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Research Article
Argument Property for Certain Analytic Functions

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Let P be the class of functions $p(z)$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. The object of the present paper is to derive certain argument inequalities of analytic functions $p(z)$ in P.

1. Introduction

Let P be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For functions p and g in the class P, we say that p is subordinate to g if there exists an analytic function w in U with $w(0) = 0$, $|w(z)| < 1$ $(z \in U)$, and such that $p(z) = g(w(z))$ $(z \in U)$. We denote this subordination by

$$p \prec g \quad (z \in U). \tag{1.2}$$

If g is univalent in U, then this subordination $p \prec g$ is equivalent to $p(0) = g(0)$ and $p(U) \subset g(U)$.

Recently, several authors investigated various argument properties of analytic functions (see, e.g., [1–6]). The object of the present paper is to discuss some argument inequalities for p in the class P.

Throughout this paper, we let

$$0<\alpha_1\le 1,\quad 0<\alpha_2\le 1,\quad \beta=\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2},\qquad c=e^{\beta\pi i}. \tag{1.3}$$

In order to prove our main result, we well need the following lemma.

Lemma 1.1 (see [6]). *Let $\lambda_0,\lambda,a,b\in R$ and $\mu\in C$. Also let*

$$\begin{gathered}\lambda_0 a\ge 0,\quad \lambda(b+2)\ge 0,\qquad (b+1)\operatorname{Re}(\mu)\ge 0,\\ |b+1|\le\frac{2}{\alpha_1+\alpha_2},\qquad |a-b-1|\le\frac{1}{\max\{\alpha_1,\alpha_2\}}.\end{gathered} \tag{1.4}$$

If $q\in P$ satisfies

$$\lambda_0(q(z))^a+\lambda(q(z))^{b+2}+\mu(q(z))^{b+1}+zq'(z)(q(z))^b\prec h(z)\quad (z\in U), \tag{1.5}$$

where

$$\begin{aligned}h(z)=\lambda_0\left(\frac{1+cz}{1-z}\right)^{a((\alpha_1+\alpha_2)/2)}&+\left(\frac{1+cz}{1-z}\right)^{(1/2)(b+1)(\alpha_1+\alpha_2)}\\ \times\left(\mu+\lambda\left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2}\right.&\left.+\frac{\alpha_1+\alpha_2}{2}\left(\frac{z}{1-z}+\frac{cz}{1+cz}\right)\right)\end{aligned} \tag{1.6}$$

is (close to convex) univalent, then

$$-\frac{\alpha_2\pi}{2}<\arg(q(z))<\frac{\alpha_1\pi}{2}\quad (z\in U). \tag{1.7}$$

The bounds α_1 and α_2 in (1.7) *are sharp for the function q defined by*

$$q(z)=\left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2}. \tag{1.8}$$

Remark 1.2 (see [6]). The function q defined by (1.8) is analytic and univalently convex in U and

$$q(U)=\left\{w:w\in C,\ -\frac{\alpha_2\pi}{2}<\ \arg w<\frac{\alpha_1\pi}{2}\right\}. \tag{1.9}$$

2. Main Result

Our main theorem is given by the following.

Theorem 2.1. *Let*

$$\lambda_0 > 0, \quad 0 < a \leq \frac{1}{\max\{\alpha_1, \alpha_2\}}, \quad |b+1| \leq \frac{2}{\alpha_1 + \alpha_2}, \quad 0 \leq a - b - 1 \leq \frac{1}{\max\{\alpha_1, \alpha_2\}}. \tag{2.1}$$

If $p \in P$ satisfies

$$-\frac{\gamma_2 \pi}{2} < \arg\left(\lambda_0 (p(z))^a + z p'(z) (p(z))^b\right) < \frac{\gamma_1 \pi}{2} \quad (z \in U), \tag{2.2}$$

where

$$\begin{aligned} \gamma_j &= \gamma_j(a, b, \alpha_1, \alpha_2) \\ &= a\alpha_j + \frac{2}{\pi} \tan^{-1}\left(\frac{((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\cos((a-b-1)\alpha_j\pi/2)}{2\lambda_0 \delta_j(a,b,\alpha_1,\alpha_2) + ((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\sin((a-b-1)\alpha_j\pi/2)}\right) \\ &\qquad (j = 1, 2), \\ \delta_j&(a, b, \alpha_1, \alpha_2) \\ &= \frac{\left[\left(1 - ((\mathcal{A})\cos(\beta\pi/2))^2\right)^{1/2} + (-1)^j \sin(\beta\pi/2)\right]^{1+\mathcal{A}}}{2\left[1 - (\mathcal{A})(\cos(\beta\pi/2))^2 + (-1)^j \left(1 - ((\mathcal{A})\cos(\beta\pi/2))^2\right)^{1/2} \sin(\beta\pi/2)\right]^{\mathcal{A}}} \quad (j = 1, 2), \end{aligned} \tag{2.3}$$

where $\mathcal{A}$ denotes $(a-b-1)(\alpha_1+\alpha_2)/2$, then

$$-\frac{\alpha_2 \pi}{2} < \arg(p(z)) < \frac{\alpha_1 \pi}{2} \quad (z \in U). \tag{2.4}$$

The bounds γ_1 and γ_2 in (2.2) *are the largest numbers such that* (2.4) *holds true.*

Proof. By taking $\lambda = \mu = 0$ in Lemma 1.1, we find that if $p \in P$ satisfies

$$\lambda_0 (p(z))^a + z p'(z) (p(z))^b \prec h(z) \quad (z \in U), \tag{2.5}$$

where

$$h(z) = \left(\frac{1+cz}{1-z}\right)^{(b+1)(\alpha_1+\alpha_2)/2} \left(\lambda_0 \left(\frac{1+cz}{1-z}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} + \frac{\alpha_1+\alpha_2}{2}\left(\frac{z}{1-z} + \frac{cz}{1+cz}\right)\right), \tag{2.6}$$

then (2.4) holds true.

For $z = e^{i\theta}$, $z \neq 1$ and $z \neq -1/c$, we get

$$\frac{z}{1-z} = -\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}, \qquad \frac{cz}{1+cz} = \frac{1}{2} + \frac{i}{2}\tan\frac{\theta+\beta\pi}{2},$$
$$\frac{1+cz}{1-z} = \frac{1+e^{i(\theta+\beta\pi)}}{1-e^{i\theta}} = \frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)} e^{\alpha_1\pi i/(\alpha_1+\alpha_2)} \neq 0. \tag{2.7}$$

We consider the following two cases.
(i) If

$$k(\theta) = \cos\frac{\theta+\beta\pi}{2}\sin\frac{\theta}{2} > 0, \tag{2.8}$$

then from (2.7), and (2.6), we have

$$h\left(e^{i\theta}\right) = \left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(b+1)(\alpha_1+\alpha_2)/2} \cdot e^{(b+1)\alpha_1\pi i/2}$$
$$\times\left(\lambda_0\left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot e^{(a-b-1)\alpha_1\pi i/2} + i\frac{(\alpha_1+\alpha_2)\cos(\beta\pi/2)}{4k(\theta)}\right), \tag{2.9}$$

and so

$$\arg\left(h\left(e^{i\theta}\right)\right) = \frac{1}{2}a\alpha_1\pi + \tan^{-1}\left(\frac{((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\cos((a-b-1)\alpha_1\pi/2)}{2\lambda_0 k_1(\theta) + ((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\sin((a-b-1)\alpha_1\pi/2)}\right), \tag{2.10}$$

where $\lambda_0 > 0$, $0 \leq (a-b-1)(\alpha_1+\alpha_2) \leq 2$, $e^{i\theta} \neq 1$, $e^{i\theta} \neq -1/c$,

$$k_1(\theta) = k(\theta)\left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} > 0. \tag{2.11}$$

We now calculate the maximum value of $k_1(\theta)$. It is easy to verify that

$$\lim_{\theta\to 0} k_1(\theta) = \lim_{e^{i\theta}\to -1/c} k_1(\theta) = 0 \tag{2.12}$$

and that

$$\begin{aligned}
k_1'(\theta) &= -\frac{(a-b-1)(\alpha_1+\alpha_2)}{4}\left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2-1} \cdot \frac{\cos(\beta\pi/2)}{(\sin(\theta/2))^2}k(\theta) \\
&\quad + \frac{1}{2}\left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot \cos\left(\theta+\frac{\beta\pi}{2}\right) \\
&= \frac{1}{2}\left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot \\
&\quad \times \left(\cos\left(\theta+\frac{\beta\pi}{2}\right) - \frac{(a-b-1)(\alpha_1+\alpha_2)}{2}\cos\frac{\beta\pi}{2}\right).
\end{aligned} \tag{2.13}$$

Set

$$\theta_1 = \cos^{-1}\left(\frac{(a-b-1)(\alpha_1+\alpha_2)}{2}\cos\frac{\beta\pi}{2}\right) - \frac{\beta\pi}{2}, \tag{2.14}$$

then $k_1'(\theta_1) = 0$. Noting that

$$\begin{gathered}
0 \le (a-b-1)(\alpha_1+\alpha_2) \le 2, \quad -1 < \beta < 1, \\
\frac{|\beta|\pi}{2} < \cos^{-1}\left(\frac{(a-b-1)(\alpha_1+\alpha_2)}{2}\cos\frac{\beta\pi}{2}\right) < \frac{\pi}{2},
\end{gathered} \tag{2.15}$$

we easily have

$$0 < \theta_1 < \pi, \quad 0 < \theta_1 + \frac{\beta\pi}{2} < \frac{\pi}{2}, \quad 0 < \frac{\theta_1+\beta\pi}{2} < \frac{\pi}{2}. \tag{2.16}$$

Hence, $k(\theta_1) > 0$, and it follows from (2.11) to (2.16) that

$$\begin{aligned}
0 &< k_1(\theta) \le k_1(\theta_1) \\
&= \left(\sin\frac{\theta_1}{2}\right)^{-(a-b-1)(\alpha_1+\alpha_2)} \cdot \left(\cos\frac{\theta_1+\beta\pi}{2}\sin\frac{\theta_1}{2}\right)^{1+\mathcal{A}} \\
&= \left(\frac{1-\cos\theta_1}{2}\right)^{-\mathcal{A}} \cdot \left(\frac{1}{2}\left(\sin\left(\theta_1+\frac{\beta\pi}{2}\right) - \sin\frac{\beta\pi}{2}\right)\right)^{1+\mathcal{A}} \\
&= \frac{\left[\left(1-((\mathcal{A})\cos(\beta\pi/2))^2\right)^{1/2} - \sin(\beta\pi/2)\right]^{1+\mathcal{A}}}{2\left[1-(\mathcal{A})(\cos(\beta\pi/2))^2 - \left(1-((\mathcal{A})\cos(\beta\pi/2))^2\right)^{1/2}\sin(\beta\pi/2)\right]^{\mathcal{A}}} \\
&= \delta_1(a,b,\alpha_1,\alpha_2),
\end{aligned} \tag{2.17}$$

where $\mathscr{A}$ denotes $(a-b-1)(\alpha_1+\alpha_2)/2$. Thus, by using (2.1), (2.10), and (2.17), we arrive at

$$\begin{aligned}\pi &> \arg\left(h\left(e^{i\theta}\right)\right) \geq \arg\left(h\left(e^{i\theta_1}\right)\right)\\ &= \frac{1}{2}a\alpha_1\pi + \tan^{-1}\left(\frac{((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\cos((a-b-1)\alpha_1\pi/2)}{2\lambda_0\delta_1(a,b,\alpha_1,\alpha_2)+((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\sin((a-b-1)\alpha_1\pi/2)}\right)\\ &= \frac{\gamma_1\pi}{2} > 0.\end{aligned} \tag{2.18}$$

(ii) If $k(\theta) < 0$, then we obtain

$$\begin{aligned}h\left(e^{i\theta}\right) &= \left(-\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(b+1)(\alpha_1+\alpha_2)/2} \cdot e^{-(b+1)\alpha_2\pi i/2}\\ &\quad\times\left(\lambda_0\left(-\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot e^{-(a-b-1)\alpha_2\pi i/2} + i\frac{(\alpha_1+\alpha_2)\cos(\beta\pi/2)}{4k(\theta)}\right),\end{aligned} \tag{2.19}$$

which leads to

$$\arg\left(h\left(e^{i\theta}\right)\right) = -\frac{1}{2}a\alpha_2\pi - \tan^{-1}\left(\frac{((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\cos((a-b-1)\alpha_2\pi/2)}{2\lambda_0 k_2(\theta)+((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\cos((a-b-1)\alpha_2\pi/2)}\right), \tag{2.20}$$

where $\lambda_0 > 0$, $0 \leq (a-b-1)(\alpha_1+\alpha_2) \leq 2$, $e^{i\theta} \neq 1$, $e^{i\theta} \neq -1/c$,

$$k_2(\theta) = -k(\theta)\left(-\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} > 0. \tag{2.21}$$

Now, we have

$$\begin{aligned}&\lim_{\theta\to 0} k_2(\theta) = \lim_{e^{i\theta}\to -1/c} k_2(\theta) = 0,\\ &k_2'(\theta) = \frac{1}{2}\left(-\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2}\left(\frac{(a-b-1)(\alpha_1+\alpha_2)}{2}\cos\frac{\beta\pi}{2} - \cos\left(-\theta-\frac{\beta\pi}{2}\right)\right).\end{aligned} \tag{2.22}$$

Let

$$\theta_2 = -\cos^{-1}\left(\frac{(a-b-1)(\alpha_1+\alpha_2)}{2}\cos\frac{\beta\pi}{2}\right) - \frac{\beta\pi}{2}, \tag{2.23}$$

then $k_2'(\theta_2) = 0$, $\theta_1 + \theta_2 = -\beta\pi$,

$$-\pi < \theta_2 < 0, \quad -\frac{\pi}{2} < \theta_2 + \frac{\beta\pi}{2} < 0, \quad -\frac{\pi}{2} < \frac{\theta_2 + \beta\pi}{2} < 0. \tag{2.24}$$

Hence, we deduce that $k(\theta_2) < 0$ and

$$\begin{aligned}
0 &< k_2(\theta) \le k_2(\theta_2) \\
&= \left(-\sin\frac{\theta_2}{2}\right)^{-(a-b-1)(\alpha_1+\alpha_2)} \cdot \left(-\cos\frac{\theta_2+\beta\pi}{2}\sin\frac{\theta_2}{2}\right)^{1+\mathcal{A}} \\
&= \left(\frac{1-\cos\theta_2}{2}\right)^{-\mathcal{A}} \cdot \left(\frac{1}{2}\left(\sin\frac{\beta\pi}{2} - \sin\left(\theta_2 + \frac{\beta\pi}{2}\right)\right)\right)^{1+\mathcal{A}} \\
&= \frac{\left[\left(1-((\mathcal{A})\cos(\beta\pi/2))^2\right)^{1/2} + \sin(\beta\pi/2)\right]^{1+\mathcal{A}}}{2\left[1-(\mathcal{A})(\cos(\beta\pi/2))^2 + \left(1-((\mathcal{A})\cos(\beta\pi/2))^2\right)^{1/2}\sin(\beta\pi/2)\right]^{\mathcal{A}}} \\
&= \delta_2(a, b, \alpha_1, \alpha_2),
\end{aligned} \tag{2.25}$$

where $\mathcal{A} = (a-b-1)(\alpha_1+\alpha_2)/2$. Further, by using (2.1), (2.20), and (2.25), we find that

$$\begin{aligned}
-\pi &< \arg\left(h\left(e^{i\theta}\right)\right) \le \arg\left(h\left(e^{i\theta_2}\right)\right) \\
&= -\frac{1}{2}a\alpha_2\pi - \tan^{-1}\left(\frac{((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\cos((a-b-1)\alpha_2\pi/2)}{2\lambda_0\delta_2(a,b,\alpha_1,\alpha_2) + ((\alpha_1+\alpha_2)/2)\cos(\beta\pi/2)\cos((a-b-1)\alpha_2\pi/2)}\right) \\
&= -\frac{\gamma_2\pi}{2} < 0.
\end{aligned} \tag{2.26}$$

In view of $h(0) = 1 > 0$, we conclude from (2.18) and (2.26) that $h(U)$ properly contains the angular region $-\gamma_2\pi/2 < \arg w < \gamma_1\pi/2$ in the complex w-plane. Therefore, if $p \in P$ satisfies (2.2), then the subordination relation (2.5) holds true, and thus we arrive at (2.4).

Furthermore, for the function q defined by (1.8), we have

$$\begin{gathered}
-\frac{\alpha_2\pi}{2} < \arg(q(z)) < \frac{\alpha_1\pi}{2} \quad (z \in U), \\
\lambda_0(q(z))^a + zq'(z)(q(z))^b = h(z).
\end{gathered} \tag{2.27}$$

Hence, by using (2.18) and (2.25), we see that the bounds γ_1 and γ_2 in (2.2) are best possible. □

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