmatrix} 1,383.2 & -0.1 \\ -0.1 & 1,874.5 \end{bmatrix},$$

$$Q_1^3 = \begin{bmatrix} -1,217.0 & -623.5 \\ -623.5 & -135.3 \end{bmatrix}, \quad Q_2^3 = \begin{bmatrix} -9.6 & -1,232.0 \\ -1,232.0 & 91.0 \end{bmatrix}, \quad Q_3^3 = \begin{bmatrix} 564.8910 & -1.6044 \\ -1.6044 & -90.0137 \end{bmatrix},$$

$$Q_1^4 = \begin{bmatrix} -61.6 & -1,470.3 \\ -1,470.3 & -4,166.8 \end{bmatrix}, \quad Q_2^4 = \begin{bmatrix} -4,191.2 & 13.6 \\ 13.6 & -4,402.9 \end{bmatrix}, \quad Q_3^4 = \begin{bmatrix} 3,768.2 & -0.5 \\ -0.5 & -89.7 \end{bmatrix},$$

$$Q_1^5 = \begin{bmatrix} 214.0 & -15,185.0 \\ -15,185.0 & 15,214.0 \end{bmatrix}, \quad Q_2^5 = \begin{bmatrix} 7,593.6 & 7,529.0 \\ 7,529.0 & 1,325.3 \end{bmatrix}, \quad Q_3^5 = \begin{bmatrix} 3,542.1 & 0.0 \\ 0.0 & -3,023.0 \end{bmatrix},$$

$$Q_1^6 = \begin{bmatrix} -40.4375 & 9.9389 \\ 9.9389 & 28.8514 \end{bmatrix}, \quad Q_2^6 = \begin{bmatrix} 54.7453 & -6.5632 \\ -6.5632 & 45.3682 \end{bmatrix}, \quad Q_3^6 = \begin{bmatrix} 53.4114 & -0.2701 \\ -0.2701 & 42.9778 \end{bmatrix},$$

$$N_1^1 = \begin{bmatrix} -123.7682 & 9.3117 \\ 9.3117 & -461.4212 \end{bmatrix}, \quad N_2^1 = \begin{bmatrix} -1,007.0 & -119.6 \\ -119.6 & -1,067.4 \end{bmatrix}, \quad N_3^1 = \begin{bmatrix} 483.1 & -2.9 \\ -2.9 & 1,962.9 \end{bmatrix},$$

$$N_1^2 = \begin{bmatrix} 1,415.9 & -464.4 \\ -464.4 & -534.9 \end{bmatrix}, \quad N_2^2 = \begin{bmatrix} 3,122.6 & 434.1 \\ 434.1 & 64.8 \end{bmatrix}, \quad N_3^2 = \begin{bmatrix} -1,429.4 & 0.1 \\ 0.1 & -1,922.0 \end{bmatrix},$$

$$N_1^3 = \begin{bmatrix} 38.2260 & -6.0579 \\ -6.0579 & -13.3891 \end{bmatrix}, \quad N_2^3 = \begin{bmatrix} -13.4412 & 6.7579 \\ 6.7579 & 9.3172 \end{bmatrix}, \quad N_3^3 = \begin{bmatrix} -9.1322 & -0.0352 \\ -0.0352 & -24.2347 \end{bmatrix},$$

$$N_1^4 = \begin{bmatrix} 31.0135 & -6.3425 \\ -6.3425 & -14.6572 \end{bmatrix}, \quad N_2^4 = \begin{bmatrix} -14.1576 & 6.0573 \\ 6.0573 & 7.3776 \end{bmatrix}, \quad N_3^4 = \begin{bmatrix} -9.6531 & 0.0282 \\ 0.0282 & -23.7517 \end{bmatrix},$$

$$N_1^5 = \begin{bmatrix} 1,234.7 & 657.0 \\ 657.0 & 110.7 \end{bmatrix}, \quad N_2^5 = \begin{bmatrix} -18.2 & 1,241.1 \\ 1,241.1 & -134.2 \end{bmatrix}, \quad N_3^5 = \begin{bmatrix} -604.6748 & 1.6013 \\ 1.6013 & 49.2281 \end{bmatrix},$$

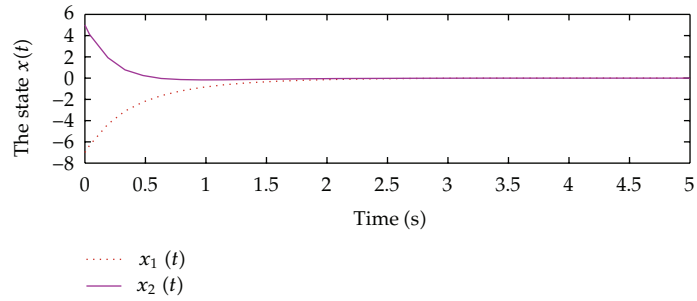


**Figure 1:** State trajectories $x_1(t)$ and $x_2(t)$ of LPD time-delay system (2.1) with (4.1), $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ by using program dde45lin with Matlab.

$$N_1^6 = \begin{bmatrix} 14.9282 & -0.7772 \\ -0.7772 & 33.2457 \end{bmatrix}, \quad N_2^6 = \begin{bmatrix} 36.9436 & 0.4195 \\ 0.4195 & 33.4742 \end{bmatrix}, \quad N_3^6 = \begin{bmatrix} 33.2443 & -0.0821 \\ -0.0821 & 25.6059 \end{bmatrix},$$

$$M_1^1 = \begin{bmatrix} 222.2533 & 104.5561 \\ 104.5561 & 54.1789 \end{bmatrix}, \quad M_2^1 = \begin{bmatrix} 2.1314 & 0.0476 \\ 0.0476 & 3.1486 \end{bmatrix}, \quad M_3^1 = \begin{bmatrix} 2.0463 & -0.0108 \\ -0.0108 & 1.4850 \end{bmatrix},$$

$$M_1^2 = \begin{bmatrix} -11.9251 & -13.1913 \\ -13.1913 & -5.6759 \end{bmatrix}, \quad M_2^2 = \begin{bmatrix} 1.4344 & -2.6047 \\ -2.6047 & 3.3445 \end{bmatrix}, \quad M_3^2 = \begin{bmatrix} 1.0670 & -0.0153 \\ -0.0153 & 0.0826 \end{bmatrix},$$

$$M_1^3 = \begin{bmatrix} 393.2821 & 56.3348 \\ 56.3348 & 295.3780 \end{bmatrix}, \quad M_2^3 = \begin{bmatrix} 269.8280 & -0.1703 \\ -0.1703 & 270.2562 \end{bmatrix}, \quad M_3^3 = \begin{bmatrix} 270.0284 & -0.0192 \\ -0.0192 & 268.5208 \end{bmatrix},$$

$$M_1^4 = \begin{bmatrix} -63.2823 & -34.4679 \\ -34.4679 & -20.2534 \end{bmatrix}, \quad M_2^4 = \begin{bmatrix} -0.8017 & -0.1312 \\ -0.1312 & -0.8706 \end{bmatrix}, \quad M_3^4 = \begin{bmatrix} -0.6584 & -0.0064 \\ -0.0064 & -1.2408 \end{bmatrix},$$

$$M_1^5 = \begin{bmatrix} 208.1212 & -29.9392 \\ -29.9392 & 252.5413 \end{bmatrix}, \quad M_2^5 = \begin{bmatrix} 268.6888 & 0.6640 \\ 0.6640 & 268.0517 \end{bmatrix}, \quad M_3^5 = \begin{bmatrix} 268.9342 & -0.0012 \\ -0.0012 & 268.6650 \end{bmatrix},$$

$$R_1^1 = \begin{bmatrix} 196.3135 & -36.7812 \\ -36.7812 & 248.7288 \end{bmatrix}, \quad R_2^1 = \begin{bmatrix} 268.6863 & 0.6631 \\ 0.6631 & 268.0488 \end{bmatrix}, \quad R_3^1 = \begin{bmatrix} 268.9326 & -0.0013 \\ -0.0013 & 268.6593 \end{bmatrix},$$

$$R_1^2 = \begin{bmatrix} 28.5196 & 7.4817 \\ 7.4817 & 4.9413 \end{bmatrix}, \quad R_2^2 = \begin{bmatrix} 1.4407 & -2.6031 \\ -2.6031 & 3.3546 \end{bmatrix}, \quad R_3^2 = \begin{bmatrix} 1.0720 & -0.0153 \\ -0.0153 & 0.0894 \end{bmatrix},$$

$$R_1^3 = \begin{bmatrix} 158.8158 & -17.7934 \\ -17.7934 & 171.3160 \end{bmatrix}, \quad R_2^3 = \begin{bmatrix} 178.2539 & 4.6824 \\ 4.6824 & 181.6405 \end{bmatrix}, \quad R_3^3 = \begin{bmatrix} 183.6198 & -0.0347 \\ -0.0347 & 184.1092 \end{bmatrix}, \tag{4.2}$$

and $\epsilon_1 = 414.9151$ and $\epsilon_2 = 381.9944$. It is known that the maximum value of $h$ for the stability of this system is $h = 0.6246$. The stability is also assured for $h < 0.6246$. The numerical solution $x_1(t)$ and $x_2(t)$ of (2.1) with (4.1) are plotted in Figure 1.

*Example 4.2.* Consider the following linear systems, which are considered in [17]:

**Table 1:** Upper bounds of time delays in Example 4.2 for various conditions.

| | $\eta = 0, \rho = 0.1$ $h_d = 0.5$ | $\eta = 0, \rho = 0.1$ $h_d \geq 1$ | $\eta = 0.1, \rho = 0.1$ $h_d = 0.5$ | $\eta = 0.1, \rho = 0.1$ $h_d \geq 1$ |
|---|---|---|---|---|
| Cao and Lam [18] (2000) | 0.5467 | — | 0.4950 | — |
| Zuo and Wang [3] (2006) | 1.1424 | — | 1.0097 | — |
| Chen et al. [17] (2008) | 1.1425 | 0.7355 | 1.0097 | 0.7147 |
| Corollary 3.2 | 2.0925 | 0.8412 | 1.8235 | 0.8406 |

**Table 2:** Upper bounds of time delays in Example 4.3 for various conditions.

| | $\eta = 0.05, \rho = 0.1$ | | | |
|---|---|---|---|---|
| | $\beta = 0$ | $\beta = 0.1$ | $\beta = 0.3$ | $\beta = 0.5$ |
| Kwon and Park [19] (2008) | 3.40 | 1.36 | 0.76 | 0.55 |
| Corollary 3.2 | 2.87 | 1.53 | 0.97 | 0.75 |

**Table 3:** Comparison of convergence rate obtained for Corollary 3.2 and from [8, 20] in Example 4.4.

| Method | Year | Convergence rate $\beta$ |
|---|---|---|
| Mondié and Kharitonov [20] | 2005 | 0.470 |
| Nam [8] | 2009 | 1.153 |
| Corollary 3.2 | 2012 | 1.410 |

$$\dot{x}(t) = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix} x(t - h(t)) + f(t, x(t)) + g(t, x(t - h(t))), \tag{4.3}$$

where $\|f(t, x(t))\| \leq \eta \|x(t)\|$, $\|g(t, x(t - h(t)))\| \leq \rho \|x(t - h(t))\|$.

By Corollary 3.2 to the system (4.3), we can obtain the maximum upper bounds of the time delay under different values of $\eta$, $\rho$, and $h_d$ as shown in Table 1. From Table 1, we see that Corollary 3.2 gives larger delay bounds than some of the recent results in literatures.

*Example 4.3.* Consider the following linear systems, which are considered in [19]:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h) + f(t, x(t)) + g(t, x(t - h)), \tag{4.4}$$

where $\|f(t, x(t))\| \leq \eta \|x(t)\|$, $\|g(t, x(t - h(t)))\| \leq \rho \|x(t - h(t))\|$. By using Corollary 3.2 to the system (4.4), we obtain the maximum upper bounds of the time delay for different values of $\eta$, $\rho$, and $h_d$ as shown in Table 2. From Table 2, it can be seen that Corollary 3.2 gives larger delay bounds than the recent results in [19].

*Example 4.4.* Consider the following linear systems, which is considered in [8]:

$$\dot{x}(t) = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0 \\ 4 & 0.1 \end{bmatrix} x(t - h) + f(t, x(t)) + g(t, x(t - h)), \tag{4.5}$$

where $\|f(t,x(t))\| \leq 0.2\|x(t)\|$, $\|g(t,x(t-h))\| \leq 0.2\|x(t-h)\|$. The maximum value of convergence rate is 1.410 by using Corollary 3.2 for system (4.5). From Table 3, we can see that Corollary 3.2 gives larger convergence rate than the results in [8, 20].

## 5. Conclusions

The problem of robust exponential stability for LPD systems with time-varying delays and nonlinear perturbations was studied. Based on the combination of Leibniz-Newton formula and linear matrix inequality, the use of suitable Lyapunov-Krasovskii functional, new delay-dependent exponential stability criteria are formulated in terms of LMIs. Numerical examples have shown significant improvements over some existing results.

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