Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 386253, 15 pages
doi:10.1155/2012/386253

Research Article

Implicit-Relation-Type Cyclic Contractive Mappings and Applications to Integral Equations

Hemant Kumar Nashine,[1] Zoran Kadelburg,[2] and Poom Kumam[3]

[1] *Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud, Chhattisgarh, Raipur 492101, India*
[2] *Faculty of Mathematics, University of Belgrade, Studentski Trg 16, 11000 Beograd, Serbia*
[3] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand*

Correspondence should be addressed to Poom Kumam, poom.kum@kmutt.ac.th

Received 28 July 2012; Accepted 5 September 2012

Academic Editor: Sehie Park

We introduce an implicit-relation-type cyclic contractive condition for a map in a metric space and derive existence and uniqueness results of fixed points for such mappings. Examples are given to support the usability of our results. At the end of the paper, an application to the study of existence and uniqueness of solutions for a class of nonlinear integral equations is presented.

1. Introduction and Preliminaries

It is well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920, which was published in 1922 [1], is one of the most important theorems in classical functional analysis. The Banach contraction principle is a very popular tool which is used to solve existence problems in many branches of mathematical analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions modifying the basic contractive condition or changing the ambient space. This celebrated theorem can be stated as follows.

Theorem 1.1 (see [1]). *Let (X, d) be a complete metric space and let T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \le k d(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where k is a constant in $(0, 1)$. Then, T has a unique fixed point $x^* \in X$.

There is in the literature a great number of generalizations of the Banach contraction principle (see, e.g., [2] and references cited therein).

Inequality (1.1) implies continuity of T. A natural question is whether we can find contractive conditions which will imply existence of a fixed point in a complete metric space but will not imply continuity.

On the other hand, cyclic representations and cyclic contractions were introduced by Kirk et al. [3].

Definition 1.2 (see [3, 4]). Let (X, d) be a metric space. Let p be a positive integer and let $A_1, A_2, \ldots, A_p$ be nonempty subsets of X. Then $Y = \bigcup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to $T : Y \to Y$ if

(i) A_i, $i = 1, 2, \ldots, p$ are nonempty closed sets, and

(ii) $T(A_1) \subseteq A_2, \ldots, T(A_{p-1}) \subseteq A_p$, $T(A_p) \subseteq A_1$.

Kirk et al. [3] proved the following result.

Theorem 1.3 (see [3]). *Let (X, d) be a metric metric space and let $Y = \bigcup_{i=1}^{p} A_i$ be a cyclic representation of Y with respect to $T : Y \to Y$. If*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.2}$$

holds for all $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$ (where $A_{p+1} = A_1$), and $0 \leq k < 1$, then T has a unique fixed point x^* and $x^* \in \bigcap_{i=1}^{p} A_i$.

Notice that, while contractions are always continuous, cyclic contractions might not be.

Following [3], a number of fixed point theorems on cyclic representations of Y with respect to a self-mapping T have appeared (see, e.g., [4–12]).

In this paper, we introduce a new class of cyclic contractive mappings satisfying an implicit relation in the framework of metric spaces and then derive the existence and uniqueness of fixed points for such mappings. Suitable examples are provided to demonstrate the validity of our results. Our main result generalizes and improves many existing theorems in the literature. We also give an application of the presented results in the area of integral equations and prove an existence theorem for solutions of a system of integral equations in the last section.

2. Notation and Definitions

First, we introduce some further notations and definitions that will be used later.

2.1. Implicit Relation and Related Concepts

In recent years, Popa [13] used implicit functions rather than contraction conditions to prove fixed point theorems in metric spaces whose strength lies in its unifying power. Namely, an implicit function can cover several contraction conditions which include known as well as some new conditions. This fact is evident from examples furnished in Popa [13]. Implicit relations on metric spaces have been used in many articles (for details see [14–19] and references cited therein).

In this section, we define a suitable implicit function involving six real nonnegative arguments to prove our results, that was given in [20].

Let $\mathbb{R}_+$ denote the nonnegative real numbers and let $\mathcal{T}$ be the set of all continuous functions $T : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions: T_1: $T(t_1, \dots, t_6)$ is non-increasing in variables $t_2, \dots, t_6$; T_2: there exists a right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that for $u \geq 0$,

$$T(u, v, u, v, 0, u + v) \leq 0 \tag{2.1}$$

or

$$T(u, v, 0, 0, v, v) \leq 0 \tag{2.2}$$

implies $u \leq f(v)$; T_3: $T(u, 0, u, 0, 0, u) > 0$, $T(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

Example 2.1. $T(t_1, \dots, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)[at_5 + bt_6]$, where $0 \leq \alpha < 1$, $0 \leq a < 1/2$, $0 \leq b < 1/2$.

Example 2.2. $T(t_1, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_4, (1/2)(t_5 + t_6)\}$, where $k \in (0, 1)$.

Example 2.3. $T(t_1, \dots, t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, (1/2)(t_5 + t_6)\})$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is right continuous and $\phi(0) = 0$, $\phi(t) < t$ for $t > 0$.

Example 2.4. $T(t_1, \dots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, where $a > 0$, $b, c, d \geq 0$, $a + b + c < 1$ and $a + d < 1$.

We need the following lemma for the proof of our theorems.

Lemma 2.5 (see [21]). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a right continuous function such that $f(t) < t$ for every $t > 0$. Then $\lim_{n \to \infty} f^n(t) = 0$, where f^n denotes the n times repeated composition of f with itself.*

Next, we introduce a new notion of cyclic contractive mapping and establish a new results for such mappings.

Definition 2.6. Let (X, d) be a metric space. Let p be a positive integer, let $A_1, A_2, \dots, A_p$ be nonempty subsets of X, and $Y = \bigcup_{i=1}^{p} A_i$. An operator $\mathcal{F} : Y \to Y$ is called an implicit relation type cyclic contractive mapping if

(∗) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to $\mathcal{F}$;

(∗∗) for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ (with $A_{p+1} = A_1$),

$$T(d(\mathcal{F}x, \mathcal{F}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), d(x, \mathcal{F}y), d(y, \mathcal{F}x)) \leq 0, \tag{2.3}$$

for some $T \in \mathcal{T}$.

Using Example 2.2, we present an example of an implicit relation type cyclic contractive mapping.

Example 2.7. Let $\mathcal{X} = [0,1]$ with the usual metric. Suppose $\mathcal{A}_1 = [0,1/2]$, $\mathcal{A}_2 = [1/2,1]$, and $\mathcal{A}_3 = \mathcal{A}_1$; note that $\mathcal{X} = \bigcup_{i=1}^{2} \mathcal{A}_i$. Define $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ such that

$$\mathcal{F}x = \begin{cases} \dfrac{1}{2}, & x \in [0,1), \\ 0, & x = 1. \end{cases} \tag{2.4}$$

Clearly, $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed subsets of $\mathcal{X}$. Moreover, $\mathcal{F}(\mathcal{A}_i) \subset \mathcal{A}_{i+1}$ for $i = 1,2$, so that $\bigcup_{i=1}^{2} \mathcal{A}_i$ is a cyclic representation of $\mathcal{X}$ with respect to $\mathcal{F}$. Furthermore, if $T : \mathbb{R}^{+^6} \to \mathbb{R}^+$ is given by

$$T(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{3}{4}\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}, \tag{2.5}$$

then $T \in \mathcal{T}$. We will show that implicit relation type cyclic contractive conditions are verified. We will distinguish the following cases:

(1) $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$.

(i) When $x \in [0,1/2]$ and $y \in [1/2,1)$, we deduce $d(\mathcal{F}x,\mathcal{F}y) = 0$ and inequality (2.3) is trivially satisfied.

(ii) When $x \in [0,1/2]$ and $y = 1$, we deduce $d(\mathcal{F}x,\mathcal{F}y) = 1/2$ and

$$t_2 = |x-1|, \qquad t_3 = \left|x - \frac{1}{2}\right|, \qquad t_4 = 1, \qquad t_5 = x, \qquad t_6 = \frac{1}{2}, \tag{2.6}$$

then $T(t_1,t_2,t_3,t_4,t_5,t_6) = 1/2 - 3/4$. Inequality (2.3) holds as it reduces to $1/2 < 3/4$.

(2) $x \in \mathcal{A}_2$, $y \in \mathcal{A}_1$.

(i) When $x \in [1/2,1)$ and $y \in [0,1/2]$, we deduce $d(\mathcal{F}x,\mathcal{F}y) = 0$ and inequality (2.3) is trivially satisfied.

(ii) When $x \in 1$ and $y = [0,1/2]$, we deduce $d(\mathcal{F}x,\mathcal{F}y) = 1/2$ and

$$t_2 = |1-y|, \qquad t_3 = 1, \qquad t_4 = \left|y - \frac{1}{2}\right|, \qquad t_5 = \frac{1}{2}, \qquad t_6 = y. \tag{2.7}$$

Then $T(t_1,t_2,t_3,t_4,t_5,t_6) = 1/2 - 3/4$. Inequality (2.3) holds as it reduces to $1/2 < 3/4$.

Hence, $\mathcal{F}$ is an implicit relation type cyclic contractive mapping.

3. Main Result

Our main result is the following.

Theorem 3.1. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X, and $Y = \bigcup_{i=1}^{p} A_i$. Suppose $\mathcal{F} : Y \to Y$ is an implicit relation type cyclic contractive mapping, for some $T \in \mathcal{T}$. Then $\mathcal{F}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{F}$ belongs to $\bigcap_{i=1}^{p} A_i$.*

Proof. Let $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$). Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = \mathcal{F}x_n, \quad n = 0,1,2,\ldots. \tag{3.1}$$

We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.2}$$

If for some k, we have $x_{k+1} = x_k$, then (3.2) follows immediately. So, we can suppose that $d(x_n, x_{n+1}) > 0$ for all n. From the condition $(*)$, we observe that for all n, there exists $i = i(n) \in \{1,2,\ldots,p\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$. Then, from the condition $(**)$, we have

$$T(d(\mathcal{F}x_n, \mathcal{F}x_{n-1}), d(x_n, x_{n-1}), d(x_n, \mathcal{F}x_n), d(x_{n-1}, \mathcal{F}x_{n-1}), d(x_n, \mathcal{F}x_{n-1}), d(x_{n-1}, \mathcal{F}x_n)) \leq 0 \tag{3.3}$$

and so

$$T(d(x_{n+1}, x_n), d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_{n+1})) \leq 0. \tag{3.4}$$

Now using T_1, we have

$$T(d(x_{n+1}, x_n), d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq 0 \tag{3.5}$$

and from T_2, there exists a right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$, for $t > 0$, such that for all $n \in \{1,2,\ldots\}$,

$$d(x_{n+1}, x_n) \leq f(d(x_n, x_{n-1})). \tag{3.6}$$

If we continue this procedure, we can have

$$d(x_{n+1}, x_n) \leq f^n(d(x_1, x_0)) \tag{3.7}$$

and so from Lemma 2.5,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.8}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Suppose it is not true. Then we can find a $\delta > 0$ and two sequences of integers $\{m(k)\}$, $\{n(k)\}$, $n(k) > m(k) \geq k$ with

$$r_k = d\left(x_{m(k)}, x_{n(k)}\right) \geq \delta \quad \text{for } k \in \{1,2,\ldots\}. \tag{3.9}$$

We may also assume

$$d(x_{m(k)}, x_{n(k)-1}) < \delta \tag{3.10}$$

by choosing $n(k)$ to be the smallest number exceeding $m(k)$ for which (3.9) holds. Now (3.7), (3.9), and (3.10) imply

$$\begin{aligned} \delta \le r_k &\le d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &< \delta + f^{n(k)-1}(d(x_0, x_1)) \end{aligned} \tag{3.11}$$

and so

$$\lim_{k \to \infty} r_k = \delta. \tag{3.12}$$

On the other hand, for all k, there exists $j(k) \in \{1, \ldots, p\}$ such that $n(k) - m(k) + j(k) \equiv 1[p]$. Then $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labelled sets A_i and A_{i+1} for certain $i \in \{1, \ldots, p\}$. Using the triangle inequality, we get

$$\begin{aligned} &\left| d(x_{m(k)-j(k)}, x_{n(k)}) - d(x_{n(k)}, x_{m(k)}) \right| \\ &\quad \le d(x_{m(k)-j(k)}, x_{m(k)}) \\ &\quad \le \sum_{l=0}^{j(k)-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \\ &\quad \le \sum_{l=0}^{p-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \text{ (from (3.2))}, \end{aligned} \tag{3.13}$$

which, by (3.12), implies that

$$\lim_{k \to \infty} d(x_{m(k)-j(k)}, x_{n(k)}) = \delta. \tag{3.14}$$

Using (3.2), we have

$$\lim_{k \to \infty} d(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}) = 0, \tag{3.15}$$

$$\lim_{k \to \infty} d(x_{n(k)+1}, x_{n(k)}) = 0. \tag{3.16}$$

Again, using the triangle inequality, we get

$$\left| d(x_{m(k)-j(k)}, x_{n(k)+1}) - d(x_{m(k)-j(k)}, x_{n(k)}) \right| \le d(x_{n(k)}, x_{n(k)+1}). \tag{3.17}$$

Passing to the limit as $k \to \infty$ in the above inequality and using (3.16) and (3.14), we get

$$\lim_{k\to\infty} d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right) = \delta. \tag{3.18}$$

Similarly, we have

$$\left|d\left(x_{n(k)}, x_{m(k)-j(k)+1}\right) - d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right| \leq d\left(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}\right). \tag{3.19}$$

Passing to the limit as $k \to \infty$ and using (3.2) and (3.14), we obtain

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)-j(k)+1}\right) = \delta. \tag{3.20}$$

Similarly, we have

$$\lim_{k\to\infty} d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right) = \delta. \tag{3.21}$$

Using the condition (2.3) for $x = x_{m(k)-j(k)}$ and $y = x_{n(k)}$, we have

$$\begin{aligned} T\big(&d\left(\digamma x_{m(k)-j(k)}, \digamma x_{n(k)}\right), d\left(x_{m(k)-j(k)}, x_{n(k)}\right), d\left(x_{m(k)-j(k)}, \digamma x_{m(k)-j(k)}\right), \\ &d\left(x_{n(k)}, \digamma x_{n(k)}\right), d\left(x_{m(k)-j(k)}, \digamma x_{n(k)}\right), d\left(x_{n(k)}, \digamma x_{m(k)-j(k)}\right)\big) \leq 0 \end{aligned} \tag{3.22}$$

and so

$$\begin{aligned} T\big(&d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right), d\left(x_{m(k)-j(k)}, x_{n(k)}\right), d\left(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}\right), \\ &d\left(x_{n(k)}, x_{n(k)+1}\right), d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right), d\left(x_{n(k)}, x_{m(k)-j(k)+1}\right)\big) \leq 0. \end{aligned} \tag{3.23}$$

Now letting $k \to \infty$ and using (3.12), (3.14), and (3.18)–(3.21), we have, by continuity of T, that

$$T(\delta, \delta, 0, 0, \delta, \delta) \leq 0, \tag{3.24}$$

a contradiction with T_3 since we have supposed that $\delta > 0$. Thus, $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{3.25}$$

We will prove that

$$x^* \in \bigcap_{i=1}^{p} A_i. \tag{3.26}$$

From condition $(*)$, and since $x_0 \in A_1$, we have $\{x_{np}\}_{n\geq 0} \subseteq A_1$. Since A_1 is closed, from (3.25), we get that $x^* \in A_1$. Again, from the condition $(*)$, we have $\{x_{np+1}\}_{n\geq 0} \subseteq A_2$. Since A_2 is closed, from (3.25), we get that $x^* \in A_2$. Continuing this process, we obtain (3.26).

Now, we will prove that x^* is a fixed point of T. Indeed, from (3.26), for all n, there exists $i(n) \in \{1, 2, \ldots, p\}$ such that $x_n \in A_{i(n)}$. Applying $(**)$ with $x = x^*$ and $y = x_n$, we obtain

$$T(d(\digamma x^*, \digamma x_n), d(x^*, x_n), d(x^*, \digamma x^*), d(x_n, \digamma x_n), d(x^*, \digamma x_n), d(x_n, \digamma x^*)) \leq 0 \tag{3.27}$$

and so letting $n \to \infty$ from the last inequality, we also have

$$T(d(\digamma x^*, x^*), 0, d(x^*, \digamma x^*), 0, 0, d(x^*, \digamma x^*)) \leq 0, \tag{3.28}$$

which is a contradiction to T_3. Thus, $d(x^*, \digamma x^*) = 0$ and so $x^* = \digamma x^*$; that is, x^* is a fixed point of T.

Finally, we prove that x^* is the unique fixed point of $\digamma$. Assume that y^* is another fixed point of $\digamma$, that is, $\digamma y^* = y^*$. By the condition $(*)$, this implies that $y^* \in \bigcap_{i=1}^{p} A_i$. Then we can apply $(**)$ for $x = x^*$ and $y = y^*$. Hence, we obtain

$$T(d(\digamma x^*, \digamma y^*), d(x^*, y^*), d(x^*, \digamma x^*), d(y^*, \digamma y^*), d(x^*, \digamma y^*), d(y^*, \digamma x^*)) \leq 0. \tag{3.29}$$

Since x^* and y^* are fixed points of $\digamma$, we can show easily that $x^* \neq y^*$. If $d(x^*, y^*) > 0$, we get

$$T(d(x^*, y^*), d(x^*, y^*), 0, 0, d(x^*, y^*), d(y^*, x^*)) \leq 0, \tag{3.30}$$

which is a contradiction to T_3. Then we have $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Thus, we have proved the uniqueness of the fixed point. □

In what follows, we deduce some fixed point theorems from our main result given by Theorem 3.1.

If we take $p = 1$ and $A_1 = X$ in Theorem 3.1, then we get immediately the following fixed point theorem.

Corollary 3.2. *Let (X, d) be a complete metric space and let $\digamma : X \to X$ satisfy the following condition: there exists $T \in \mathcal{T}$ such that*

$$T(d(\digamma x, \digamma y), d(x, y), d(x, \digamma x), d(y, \digamma y), d(x, \digamma y), d(y, \digamma x)) \leq 0, \tag{3.31}$$

for all $x, y \in X$. Then $\digamma$ has a unique fixed point.

Corollary 3.3. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$, and $\mathcal{F} : Y \to Y$. Suppose that there exists $T \in \mathcal{T}$ such that*

$(*)'$ $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to $\mathcal{F}$;

$(**)'$ for any $(x,y) \in A_i \times A_{i+1}$, $i = 1,2,\ldots,p$(with $A_{p+1} = A_1$),

$$d(\mathcal{F}x,\mathcal{F}y) \le k \max\left\{ d(x,y), d(x,\mathcal{F}x), d(y,\mathcal{F}y), \frac{d(x,\mathcal{F}y)+d(y,\mathcal{F}x)}{2} \right\}, \tag{3.32}$$

where $k \in (0,1)$. Then $\mathcal{F}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{F}$ belongs to $\bigcap_{i=1}^{p} A_i$.

Remark 3.4. Corollary 3.3 is an extension to Theorem 2.1 in [3, 4].

Corollary 3.5. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$, and $\mathcal{F} : Y \to Y$. Suppose that there exists $T \in \mathcal{T}$ such that*

$(*)'$ $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to $\mathcal{F}$;

$(**)'$ for any $(x,y) \in A_i \times A_{i+1}$, $i = 1,2,\ldots,p$(with $A_{p+1} = A_1$),

$$d(\mathcal{F}x,\mathcal{F}y) \le \phi\left(\max\left\{ d(x,y), d(x,\mathcal{F}x), d(y,\mathcal{F}y), \frac{d(x,\mathcal{F}y)+d(y,\mathcal{F}x)}{2} \right\}\right), \tag{3.33}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is right continuous and $\phi(0) = 0, \phi(t) < t$ for $t > 0$. Then $\mathcal{F}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{F}$ belongs to $\bigcap_{i=1}^{p} A_i$.

Remark 3.6. Taking in Corollary 3.5, $\phi(t) = (1-k)t$ with $k \in (0,1)$, we obtain a generalized version of Theorem 3 in [3, 8].

Corollary 3.7. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$, and $\mathcal{F} : Y \to Y$. Suppose that there exists $T \in \mathcal{T}$ such that*

$(*)'$ $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to $\mathcal{F}$;

$(**)'$ for any $(x,y) \in A_i \times A_{i+1}$, $i = 1,2,\ldots,p$(with $A_{p+1} = A_1$),

$$d(\mathcal{F}x,\mathcal{F}y) \le \alpha \max\{d(x,y), d(x,\mathcal{F}x), d(y,\mathcal{F}y)\} + (1-\alpha)\left[ad(x,\mathcal{F}y) + bd(y,\mathcal{F}x)\right], \tag{3.34}$$

where $0 \le \alpha < 1$, $0 \le a < 1/2$, $0 \le b < 1/2$.

Then $\mathcal{F}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{F}$ belongs to $\bigcap_{i=1}^{p} A_i$.

The following example demonstrates the validity of Theorem 3.1.

Example 3.8. Let $\mathcal{X} = \mathbb{R}$ with the usual metric. Suppose $\mathcal{A}_1 = [-2,0] = \mathcal{A}_3$, $\mathcal{A}_2 = [0,2] = \mathcal{A}_4$, and $\mathcal{Y} = \bigcup_{i=1}^{4} \mathcal{A}_i$. Define $\mathcal{F} : \mathcal{Y} \to \mathcal{Y}$ by $\mathcal{F}x = -x/6$, for all $x \in \mathcal{Y}$. Clearly, $\mathcal{A}_i (i = 1,2,3,4)$ are closed subsets of $\mathcal{X}$. Moreover, $\mathcal{F}(\mathcal{A}_i) \subset \mathcal{A}_{i+1}$ for $i = 1,2,3,4$ so that $\bigcup_{i=1}^{4} \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{F}$. Moreover, mapping $\mathcal{F}$ is implicit relation type cyclic contractive, with $T : \mathbb{R}^{+^6} \to \mathbb{R}^+$ defined by

$$T(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{1}{2} \max\left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\}. \tag{3.35}$$

Indeed, to see this fact we examine the following cases.

Inequality (2.3) reduces to

$$d(\mathcal{F}x, \mathcal{F}y) = \frac{|x - y|}{6} \leq \frac{1}{2} \max\left\{ |x - y|, \frac{5|x|}{6}, \frac{5|y|}{6}, \frac{|x + y/6| + |y + x/6|}{2} \right\}. \tag{3.36}$$

(I) For $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$:

(i) suppose $x = -1$ and $y = 0$. Then inequality (2.3) holds as it reduces to $1/6 < 7/18$;

(ii) suppose $x = 0$ and $y = 1$. Then inequality (2.3) holds as it reduces to $1/6 < 7/18$;

(iii) suppose $x = -1$ and $y = 1$. Then inequality (2.3) holds as it reduces to $1/3 < 2/3$;

(iv) suppose $x = -2$ and $y = 1$. Then inequality (2.3) holds as it reduces to $1/2 < 1$;

(v) suppose $x = -2$ and $y = 2$. Then inequality (2.3) holds as it reduces to $2/3 \leq 2/3$.

(II) For $x \in \mathcal{A}_2$, $y \in \mathcal{A}_1$:

(i) suppose $x = 1/2$ and $y = -1/2$. Then inequality (2.3) holds as it reduces to $1/6 < 1/3$;

(ii) suppose $x = 2$ and $y = -1$. Then inequality (2.3) holds as it reduces to $1/2 < 1$;

(iii) suppose $x = 1$ and $y = -1$. Then inequality (2.3) holds as it reduces to $1/3 < 2/3$.

(III) For $x = y$, $d(\mathcal{F}x, \mathcal{F}y) = 0$, inequality (2.3) trivially holds.

Similarly other cases can be verified. Hence, $\mathcal{F}$ is an implicit relation type cyclic contractive mapping. Therefore, all conditions of Theorem 3.1 are satisfied and so $\mathcal{F}$ has a fixed point (which is $z = 0 \in \bigcap_{i=1}^{4} \mathcal{A}_i$).

We illustrate Theorem 3.1 by another example which is obtained by modifying the one from [22].

Example 3.9. Let $\mathcal{X} = \mathbb{R}^{+^3}$ and we define $d : \mathcal{X} \times \mathcal{X} \to [0, 1)$ by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|, \quad \text{for } x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathcal{X}, \tag{3.37}$$

and let $\mathcal{A} = \{(x, 0, 0) : x \in \mathbb{R}^+\}$, $\mathcal{B} = \{(0, y, 0) : y \in \mathbb{R}^+\}$, and $\mathcal{C} = \{(0, 0, z) : z \in \mathbb{R}^+\}$ be three subsets of $\mathcal{X}$.

Define $\mathcal{F} : \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \to \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ by

$$\begin{aligned}
\mathcal{F}((x,0,0)) &= \left(0, \frac{1}{6}x, 0\right); \quad \forall x \in \mathbb{R}^+; \\
\mathcal{F}((0,y,0)) &= \left(0, 0, \frac{1}{6}y\right); \quad \forall y \in \mathbb{R}^+; \\
\mathcal{F}((0,0,z)) &= \left(\frac{1}{6}z, 0, 0\right); \quad \forall x \in \mathbb{R}^+.
\end{aligned} \tag{3.38}$$

Let the function $T : \mathbb{R}^{+^6} \to \mathbb{R}^+$ be defined by

$$T(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi\left(\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}\right), \tag{3.39}$$

where $t_1 = d(\mathcal{F}x, \mathcal{F}y)$, $t_2 = d(x,y)$, $t_3 = d(x, \mathcal{F}x)$, $t_4 = d(y, \mathcal{F}y)$, $t_5 = d(x, \mathcal{F}y)$, and $t_6 = d(y, \mathcal{F}x)$, for all $x, y \in \mathcal{X}$. Then $\mathcal{F}$ is an implicit type cyclic contractive mapping for $\phi(t) = (1/4)t$ for $t \geq 0$. Therefore, all conditions of Theorem 3.1 are satisfied and so $\mathcal{F}$ has a fixed point (which is $(0,0,0) \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$).

4. An Application to Integral Equations

In this section, we apply Theorem 3.1 to study the existence and uniqueness of solutions to a class of nonlinear integral equations.

We consider the following nonlinear integral equation,

$$u(t) = \int_0^{\mathbf{T}} G(t,s) f(s, u(s)) ds, \quad \forall t \in [0, \mathbf{T}], \tag{4.1}$$

where $\mathbf{T} > 0$, $f : [0, \mathbf{T}] \times \mathbb{R} \to \mathbb{R}$ and $G : [0, \mathbf{T}] \times [0, \mathbf{T}] \to [0, \infty)$ are continuous functions.

Let $\mathcal{X} = C([0, \mathbf{T}])$ be the set of real continuous functions on $[0, \mathbf{T}]$. We endow $\mathcal{X}$ with the standard metric

$$d_\infty(u, v) = \max_{t \in [0, \mathbf{T}]} |u(t) - v(t)|, \quad \forall u, v \in \mathcal{X}. \tag{4.2}$$

It is well known that $(\mathcal{X}, d_\infty)$ is a complete metric space. Define the mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{F}u(t) = \int_0^{\mathbf{T}} G(t,s) f(s, u(s)) ds, \quad \forall t \in [0, \mathbf{T}]. \tag{4.3}$$

Let $(\alpha, \beta) \in \mathcal{X}^2$, $(\alpha_0, \beta_0) \in \mathbb{R}^2$ such that

$$\alpha_0 \leq \alpha \leq \beta \leq \beta_0. \tag{4.4}$$

We suppose that for all $t \in [0, \mathbf{T}]$, we have

$$\alpha(t) \le \int_0^{\mathbf{T}} G(t,s) f(s, \beta(s)) ds, \tag{4.5}$$

$$\beta(t) \ge \int_0^{\mathbf{T}} G(t,s) f(s, \alpha(s)) ds. \tag{4.6}$$

We suppose that for all $s \in [0, \mathbf{T}]$, $f(s, \cdot)$ is a decreasing function, that is,

$$x, y \in \mathbb{R}, x \ge y \Longrightarrow f(s,x) \le f(s,y). \tag{4.7}$$

We suppose that

$$\sup_{t \in [0,\mathbf{T}]} \int_0^{\mathbf{T}} G(t,s) \; ds \le 1. \tag{4.8}$$

Finally, we suppose that for all $s \in [0,1]$, for all $x, y \in \mathbb{R}$ with $x \le \beta_0$ and $y \ge \alpha_0$ or $x \ge \alpha_0$ and $y \le \beta_0$,

$$\left| f(s,x) - f(s,y) \right| \le k \max\{d(x,y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), d(x, \mathcal{F}y), d(y, \mathcal{F}x)\}, \tag{4.9}$$

where $k \in (0,1)$.

Now, define the set

$$\mathcal{C} = \{u \in C([0,\mathbf{T}]) : \alpha \le u \le \beta\}. \tag{4.10}$$

We have the following result.

Theorem 4.1. *Under the assumptions* (4.4)–(4.9), *Problem* (4.1) *has one and only one solution* $u^* \in \mathcal{C}$.

Proof. Define the closed subsets of $\mathcal{X}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ by

$$\begin{aligned} \mathcal{A}_1 &= \{u \in \mathcal{X} : u \le \beta\}, \\ \mathcal{A}_2 &= \{u \in \mathcal{X} : u \ge \alpha\}. \end{aligned} \tag{4.11}$$

We will prove that

$$\mathcal{F}(\mathcal{A}_1) \subseteq \mathcal{A}_2, \qquad \mathcal{F}(A_2) \subseteq A_1. \tag{4.12}$$

Let $u \in \mathcal{A}_1$, that is,

$$u(s) \le \beta(s), \quad \forall s \in [0, \mathbf{T}]. \tag{4.13}$$

Using condition (4.7), since $G(t,s) \geq 0$ for all $t,s \in [0,\mathbf{T}]$, we obtain that

$$G(t,s)f(s,u(s)) \geq G(t,s)f\left(s,\beta(s)\right), \quad \forall t,s \in [0,\mathbf{T}]. \tag{4.14}$$

The above inequality with condition (4.5) imply that

$$\int_0^{\mathbf{T}} G(t,s)f(s,u(s))ds \geq \int_0^{\mathbf{T}} G(t,s)f\left(s,\beta(s)\right)ds \geq \alpha(t), \tag{4.15}$$

for all $t \in [0,\mathbf{T}]$. Then we have $\mathcal{F}u \in \mathcal{A}_2$.

Similarly, let $u \in \mathcal{A}_2$, that is,

$$u(s) \geq \alpha(s), \quad \forall s \in [0,\mathbf{T}]. \tag{4.16}$$

Using condition (4.7), since $G(t,s) \geq 0$ for all $t,s \in [0,\mathbf{T}]$, we obtain that

$$G(t,s)f(s,u(s)) \leq G(t,s)f(s,\alpha(s)), \quad \forall t,s \in [0,\mathbf{T}]. \tag{4.17}$$

The above inequality with condition (4.6) imply that

$$\int_0^{\mathbf{T}} G(t,s)f(s,u(s))ds \leq \int_0^{\mathbf{T}} G(t,s)f(s,\alpha(s))ds \leq \beta(t), \tag{4.18}$$

for all $t \in [0,\mathbf{T}]$. Then we have $\mathcal{F}u \in \mathcal{A}_1$. Finally, we deduce that (4.12) holds.

Now, let $(u,v) \in \mathcal{A}_1 \times \mathcal{A}_2$, that is, for all $t \in [0,\mathbf{T}]$,

$$u(t) \leq \beta(t), \qquad v(t) \geq \alpha(t). \tag{4.19}$$

This implies from condition (4.4) that for all $t \in [0,\mathbf{T}]$,

$$u(t) \leq \beta_0, \qquad v(t) \geq \alpha_0. \tag{4.20}$$

Now, using conditions (4.8) and (4.9), we can write that for all $t \in [0, \mathbf{T}]$, we have

$$\begin{aligned}|\mathcal{F}u - \mathcal{F}v|(t) &\leq \int_0^{\mathbf{T}} G(t,s)\left|f(s,u(s)) - f(s,v(s))\right|ds \\ &\leq \int_0^{\mathbf{T}} G(t,s)k\max\{|u(s)-v(s)|,|u(s)-\mathcal{F}u(s)|, \\ &\qquad |v(s)-\mathcal{F}v(s)|,|u(s)-\mathcal{F}v(s)|,|v(s)-\mathcal{F}u(s)|\}ds \\ &\leq k\max\{d_\infty(u,v),d_\infty(u,\mathcal{F}u),d_\infty(v,\mathcal{F}v),d_\infty(u,\mathcal{F}v),d_\infty(v,\mathcal{F}u)\}\int_0^{\mathbf{T}} G(t,s)ds \\ &\leq k\max\{d_\infty(u,v),d_\infty(u,\mathcal{F}u),d_\infty(v,\mathcal{F}v),d_\infty(u,\mathcal{F}v),d_\infty(v,\mathcal{F}u)\}.\end{aligned} \tag{4.21}$$

This implies that

$$d_\infty(\mathcal{F}u,\mathcal{F}v) \leq k\max\{d_\infty(u,v),d_\infty(u,\mathcal{F}u),d_\infty(v,\mathcal{F}v),d_\infty(u,\mathcal{F}v),d_\infty(v,\mathcal{F}u)\}. \tag{4.22}$$

Using the same technique, we can show that the above inequality holds also if we take $(u, v) \in \mathcal{A}_2 \times \mathcal{A}_1$.

Now, all the conditions of Corollary 3.3 are satisfied (with $p = 2$) and we deduce that $\mathcal{F}$ has a unique fixed point $u^* \in \mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{C}$; that is, $u^* \in \mathcal{C}$ is the unique solution to (4.1). □

Acknowledgments

This work was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC Grant no. 55000613). Moreover, the second author is grateful to the Ministry of Science and Technological Development of Serbia and the third author gratefully acknowledges the support provided visitor project by the Department of Mathematic and Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT) during his stay at the Department of Mathematical and Statistical Sciences, University of Alberta, Canada' as a visitor for the short-term research.

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