Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 386359, 17 pages
doi:10.1155/2012/386359

Research Article

The Zeros of Orthogonal Polynomials for Jacobi-Exponential Weights

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Received 13 July 2012; Revised 14 October 2012; Accepted 19 October 2012

Academic Editor: Patricia J. Y. Wong

This paper gives the estimates of the zeros of orthogonal polynomials for Jacobi-exponential weights.

1. Introduction and Results

This paper deals with the zeros of orthogonal polynomials for Jacobi-exponential weights. Let w be a weight in $\mathbf{I} := (a,b)$, $-\infty \le a < 0 < b \le \infty$, for which the moment problem possesses a unique solution. Denote by $\mathbf{N}$ the set of positive integers. $\mathbf{P}_n$ stands for the set of polynomials of degree at most n.

Assume that $W = e^{-Q}$ where $Q : \mathbf{I} \to [0,\infty)$ is continuous. Also, let $0 < p < \infty$,

$$\begin{gathered} a \le t_r < t_{r-1} < \cdots < t_2 < t_1 \le b, \\ p_i > \frac{-1}{p}, \quad i = 1, 2, \ldots, r, \\ U(x) = \prod_{i=1}^{r} |x - t_i|^{p_i}. \end{gathered} \tag{1.1}$$

The letters $c, C_0, C_1, \ldots$ stand for positive constants independent of variables and indices, unless otherwise indicated, and their values may be different at different occurrences, even in subsequent formulas. Moreover, $C_n \sim D_n$ means that there are two constants c_1 and

c_2 such that $c_1 \leq C_n/D_n \leq c_2$ for the relevant range of n. We write $c = c(\lambda)$ or $c \neq c(\lambda)$ to indicate dependence on or independence of a parameter λ.

Definition 1.1 (see [1, Definition 1.7, page 14]). Given $c, t \geq 0$ and a nonnegative Borel measure ν with compact support in $\mathbf{C}$ and total mass $\leq t$, one says that

$$P(z) := c\ \exp\left(\int \ln|z-t|d\nu(t)\right) \tag{1.2}$$

is an exponential of a potential of mass $\leq t$. One denotes the set of all such P by $\mathcal{P}_t$.

One notes that, for $P \in \mathbf{P}_n$,

$$|P| \in \mathcal{P}_t, \quad t \geq n. \tag{1.3}$$

Definition 1.2 (see [1, page 19]). Let w be a weight in $\mathbf{I}$. For $0 < p < \infty$, generalized Christoffel functions with respect to w for $z \in \mathbf{C}$ are defined by

$$\lambda_{p,n}(w;z) = \inf_{P \in \mathbf{P}_n} \left(\frac{\|Pw\|_{L_p(\mathbf{I})}}{|P(z)|}\right)^p. \tag{1.4}$$

For $p = \infty$, generalized Christoffel functions with respect to w for $z \in \mathbf{C}$ are defined by

$$\lambda_{\infty,n}(w;z) = \inf_{P \in \mathbf{P}_n} \frac{\|Pw\|_{L_\infty(\mathbf{I})}}{|P(z)|}. \tag{1.5}$$

Obviously, for the classical Christoffel function $\lambda_n(w^2;x)$ with respect to w^2, we have

$$\lambda_n\left(w^2;x\right) = \lambda_{2,n-1}(w;x). \tag{1.6}$$

A function $f : (c,d) \rightarrow (0,\infty)$ is said to be *quasi-increasing* (or *quasi-decreasing*) if there exists $C > 0$ such that

$$f(x) \leq (\text{or } \geq) C f(y), \quad c < x \leq y < d. \tag{1.7}$$

Definition 1.3 (see [1, pages 10–12]). Let $a < 0 < b$. Assume that $W = e^{-Q}$ where $Q : \mathbf{I} \rightarrow [0,\infty)$ satisfies the following properties

(a) $Q' \in C(\mathbf{I})$ and $Q(0) = 0$.

(b) Q' is nondecreasing in $\mathbf{I}$.

(c) We have

$$\lim_{t \rightarrow a+} Q(t) = \lim_{t \rightarrow b-} Q(t) = \infty. \tag{1.8}$$

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, \quad t \neq 0, \tag{1.9}$$

is quasi-decreasing in $(a, 0)$ and quasi-increasing in $(0, b)$, respectively. Moreover

$$T(t) \geq \Lambda > 1, \quad t \in \mathbf{I} \setminus \{0\}. \tag{1.10}$$

(e) There exists $\epsilon_0 \in (0, 1)$ such that, for $y \in \mathbf{I} \setminus \{0\}$,

$$T(y) \sim T\left(y\left[1 - \frac{\epsilon_0}{T(y)}\right]\right). \tag{1.11}$$

Then we write $W \in \mathcal{F}$.

(f) In addition, assume that there exist $C, \epsilon_1 > 0$ such that, for all $x \in \mathbf{I} \setminus \{0\}$,

$$\int_{x-\epsilon_1|x|/T(x)}^{x} \frac{|Q'(t) - Q'(x)|}{|t-x|^{3/2}} dt \leq C|Q'(x)| \left[\frac{T(x)}{|x|}\right]^{1/2}. \tag{1.12}$$

Then we write $W \in \mathcal{F}(\mathrm{Lip}(1/2))$.

For $W \in \mathcal{F}$ and $t > 0$, the Mhaskar-Rahmanov-Saff numbers $a_{-t} := a_{-t}(Q) < 0 < a_t := a_t(Q)$ are defined by the equations

$$\begin{aligned} t &= \frac{1}{\pi} \int_{a_{-t}}^{a_t} \frac{xQ'(x)}{[(x-a_{-t})(a_t - x)]^{1/2}} dx, \\ 0 &= \frac{1}{\pi} \int_{a_{-t}}^{a_t} \frac{Q'(x)}{[(x-a_{-t})(a_t - x)]^{1/2}} dx. \end{aligned} \tag{1.13}$$

Put for $t > 0$,

$$\Delta_t := \Delta_t(Q) := [a_{-t}, a_t],$$

$$\delta_t := \delta_t(Q) := \frac{1}{2}(a_t + |a_{-t}|), \qquad \eta_{\pm t} := \eta_{\pm t}(Q) := \left[tT(a_{\pm t})\sqrt{\frac{|a_{\pm t}|}{\delta_t}} \right]^{-2/3},$$

$$\varphi_t(x) := \varphi_t(Q;x) := \begin{cases} \dfrac{|x-a_{-2t}||x-a_{2t}|}{t\sqrt{[|x-a_{-t}|+|a_{-t}|\eta_{-t}][|x-a_t|+a_t\eta_t]}}, & x \in [a_{-t}, a_t], \\ \varphi_t(a_t), & x \in (a_t, b), \\ \varphi_t(a_{-t}), & x \in (a, a_{-t}), \end{cases} \tag{1.14}$$

$$\mathbf{J}_{L,t} := \mathbf{J}_{L,t}(Q) := [a_{-t}(1+L\eta_{-t}), a_t(1+L\eta_t)], \quad L > 0,$$

$$\mathbf{K}_{L,t} := \mathbf{K}_{L,t}(Q) := [-1+L(1+a_{-t}), 1-L(1-a_t)], \quad L > 1.$$

Let

$$U_t(x) := \prod_{i=1}^{r}\left(|x-t_i| + \frac{\delta_t}{t}\right)^{p_i}, \qquad \rho := \rho(U) := \sum_{i=1}^{r}\max\{p_i, 0\}. \tag{1.15}$$

In 1994 and 2001, Levin and Lubinsky [1, 2] published their monographs on orthogonal polynomials for exponential weights W^2. Then they [3, 4] discussed orthogonal polynomials for exponential weights $x^{2\alpha}W(x)^2, \alpha > -1/2$, in $[0, b)$, since the results of [1, 2] cannot be applied to such weights. Kasuga and Sakai [5] considered generalized Freud weights $|x|^{2\alpha}W(x)^2$ in $(-\infty, \infty)$. Recently the second author [6] obtained the L_p Christoffel functions for Jacobi-exponential weights UW, which are the combination of the two best important weights: Jacobi weight and the exponential weight, and restricted range inequalities.

Theorem 1.4 (see [6, Theorem 1.1]). *Let $W \in \mathcal{F}(\mathrm{Lip}\,(1/2))$, $L > 0$, and $0 < p < \infty$. Assume that*

$$\lim_{t\to\infty}\frac{|a_{-t}|}{a_t} = \gamma, \quad 0 < \gamma < \infty. \tag{1.16}$$

Then there exists $n_0 > 0$ such that, for $n \geq n_0$ and $x \in \mathbf{J}_{L,n}$, the relation

$$\lambda_{p,n}(UW;x) \sim \varphi_n(x)U_n(x)^p W(x)^p \tag{1.17}$$

uniformly holds.

Theorem 1.5 (see [6, Theorem 1.2]). *Let $W = e^{-Q(x)}$, where $Q : \mathbf{I} \to [0, \infty)$ is convex with $Q(a+) = Q(b-) = \infty$ and $Q(x) > Q(0) = 0$, $x \in \mathbf{I} \setminus \{0\}$. Let $0 < p \leq \infty$. Assume that relation* (1.16) *is valid. Then there exist $C, t_0 > 0$ such that, for $t \geq t_0$ and $P \in \mathcal{P}_{t-\rho-2/p}$,*

$$\|PUW\|_{L_p(\mathbf{I})} \leq C\|PUW\|_{L_p(\Delta_t)},$$
$$\|PU_tW\|_{L_p(\mathbf{I})} \leq C\|PU_tW\|_{L_p(\Delta_t)}. \tag{1.18}$$

Theorem 1.6 (see [6, Theorem 1.3]). *Let $W \in \mathcal{F}(\mathrm{Lip}\,(1/2))$, $L > 0$, and $0 < p < \infty$. Assume that relation* (1.16) *is valid. Then there exist $C, t_0 > 0$ such that, for $t \geq t_0$ and $P \in \mathcal{P}_t$,*

$$\|PUW\|_{L_p(\mathbf{I})} \leq C\|PUW\|_{L_p([a_{-t}(1-L\eta_{-t}),a_t(1-L\eta_t)])}. \tag{1.19}$$

In this paper we discuss the zeros of orthogonal polynomials for Jacobi-exponential weights UW and restricted range inequalities.

Theorem 1.7. *Let $W \in \mathcal{F}(\mathrm{Lip}\,(1/2))$. Assume that* (1.16) *is valid, and*

$$a < t_r < \cdots < t_1 < b, \tag{1.20}$$

$$\varphi_t(x) = O(1), \quad t \longrightarrow \infty. \tag{1.21}$$

Then

$$x_{kn} - x_{k+1,n} \leq c\varphi_n(x_{kn}), \quad k = 1, 2, \ldots, n-1. \tag{1.22}$$

Theorem 1.8. *Let $W = e^{-Q(x)}$, where $Q : \mathbf{I} \to [0, \infty)$ is convex with $Q(a+) = Q(b-) = \infty$ and $Q(x) > Q(0) = 0$, $x \in \mathbf{I} \setminus \{0\}$. Let $0 < p \leq \infty$. Assume that all p_i are positive and relation* (1.16) *is valid. Then there exist $t_0 > 0$ such that, for $t \geq t_0$ and $P \in \mathcal{P}_{t-\rho-2/p}$,*

$$\|PUW\|_{L_p(\mathbf{I}\setminus\Delta_t)} \leq \|PUW\|_{L_p(\Delta_t)}. \tag{1.23}$$

Theorem 1.9. *Let the assumptions of Theorem 1.8 prevail. Then*

$$x_{1n} < a_{n+\rho+1/2}, \tag{1.24}$$

$$x_{nn} > a_{-n-\rho-1/2}. \tag{1.25}$$

Theorem 1.10. *Let $W \in \mathcal{F}(\mathrm{Lip}(1/2))$. Then*

$$x_{1n} \geq a_n(1 - c\eta_n), \tag{1.26}$$

$$x_{nn} \leq a_{-n}(1 - c\eta_{-n}). \tag{1.27}$$

If all $p_i \geq 0$, then

$$1 - \frac{x_{1n}}{a_n} \sim \eta_n, \tag{1.28}$$

$$1 - \frac{x_{nn}}{a_{-n}} \sim \eta_{-n}. \tag{1.29}$$

Here we should point out that our main result (Theorem 1.7) cannot follow from [7] given by Mastroianni and Totik, because in general Jacobi-exponential weights UW are not doubling weights, although Jacobi weights U are doubling weights. A doubling weight means that the measure of a twice enlarged interval is less than a constant times the measure of the original interval. For example, for $W(t) = \exp(-t^2)$, by L'Hospital rule

$$\begin{aligned}\lim_{d\to\infty}\frac{\int_{d/2}^{5d/2}\exp(-t^2)dt}{\int_{d}^{2d}\exp(-t^2)dt} &= \lim_{d\to\infty}\frac{\exp\left(-(5d/2)^2\right)-\exp\left(-(d/2)^2\right)}{\exp\left(-(2d)^2\right)-\exp(-d^2)}\\ &= \lim_{d\to\infty}\frac{\exp(3d^2/4)-\exp(-21d^2/4)}{1-\exp(-3d^2)}=\infty,\end{aligned} \tag{1.30}$$

which implies that $W(t) = \exp(-t^2)$ is not a doubling weight.

We will give some auxiliary lemmas in Section 2 and the proofs of Theorems 1.7–1.10 in Section 3, respectively.

2. Auxiliary Lemmas

Lemma 2.1 (Levin and Lubinsky [1, Lemma 3.5, pages 71-72]). *Let $W \in \mathcal{F}$. Then for fixed $L > 1$ and uniformly for $t > 0$,*

$$a_{Lt} \sim a_t. \tag{2.1}$$

Moreover, there exists $\tau_0 > 0$ such that, for $t \ge \tau \ge \tau_0$, the inequalities

$$1 \le \frac{\delta_t}{\delta_\tau} \le c\left(\frac{t}{\tau}\right)^{1/\Lambda} \tag{2.2}$$

hold.

Lemma 2.2 (Shi [6]). *Let $W \in \mathcal{F}$. Then, for large enough t,*

$$a_{2t} \ge a_t(1+\eta_t). \tag{2.3}$$

Lemma 2.3. *Let $\mathbf{I} = (-1,1)$, $W \in \mathcal{F}$, and $L > 1$. Then, for $x \in \mathbf{K}_{L,t}$,*

$$\varphi_t(x) \sim \frac{1}{t}\left[(a_{2t}-x)(x-a_{-2t})\right]^{1/2}, \tag{2.4}$$

$$\varphi_t(x) \le \frac{c\delta_t}{t}. \tag{2.5}$$

Proof. By the same argument as that of [8, (2.25)] we can prove (2.4). By (2.4) and (2.1) for $x \in \mathbf{K}_{L,t}$,

$$\varphi_t(x) \le \frac{c}{t}\cdot\frac{1}{2}(a_{2t}-a_{-2t}) \le \frac{c\delta_t}{t}. \tag{2.6}$$

□

Lemma 2.4. *Let $W \in \mathcal{F}$. Then, for $x \in \mathbf{I}$,*

$$\varphi_t(x) \leq \frac{c\delta_t}{t^{2/3}T(a_t)^{1/6}}. \tag{2.7}$$

Proof. By the definition of φ_t it is enough to prove (2.7) for $x \in \Delta_t$. Without loss of generality we can assume that $0 \leq x \leq a_t$. By Lemma 3.11(b) in [1, page 81] for $t > 0$,

$$\left|\frac{a_{2t}}{a_t} - 1\right| \sim \frac{1}{T(a_t)}. \tag{2.8}$$

By Lemma 2.12 in [8], (2.3), (2.1), and (2.8),

$$\begin{aligned} S(x) &= \frac{a_{2t} - x}{a_t(1+\eta_t) - x} \cdot \frac{a_{-2t} - x}{a_{-t}(1+\eta_{-t}) - x} \\ &\leq \frac{a_{2t} - a_t}{a_t\eta_t} \cdot \frac{a_{-2t}}{a_{-t}(1+\eta_{-t})} \leq c\frac{a_{2t}/a_t - 1}{\eta_t} \leq \frac{c}{\eta_t T(a_t)}. \end{aligned} \tag{2.9}$$

By (1.63) in [1, page 15],

$$\eta_t T(a_t) \geq t^{-2/3}T(a_t)^{1/3} \tag{2.10}$$

and hence

$$S(x) \leq ct^{2/3}T(a_t)^{-1/3}. \tag{2.11}$$

Thus

$$\begin{aligned} \varphi_t(x) &= \frac{\left[(a_{2t} - x)(x - a_{-2t})\right]^{1/2}}{t} S(x)^{1/2} \\ &\leq \frac{c\delta_t}{t}\left[t^{2/3}T(a_t)^{-1/3}\right]^{1/2} = \frac{c\delta_t}{t^{2/3}T(a_t)^{1/6}}. \end{aligned} \tag{2.12}$$

□

Let $\mathbf{I}_k = [x_{k+1,n}, x_{kn}]$, $d_k = x_{kn} - x_{k+1,n}$, $k = 1, 2, \ldots, n-1$. Let, for $n \geq n_0$ and $d := \min_{1 \leq i \leq r-1}(t_i - t_{i+1})$,

$$\max_{1 \leq k \leq n-1} d_k \leq \frac{d}{4}. \tag{2.13}$$

Lemma 2.5. *For fixed index k, $1 \leq k \leq n-1$, let j, $1 \leq j \leq r$, satisfy*

$$\min_{x \in \mathbf{I}_k}\left|x - t_j\right| = \min_{1 \leq i \leq r}\min_{x \in \mathbf{I}_k}\left|x - t_i\right|. \tag{2.14}$$

Then

$$\prod_{i\neq j}|x_{\kappa n}-t_i|^{p_i}\sim\prod_{i\neq j}\left(|x_{\kappa n}-t_i|+\frac{\delta_n}{n}\right)^{p_i}\sim\prod_{i\neq j}|x-t_i|^{p_i},\quad x\in\mathbf{I}_k,\ \kappa=k,k+1. \tag{2.15}$$

Proof. We give the proof of (2.15) for $\kappa = k$ only, the proof of (2.15) for $\kappa = k+1$ being similar.

We claim that, for $i \neq j$,

$$|x_{kn}-t_i|\geq\frac{3}{8}d. \tag{2.16}$$

In fact, suppose without loss of generality that $x_{kn} \geq t_j$. It is enough to show (2.16) for $i = j-1$. Because $|x_{kn}-t_{j+1}| \geq t_j - t_{j+1} \geq d$.

If $t_j \in \mathbf{I}_k$ then by (2.13)

$$|x_{kn}-x_{k+1,n}|\leq\frac{d}{4}\leq t_{j-1}-t_j\leq\left|t_{j-1}-x_{kn}\right|+\left|x_{kn}-t_j\right| \tag{2.17}$$

and hence

$$\left|x_{k+1,n}-t_j\right|\leq\left|x_{kn}-t_{j-1}\right|; \tag{2.18}$$

if $t_j \notin \mathbf{I}_k$ then by (2.14)

$$\left|x_{k+1,n}-t_j\right|=\min_{x\in\mathbf{I}_k}\left|x-t_j\right|\leq\min_{x\in\mathbf{I}_k}\left|x-t_{j-1}\right|=t_{j-1}-x_{kn}, \tag{2.19}$$

which again implies (2.18). Then by (2.18)

$$\begin{aligned}d\leq\left|t_{j-1}-t_j\right|&\leq\left|t_{j-1}-x_{kn}\right|+|x_{kn}-x_{k+1,n}|+\left|x_{k+1,n}-t_j\right|\\&\leq 2\left|x_{kn}-t_{j-1}\right|+d_k\\&\leq 2\left|x_{kn}-t_{j-1}\right|+\frac{1}{4}d\end{aligned} \tag{2.20}$$

and hence $|x_{kn}-t_{j-1}| \geq 3d/8$. This proves (2.16).

With the help of (2.16) for $x \in \mathbf{I}_k$ and $i \neq j$,

$$\begin{aligned}|x-t_i|&\leq|x_{kn}-t_i|+|x-x_{kn}|\leq|x_{kn}-t_i|+\frac{d}{4}\leq\frac{5}{3}|x_{kn}-t_i|,\\|x-t_i|&\geq|x_{kn}-t_i|-|x-x_{kn}|\geq|x_{kn}-t_i|-\frac{d}{4}\geq\frac{1}{3}|x_{kn}-t_i|.\end{aligned} \tag{2.21}$$

Hence

$$|x-t_i|\sim|x_{kn}-t_i|. \tag{2.22}$$

Furthermore, by (2.2) with $\tau = 1$

$$\frac{\delta_t}{t} \le c\delta_1 t^{1/\Lambda - 1} = o(1), \quad t \longrightarrow \infty. \tag{2.23}$$

So for $i \neq j$,

$$|x_{kn} - t_i| \sim |x_{kn} - t_i| + \frac{\delta_n}{n}. \tag{2.24}$$

This proves (2.15). □

By the same argument as that of Lemma 7.2.7 in [9, page 157] replacing $1/n$ by C_n, we can get its extension.

Lemma 2.6. *Let* $p \ge 0,\ B_n \ge A_n \ge 0,\ C_n \ge 0,\ \sigma = \pm 1$, *and*

$$B_n^{p+1} + \sigma A_n^{p+1} \le C C_n \left[(B_n + C_n)^p + (A_n + C_n)^p \right]. \tag{2.25}$$

Then

$$B_n + \sigma A_n \le c C_n. \tag{2.26}$$

Lemma 2.7. *Let* $W \in \mathcal{F}$. *Let* (1.16), (1.20), *and* (1.21) *prevail. Then there exists* $t_0 > 0$ *such that, for* $t \ge t_0$ *and for each index* $j,\ 1 \le j \le r$,

$$\left|x - t_j\right| + \frac{\delta_t}{t} \sim \left|x - t_j\right| + \varphi_t(x) \tag{2.27}$$

holds uniformly for $x \in \mathbf{I}$.

Proof. Let $0 < \epsilon < \min\{b - t_1, t_r - a\}$ and $\Delta = [t_r - \epsilon, t_1 + \epsilon]$. We separate two cases.
Case 1 ($x \in \Delta$). In this case by (2.3) and (2.1),

$$\varphi_t(x) \ge \frac{1}{t}\left[(a_{2t} - x)(x - a_{-2t})\right]^{1/2} \ge \frac{1}{t}\left[(a_{2t} - t_1 - \epsilon)(t_r - \epsilon - a_{-2t})\right]^{1/2} \ge \frac{c\delta_t}{t} \tag{2.28}$$

which coupled with (2.5) gives

$$\varphi_t(x) \sim \frac{\delta_t}{t}. \tag{2.29}$$

Hence (2.27) follows.
Case 2 ($x \notin \Delta$). In this case by (2.23),

$$\left|x - t_j\right| \ge \epsilon \ge \frac{c\delta_t}{t} \tag{2.30}$$

and by (1.21)

$$\left|x - t_j\right| \geq \epsilon \geq c\varphi_t(x). \tag{2.31}$$

Again (2.27) follows. □

Corollary 2.8. *Let $W \in \mathcal{F}$. Let* (1.16) *and* (1.20) *prevail. If*

$$\frac{\delta_t}{t^{2/3}T(a_t)^{1/6}} = O(1), \quad t \longrightarrow \infty \tag{2.32}$$

then (2.27) *holds.*

In particular, if $\Lambda \geq 3/2$ then (2.32), (1.21), *and* (2.27) *hold.*

Proof. By (2.7) relation (2.32) implies (1.21). Then by Lemma 2.7 relation (2.27) is valid.

In particular, if $\Lambda \geq 3/2$ then by (2.2) with $\tau = \tau_0$ relation (2.32) is valid and hence (1.21) and (2.27) hold. □

3. Proof of Theorems

3.1. Proof of Theorem 1.7

Denote by ℓ_{kn}'s the fundamental polynomials based on the zeros x_{kn}'s. By Theorem 1.4 and Lemma 11.8 in [8, pages 320-321]

$$\begin{aligned}
&\lambda_n(WU; x_{kn})W(x_{kn})^{-2} + \lambda_n(WU; x_{k+1,n})W(x_{k+1,n})^{-2} \\
&\quad = \int_I \left[\ell_{kn}(t)^2 W(x_{kn})^{-2} + \ell_{k+1,n}(t)^2 W(x_{k+1,n})^{-2}\right] W(t)^2 U(t)^2 dt \\
&\quad \geq \int_{x_{k+1,n}}^{x_{kn}} \left[\ell_{kn}(t)^2 W(x_{kn})^{-2} + \ell_{k+1,n}(t)^2 W(x_{k+1,n})^{-2}\right] W(t)^2 U(t)^2 dt \\
&\quad \geq \frac{1}{2}\int_{x_{k+1,n}}^{x_{kn}} U(t)^2 dt.
\end{aligned} \tag{3.1}$$

On the other hand, by Theorem 1.4,

$$\begin{aligned}
&\lambda_n(WU; x_{kn})W(x_{kn})^{-2} + \lambda_n(WU; x_{k+1,n})W(x_{k+1,n})^{-2} \\
&\quad \leq c\left[\varphi_n(x_{kn})U_n(x_{kn})^2 + \varphi_n(x_{k+1,n})U_n(x_{k+1,n})^2\right].
\end{aligned} \tag{3.2}$$

Then for $\overline{\varphi}_n(x_{kn}) := \max\{\varphi_n(x_{kn}), \varphi_n(x_{k+1,n})\}$,

$$\int_{x_{k+1,n}}^{x_{kn}} U(t)^2 dt \leq c\overline{\varphi}_n(x_{kn})\left[U_n(x_{kn})^2 + U_n(x_{k+1,n})^2\right]. \tag{3.3}$$

Let j be defined by (2.14). Using Lemma 2.5 it follows from (3.3) that

$$\int_{x_{k+1,n}}^{x_{kn}} |t-t_j|^{2p_j} dt \le c\overline{\varphi}_n(x_{kn}) \left[\left(|x_{kn}-t_j| + \frac{\delta_n}{n} \right)^{2p_j} + \left(|x_{k+1,n}-t_j| + \frac{\delta_n}{n} \right)^{2p_j} \right]. \tag{3.4}$$

Further, by (2.27),

$$\int_{x_{k+1,n}}^{x_{kn}} |t-t_j|^{2p_j} dt \le c\overline{\varphi}_n(x_{kn}) \left\{ \left[|x_{kn}-t_j| + \overline{\varphi}_n(x_{kn}) \right]^{2p_j} + \left[|x_{k+1,n}-t_j| + \overline{\varphi}_n(x_{kn}) \right]^{2p_j} \right\}. \tag{3.5}$$

By calculation from (3.5) we get

$$\begin{aligned} &\frac{1}{2p_j+1}\left[|x_{kn}-t_j|^{2p_j+1} + \sigma |x_{k+1,n}-t_j|^{2p_j+1} \right] \\ &\quad = \int_{x_{k+1,n}}^{x_{kn}} |t-t_j|^{2p_j} dt \le c\overline{\varphi}_n(x_{kn}) \left\{ \left[|x_{kn}-t_j| + \overline{\varphi}_n(x_{kn}) \right]^{2p_j} + \left[|x_{k+1,n}-t_j| + \overline{\varphi}_n(x_{kn}) \right]^{2p_j} \right\}, \end{aligned} \tag{3.6}$$

where

$$\sigma = \begin{cases} 1, & t_j \in \mathbf{I}_k, \\ -1, & t_j \notin \mathbf{I}_k. \end{cases} \tag{3.7}$$

We separate two cases.

Case 1 ($p_j \ge 0$). Using Lemma 2.6 it follows from (3.6) that

$$x_{kn} - x_{k+1,n} \le c\overline{\varphi}_n(x_{kn}). \tag{3.8}$$

Case 2 ($p_j < 0$). Suppose without loss of generality that $x_{k+1,n} > t_j$ for the case when $t_j \notin \mathbf{I}_k$. By (3.6),

$$\begin{aligned} &\frac{1}{2p_j+1}\left[|x_{kn}-t_j|^{2p_j+1} + \sigma |x_{k+1,n}-t_j|^{2p_j+1} \right] \\ &\quad = \int_{x_{k+1,n}}^{x_{kn}} |t-t_j|^{2p_j} dt \le c_0\overline{\varphi}_n(x_{kn}) \min\left\{ \overline{\varphi}_n(x_{kn})^{2p_j}, |x_{k+1,n}-t_j|^{2p_j} \right\}. \end{aligned} \tag{3.9}$$

Subcase 2.1 ($t_j \in \mathbf{I}_k$). Inequality (3.9) gives

$$|x_{\kappa n}-t_j|^{2p_j+1} \le c\overline{\varphi}_n(x_{kn})^{2p_j+1}, \quad \kappa = k, k+1 \tag{3.10}$$

which yields (3.8).

Subcase 2.2 ($t_j \notin \mathbf{I}_k$). In this case we distinguish two subcases.

(1) $|x_{k+1,n} - t_j| \geq 2c_0\overline{\varphi}_n(x_{kn})$, where c_0 is given by (3.9). In this case

$$\begin{aligned}\int_{x_{k+1,n}}^{x_{kn}} (t-t_j)^{2p_j} dt &= \int_{x_{k+1,n}}^{x_{kn}} (t-t_j)(t-t_j)^{2p_j-1} dt \\ &\geq (x_{k+1,n}-t_j)\int_{x_{k+1,n}}^{x_{kn}} (t-t_j)^{2p_j-1} dt \\ &= (x_{k+1,n}-t_j)\frac{1}{2|p_j|}\left[(x_{k+1,n}-t_j)^{2p_j} - (x_{kn}-t_j)^{2p_j}\right] \\ &\geq \frac{c_0\overline{\varphi}_n(x_{kn})}{|p_j|}\left[(x_{k+1,n}-t_j)^{2p_j} - (x_{kn}-t_j)^{2p_j}\right],\end{aligned} \tag{3.11}$$

which by (3.9) gives

$$(x_{k+1,n}-t_j)^{2p_j} \leq (1-|p_j|)^{-1}(x_{kn}-t_j)^{2p_j} \leq 2(x_{kn}-t_j)^{2p_j}. \tag{3.12}$$

On the other hand, by (3.9) and (3.12),

$$\begin{aligned}c_0\overline{\varphi}_n(x_{kn})(x_{k+1,n}-t_j)^{2p_j} &\geq \int_{x_{k+1,n}}^{x_{kn}} (t-t_j)^{2p_j} dt \geq (x_{kn}-t_j)^{2p_j}(x_{kn}-x_{k+1,n}) \\ &\geq \frac{1}{2}(x_{k+1,n}-t_j)^{2p_j}(x_{kn}-x_{k+1,n})\end{aligned} \tag{3.13}$$

and hence (3.8) follows.

(2) $|x_{k+1,n} - t_j| < 2c_0\overline{\varphi}_n(x_{kn})$. By (3.9),

$$\begin{aligned}c_0\overline{\varphi}_n(x_{kn})^{2p_j+1} &\geq \frac{1}{2p_j+1}\left[(x_{kn}-t_j)^{2p_j+1} - (x_{k+1,n}-t_j)^{2p_j+1}\right] \\ &\geq \frac{1}{2p_j+1}\left[(x_{kn}-t_j)^{2p_j+1} - (2c_0\overline{\varphi}_n(x_{kn}))^{2p_j+1}\right].\end{aligned} \tag{3.14}$$

So $x_{kn} - t_j \leq c\overline{\varphi}_n(x_{kn})$ and (3.8) follows.

Finally, applying Theorem 5.7(b) in [1, page 125] we conclude $\overline{\varphi}_n(x_{kn}) \sim \varphi_n(x_{kn})$ and hence (1.22) follows from (3.8).

3.2. Proof of Theorem 1.8

For $P \in \mathcal{P}_{t-\rho-2/p}$, we have $PU \in \mathcal{P}_{t-2/p}$ and hence apply Theorem 1.8 in [1, page 15] to obtain (1.23).

3.3. Proof of Theorem 1.9

Use the same argument as that of Theorem 11.1 in [1, page 313].

3.4. Proof of Theorem 1.10

We give the proofs of (1.26) and (1.28) only, the proofs of (1.27) and (1.29) being similar.

First let us prove (1.26). Choose $\alpha, \beta > 1$ so that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad 2\beta p_i > -1, \quad i = 1, \dots, r. \tag{3.15}$$

Let L_n denote the linear map of Δ_n onto $[-1, 1]$. By Lemma 11.7 in [1, page 318] there exists $y_n \in \Delta_n$ such that

$$L_n(y_n) = \cos\frac{2\pi}{m}, \quad m = m(n), \tag{3.16}$$

and for large enough n and $R_n \in \mathbf{P}_{n-2m}$ such that

$$R_n(x)W(x)^{1/\alpha} \geq C_1, \quad x \in [0, y_n], \tag{3.17}$$

$$\left\| R_n W^{1/\alpha} \right\|_{L_\infty(\mathbf{I})} \leq C_2. \tag{3.18}$$

Using (11.7) in [1, page 318] in the form

$$1 - \frac{x_{1n}}{a_n} = \min_{P \in \mathbf{P}_{n-1}} \frac{\int_{\mathbf{I}} (1 - x/a_n)(PUW)^2(x)dx}{\int_{\mathbf{I}} (PUW)^2(x)dx}. \tag{3.19}$$

Again choose [1, page 319]

$$P(x) = R_n(x)V_{m,\cos(2\pi/m)}(L_n(x))^2 \in \mathbf{P}_{n-2}. \tag{3.20}$$

Applying Theorem 1.5 and (3.18), and using the same argument as that in [1, pages 319-320], we can get

$$\begin{aligned}
&\int_{\mathbf{I}}\left(1-\frac{x}{a_n}\right)(PUW)^2(x)dx \\
&\quad\le c\int_{\Delta_n}\left(1-\frac{x}{a_n}\right)(PUW)^2(x)dx \\
&\quad= c\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)\left(P(x)W(x)^{1/\alpha}\right)^2\right]\left[\left(U(x)W(x)^{1/\beta}\right)^2\right]dx \\
&\quad\le c\left\{\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)\left(P(x)W(x)^{1/\alpha}\right)^2\right]^{\alpha}dx\right\}^{1/\alpha}\left\{\int_{\Delta_n}\left[\left(U(x)W(x)^{1/\beta}\right)^2\right]^{\beta}dx\right\}^{1/\beta} \\
&\quad\le c\left\{\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)\left(P(x)W(x)^{1/\alpha}\right)^2\right]^{\alpha}dx\right\}^{1/\alpha} \\
&\quad\le c\left\{\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)V_{m,\cos(2\pi/m)}(L_n(x))^4\right]^{\alpha}dx\right\}^{1/\alpha} \\
&\quad= \frac{c\delta_n}{a_n}\left\{\int_{\Delta_n}\left[(1-L_n(x))V_{m,\cos(2\pi/m)}(L_n(x))^4\right]^{\alpha}dx\right\}^{1/\alpha} \\
&\quad= \frac{c\delta_n^2}{a_n}\left\{\int_{-1}^{1}\left[(1-t)V_{m,\cos(2\pi/m)}(t)^4\right]^{\alpha}dt\right\}^{1/\alpha} \\
&\quad\le \frac{c\delta_n^2}{a_n m^4}\left\{\int_{-\infty}^{\infty}\left[(1+|v|)\min\left\{1,\frac{c}{|v|}\right\}^4\right]^{\alpha}dv\right\}^{1/\alpha} \\
&\quad\le c a_n \eta_n^2.
\end{aligned} \tag{3.21}$$

On the other hand, by (3.17),

$$\begin{aligned}
\int_{\mathbf{I}}(PUW)^2(x)dx &\ge \int_{y_n(1-C_1\eta_n)}^{y_n}(PUW)^2(x)dx \\
&\ge \int_{y_n(1-C_1\eta_n)}^{y_n}V_{m,\cos(2\pi/m)}(L_n(x))^4 U(x)^2dx.
\end{aligned} \tag{3.22}$$

By (1.20) for large enough n, we have

$$U(x)\ge c>0,\quad x\in\left[y_n(1-C_1\eta_n),y_n\right]. \tag{3.23}$$

Hence (3.22) implies

$$\begin{aligned}\int_{\mathbf{I}}(PUW)^2(x)dx &\geq c\int_{y_n(1-C_1\eta_n)}^{y_n} V_{m,\cos(2\pi/m)}(L_n(x))^4dx\\ &= c\delta_n\int_{\cos(2\pi/m)-C_1y_n\eta_n/\delta_n}^{\cos(2\pi/m)} V_{m,\cos(2\pi/m)}(t)^4dt.\end{aligned}\tag{3.24}$$

But in [1, page 320] the following estimate is given:

$$\delta_n\int_{\cos(2\pi/m)-C_1y_n\eta_n/\delta_n}^{\cos(2\pi/m)} V_{m,\cos(2\pi/m)}(t)^4dt \geq ca_n\eta_n.\tag{3.25}$$

Substituting this estimate into (3.24) gives

$$\int_{\mathbf{I}}(PUW)^2(x)dx \geq ca_n\eta_n\tag{3.26}$$

which coupled with (3.21) yields (1.26).

Next let us prove (1.28). We already know that

$$a_n(1-c\eta_n) \leq x_{1n} < a_{n+\rho+1/2} = a_n(1+o(\eta_n)),\tag{3.27}$$

by (1.26) and (1.24). We must prove that, for some $c_1 > 0$, and n large enough, we have

$$x_{1n} < a_n(1-c_1\eta_n).\tag{3.28}$$

We use the idea for the proof of Corollary 13.4(b) in [1, pages 380-381] with modification. By the same argument as that proof with $A = a_{n+\rho+1/2}(1-\varepsilon\eta_n)$ instead, applying Theorem 1.8 we obtain

$$1-\frac{x_{1n}}{A} = \lambda_n\Big((UW)^2, x_{1n}\Big)^{-1}\int_{\mathbf{I}}\Big(1-\frac{x}{A}\Big)(\ell_{1n}UW)(x)^2dx,\tag{3.29}$$

$$\begin{aligned}&\int_{\mathbf{I}}\Big(1-\frac{x}{A}\Big)(\ell_{1n}UW)(x)^2dx\\ &\quad= \int_a^A\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx - \int_A^b\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx\\ &\quad\geq \int_a^A\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx\\ &\qquad - \int_A^{a_{n+\rho+1/2}}\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx - \int_{\Delta_{n+\rho+1/2}}\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx\\ &\quad\geq -2\int_A^{a_{n+\rho+1/2}}\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx,\end{aligned}\tag{3.30}$$

where ℓ_{1n} denotes the fundamental polynomial of Lagrange interpolation based on the zeros of the nth orthogonal polynomial with respect to the weight $(UW)^2$.

But

$$\int_A^{a_{n+\rho+1/2}} \left|1-\frac{x}{A}\right| (\ell_{1n}UW)(x)^2 dx \le \left(\frac{a_{n+\rho+1/2}}{A}-1\right)\int_{\mathbf{I}}(\ell_{1n}UW)(x)^2 dx = \left(\frac{a_{n+\rho+1/2}}{A}-1\right)\lambda_n\left((UW)^2, x_{1n}\right) \tag{3.31}$$

which, coupled with (3.30) and (3.29), gives

$$1-\frac{x_{1n}}{A} \ge -c\varepsilon\eta_n. \tag{3.32}$$

Thus

$$\begin{aligned}\frac{x_{1n}}{a_n} &= \frac{x_{1n}}{A}\frac{A}{a_{n+\rho+1/2}}\frac{a_{n+\rho+1/2}}{a_n} \\ &\le (1+c\varepsilon\eta_n)(1-\varepsilon\eta_n)(1+o(\eta_n)) \\ &< 1-c_1\eta_n,\end{aligned} \tag{3.33}$$

for n large enough, provided $\varepsilon > 0$ is small enough.

Acknowledgments

The authors thank the referee for carefully reading their paper, and making helpful suggestions and comments on improving their original paper. The research is supported in part by the National Natural Science Foundation of China (no. 11171100, no. 10871065, and no. 11071064) and by Hunan Provincial Innovation Foundation for Postgraduate.

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