Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 418943, 16 pages
doi:10.1155/2012/418943

Research Article

A Jacobi Dual-Petrov Galerkin-Jacobi Collocation Method for Solving Korteweg-de Vries Equations

Ali H. Bhrawy[1,2] and M. M. Al-Shomrani[1,3]

[1] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt*
[3] *Faculty of Computer Science and Information Technology, Northern Border University, Saudi Arabia*

Correspondence should be addressed to Ali H. Bhrawy, alibhrawy@yahoo.co.uk

Received 15 April 2012; Revised 22 June 2012; Accepted 29 June 2012

Academic Editor: Xiaodong Yan

The present paper is devoted to the development of a new scheme to solve the initial-boundary value Korteweg-de Vries equation which models many physical phenomena such as surface water waves in a channel. The scheme consists of Jacobi dual-Petrov Galerkin-Jacobi collocation method in space combined with Crank-Nicholson-leap-frog method in time such that at each time step only a sparse banded linear algebraic system needs to be solved. Numerical results are presented to show that the proposed numerical method is accurate and efficient for Korteweg-de Vries equations and other third-order nonlinear equations.

1. Introduction

There is a vast body of literature on various aspects of the Korteweg-de Vries (KdV) equation [1–4]. The KdV and extended KdV equations have been useful as an approximate model equations in a number of important physical situations. For example, in plasma physics, these equations give rise to the ion acoustic solitons [5, 6], in geophysical fluid dynamics, they describe a long wave in shallow seas and deep oceans [7, 8]. Their strong presence is exhibited in cluster physics, super deformed nuclei, fission, thin films, radar and rheology [9, 10], optical-fiber communications [11], and superconductors [12]. Many explicit exact methods have been introduced for the solution of KdV equations (see, eg., [13–16]).

In the spectral methods, one may use a collocation method based on a certain family of polynomials. Pavoni [17] defined a Chebyshev-collocation method. Heinrichs [18] put forward a scheme for the adjoint of the linearized problem and also presented a numerical investigation of the eigenvalues of the spectral differentiation operator. An interesting hybrid method was proposed by Ma and Sun [19], where the linear term was treated by a Legendre-Petrov method, and the nonlinear term was treated using a Chebyshev-collocation method.

Ma and Sun were able to evaluate the nonlinear term pseudospectrally. In [20], the authors focused on a collocation method based on Chebyshev polynomials, with the concrete aim of comparing it with a simple finite-difference method, the Chebyshev method is found to be more efficient than the finite-difference method. Dehghan and Shokri [21] proposed a numerical scheme to solve the third-order nonlinear KdV equation using collocation points and approximating the solution using multiquadric radial basis function.

A well-known advantage of a spectral method is that it achieves high accuracy with relatively fewer spatial grid points when compared with a finite-difference method. On the other hand, spectral methods typically give rise to full matrices, partially negating the gain in efficiency due to the fewer number of grid points. In general, the use of Jacobi polynomials ($P_n^{(\alpha,\beta)}$ with $\alpha, \beta \in (-1, \infty)$ and n is the polynomial degree) has the advantage of obtaining the solutions of differential equations in terms of the Jacobi indexes α and β (see, eg., [22–26]).

Each of these particular pairs of α and β has been used separately for solving approximately differential equations (see [18, 27–30]). Hence, to generalize and instead of developing approximation results for each particular pair of indexes, it would be very useful to carry out a systematic study on Jacobi polynomials with general indexes which can then be directly applied to other applications [31–33]. It is with this motivation that we introduce in this paper a family of Jacobi polynomials with general indexes.

The convergence of spectral projections of the KdV equation was proved by Maday and Quarteroni [34]. In particular, it was announced in [35] that if the solution of KdV equation is analytic in a strip about the real axis, then the convergence rate is in fact exponential. Moreover, it was shown in [36] that a spectral approximation of the KdV equation with periodic boundary conditions converges exponentially fast to the true solution if the Fourier basis is used and if the solution is analytic in a fixed strip about the real axis. For the rate of convergence of a collocation projection of the KdV equation, see for instance, [37].

It has been pointed out that the Chebyshev and Legendre collocation methods using the Gauss-Lobatto points may be unstable for third-order differential equations and thus are poor choices for such problems [38]. The Gauss-Lobatto points are from the generalized Gaussian quadrature rule, which uses the values $u(-1)$ and $u(1)$. With the Legendre method, these are the zeros of the Jacobi polynomial $P_{N-1}^{(1,1)}(x)$. As for the Legendre method using the values $u(-1)$, $u(1)$, and $u_x(1)$, the generalized Gaussian quadrature nodes are the zeros of $P_{N-2}^{(2,1)}(x)$ [39]. Huang and Sloan [39] have shown that, with this choice of collocation points, the pseudospectral method is stable for the linear third-order differential equation and the convergence has been established.

In [40], Don and Gottlieb first introduced the Chebyshev-Legendre (CL) method, where the Legendre method was implemented on Chebyshev points. The boundary conditions were imposed via a penalty technique, and the scheme was in a collocation form. It was shown that the method was stable in the unweighted L^2-norm, and error estimates were given for linear problems [40]. Ma [41] applied Chebyshev-Legendre viscosity methods to the nonlinear conservation laws. Moreover, A Jacobi-Jacobi dual-Petrov Galerkin method for the differential equations with variable coefficients is developed in [42]. This method is based on the Petrov-Galerkin variational form of one Jacobi polynomial class, but the nonlinear and the right-hand terms are treated by using Gauss-Lobatto quadrature form of another Jacobi class.

Moreover, a Legendre Petrov-Galerkin and Chebyshev collocation (LPG-CC) method is developed for nonlinear equations such as the Burgers-like equations, the Korteweg-de Vries equation, and the Kuramoto-Sivashinsky equation which are very important equations in physical models. The linear part in the scheme is formulated in the LPG, form while the

nonlinear term is treated with the Chebyshev collocation method. The time discretization is a classical Crank-Nicholson-leap-frog scheme. Yuan and Wu [43] extended the Legendre dual-Petrov-Galerkin method proposed by Shen [44], further developed by Yuan et al. [45] to general fifth-order KdV-type equations with various nonlinear terms.

The main aim of this paper is to propose a suitable way to approximate the third-order differential equations in space, by dual-Petrov Galerkin method-based on Jacobi polynomials such that it can be implemented efficiently and at the same time has a good convergence property. Moreover, we introduce the Jacobi dual Petrov Galerkin-Jacobi collocation (JDPG-JC) method for solving third-order differential equations with nonlinear term. The method is basically formulated in the Jacobi spectral form with general indexes α, $\beta > -1$ but the nonlinear term being treated by the Jacobi collocation method with other two general indexes θ, $\vartheta > -1$ so that the schemes can be implemented at Jacobi-Gauss-Lobatto points efficiently. Therefore, we can generalize Legendre Petrov-Galerkin and Chebyshev collocation method to Jacobi Petrov-Galerkin and Jacobi collocation method. Some other cases can be obtained directly as special cases from our proposed JDPG-JC approximations. We, therefore, motivated our interest in JDPG-JC approximations. Finally, numerical results are presented in which the usual exponential convergence behaviour of spectral approximations is exhibited.

The layout of the paper is as follows. In Section 2, we give an overview of Jacobi polynomials and their relevant properties needed hereafter. Section 3 is devoted to the theoretical derivation of the Jacobi dual-Petrov Galerkin (JDPG) method for linear third-order differential equations subject to homogeneous boundary conditions. Section 4 gives the corresponding results for those obtained in Section 3 but for the KDV equation using JDPG-JC method. In Section 5, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithms. Some concluding remarks are given in the final section.

2. Preliminaries

Let $S_N(I)$ be the space of polynomials of degree at most N on the interval $I = (-1, 1)$, we set

$$\begin{aligned} W_N &= \{u \in S_N : \ u(\pm 1) = u'(1) = 0\}, \\ W_N^* &= \{u \in S_N : \ u(\pm 1) = u'(-1) = 0\}. \end{aligned} \tag{2.1}$$

And let $P_n^{(\alpha,\beta)}(x)$ $(n = 0, 1, 2, \ldots)$ be the Jacobi polynomials orthogonal with the weight functions $w^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$, where $\alpha, \beta > -1$.

Let $x_{N,j}^{(\alpha,\beta)}$, $0 \le j \le N$, be the zeros of $(1-x^2)\partial_x P_N^{(\alpha,\beta)}$. Denote by $\varpi_{N,j}^{(\alpha,\beta)}$, $0 \le j \le N$, the weights of the corresponding Gauss-Lobatto quadrature formula. They are arranged in decreasing order. We define the discrete inner product and norm as follows:

$$(u, v)_{w^{\alpha,\beta},N} = \sum_{k=0}^{N} u\left(x_{N,k}^{(\alpha,\beta)}\right) v\left(x_{N,k}^{(\alpha,\beta)}\right) \varpi_{N,k}^{(\alpha,\beta)}, \qquad \|u\|_{w^{\alpha,\beta},N} = \sqrt{(u,u)_{w^{\alpha,\beta},N}}. \tag{2.2}$$

Obviously,

$$(u, v)_{w^{\alpha,\beta},N} = (u, v)_{w^{\alpha,\beta}} \quad \forall uv \in S_{2N-1}. \tag{2.3}$$

Thus, for any $u \in S_N$, the norms $\|u\|_{w^{\alpha,\beta},N}$ and $\|u\|_{w^{\alpha,\beta}}$ coincide.

Associating with this quadrature rule, we denote by $I_N^{P^{\alpha,\beta}}$ the Jacobi-Gauss-Lobatto interpolation (cf. [46]),

$$I_N^{P^{(\alpha,\beta)}} u\left(x_{N,k}^{(\alpha,\beta)}\right) = u\left(x_{N,k}^{(\alpha,\beta)}\right), \quad 0 \le j \le N. \tag{2.4}$$

We denote by $I_N^c = I_N^{P^{(-1/2,-1/2)}}$ and $I_N^l = I_N^{P^{(0,0)}}$ the Chebyshev-Gauss-Lobatto and Legendre-Gauss-Lobatto interpolation operators, respectively.

The following special values will be of fundamental importance in what follows:

$$\begin{aligned} P_n^{(\alpha,\beta)}(1) &= \frac{(\alpha+1)_n}{n!}, \qquad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(\beta+1)_n}{n!}, \\ D^q P_n^{(\alpha,\beta)}(1) &= \prod_{i=0}^{q-1} \frac{\Gamma(n+\alpha+1)(n+\lambda+i)}{2^q(n-q)!\Gamma(q+\alpha+1)}, \qquad D^q P_n^{(\alpha,\beta)}(-1) = (-1)^{n+q} D^q P_n^{(\beta,\alpha)}(1), \end{aligned} \tag{2.5}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ and $\lambda = 1+\beta+\alpha$.

Lemma 2.1 (see, Doha [47]). *The qth derivative of $P_n^{(\alpha,\beta)}(x)$ can be written as*

$$D^q P_k^{(\alpha,\beta)}(x) = \sum_{i=0}^{k-q} C_q(k,i,\alpha,\beta) P_i^{(\alpha,\beta)}(x), \tag{2.6}$$

where,

$$\begin{aligned} C_q(k,i,\alpha,\beta) = {} & \frac{(k+\lambda)_q (k+\lambda+q)_i (i+\alpha+q+1)_{k-i-q}\, \Gamma(i+\lambda)}{2^q (k-i-q)!\, \Gamma(2i+\lambda)} \\ & \times {}_3F_2\left(\begin{matrix} -k+i+q, & k+i+\lambda+q, & i+\alpha+1 \\ i+\alpha+q+1, & 2i+\lambda+1, & 1 \end{matrix}\right). \end{aligned} \tag{2.7}$$

3. Linear Third-Order Equation

Let us begin with a simple model problem:

$$\begin{aligned} \partial_t u + \partial_x^3 u &= f(x,t), \quad x \in I, \quad t \in (0,T], \\ u(-1,t) = u(1,t) &= \partial_x u(1,t) = 0, \quad t \in (0,T], \\ u(x,0) &= u_0(x), \quad x \in I. \end{aligned} \tag{3.1}$$

The study of Jacobi Dual-Petrov Galerkin for this linear problem is the foundation for those general third-order problems, such as the Korteweg-de Vries equation and problems arising from water wave, plasma physics, and anharmonic lattices.

For the sake of simplicity, we only consider homogeneous boundary conditions in (3.1). From a numerical point of view, problems with nonhomogeneous boundary conditions can be easily handled by introducing a lifting function (cf. [42, 44]).

The dual-Petrov Galerkin method generates a sequence of approximate solutions that satisfy a weak form of the original differential equations as tested against polynomials in a

dual space. It is worthy to note here that third-order problems lack the symmetry of second- and fourth-order ones, so we propose JDPG method, which is more reasonable and suitable than the standard Galerkin one, see [42, 44]. By choosing appropriate base functions, the system in JDPG method is sparse and can be implemented efficiently.

The semidiscrete JDPG method for (3.1) is to find $u_N(t) \in W_N$ such that

$$\begin{aligned} &(\partial_t u_N(t), v)_{w^{\alpha,\beta}} + \left(\partial_x^3 u_N(t), v\right)_{w^{\alpha,\beta}} = \left(f(t), v\right)_{w^{\alpha,\beta}}, \quad 0 < t < T, \\ &(u_N(0), v)_{w^{\alpha,\beta}} = (u_0, v)_{w^{\alpha,\beta}} \quad \forall v \in W_N^*, \end{aligned} \tag{3.2}$$

where $w^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ and $(u,v)_{w^{\alpha,\beta}} = \int_I uvw^{\alpha,\beta}dx$ are the inner products in the weighted space $L^2_{w^{\alpha,\beta}}(I)$. The norm in $L^2_{w^{\alpha,\beta}}(I)$ will be denoted by $\|\cdot\|_{w^{\alpha,\beta}}$.

Since the main differential operator in (3.1) is not symmetric, it is quite natural to employ a Jacobi dual-Petrov Galerkin (JDPG) method. To be more precise, we choose the trial functions to satisfy the underlying boundary conditions of the differential equations, and we choose the test functions to satisfy the dual boundary conditions ($u(-1,t) = u(1,t) = \partial_x u(-1,t) = 0$), let $\{P_k^{(\alpha,\beta)}\}$ be a sequence of Jacobi polynomials. We choose the test basis and trial functions of expansion $\phi_k(x)$ and $\psi_k(x)$, as given in Doha et al. [42], to be the form

$$\begin{aligned} \phi_k(x) &= \eta_k\left[P_k^{(\alpha,\beta)}(x) + \epsilon_k P_{k+1}^{(\alpha,\beta)}(x) + \varepsilon_k P_{k+2}^{(\alpha,\beta)}(x) + \zeta_k P_{k+3}^{(\alpha,\beta)}(x)\right], \\ \psi_k(x) &= \eta_k\left[P_k^{(\alpha,\beta)}(x) + \rho_k P_{k+1}^{(\alpha,\beta)}(x) + \varrho_k P_{k+2}^{(\alpha,\beta)}(x) + \sigma_k P_{k+3}^{(\alpha,\beta)}(x)\right], \end{aligned} \tag{3.3}$$

where η_k, ϵ_k, ε_k, ζ_k, ρ_k, ϱ_k, and σ_k are the unique constants such that $\phi_k(x) \in W_N$ and $\psi_k(x) \in W_N^*$, for all $k = 0, 1, \ldots, N-3$, and are given by

$$\begin{aligned} \eta_k &= \left(\frac{(k+\alpha+1)_2(k+\beta+1)\Gamma(k+1)\Gamma(k+\lambda+3)}{2^{\lambda-3}(1+k)_3((2k+\lambda+1)_2)^2(2k+\lambda+3)\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}\right)^{1/2}, \\ \epsilon_k &= \frac{-(k+1)(2k+\lambda+2)(k-\alpha+2\beta+1)}{(k+\alpha+1)(k+\beta+1)(2k+\lambda+4)}, \\ \varepsilon_k &= \frac{-(k+1)_2(2k+\lambda+1)(k-\beta+2\alpha+3)}{(k+\alpha+1)_2(k+\beta+1)(2k+\lambda+5)}, \\ \zeta_k &= \frac{(k+1)_3(2k+\lambda+1)_2}{(k+\alpha+1)_2(k+\beta+1)(2k+\lambda+4)_2}, \\ \rho_k &= \frac{(k+1)(2k+\lambda+2)(k-\beta+2\alpha+1)}{(k+\alpha+1)(k+\beta+1)(2k+\lambda+4)}, \\ \varrho_k &= \frac{-(k+1)_2(2k+\lambda+1)(k-\alpha+2\beta+3)}{(k+\beta+1)_2(k+\alpha+1)(2k+\lambda+5)}, \\ \sigma_k &= \frac{-(k+1)_3(2k+\lambda+1)_2}{(k+\beta+1)_2(k+\alpha+1)(2k+\lambda+4)_2}. \end{aligned} \tag{3.4}$$

Therefore by dimension argument for $N \geq 3$, we have

$$\begin{aligned} W_N &= \operatorname{span}\{\phi_k(x) : k = 0, 1, 2, \ldots, N-3\}, \\ W_N^* &= \operatorname{span}\{\psi_k(x) : k = 0, 1, 2, \ldots, N-3\}. \end{aligned} \tag{3.5}$$

We use the Crank-Nicolson scheme for the time advancing. Let Δt be the step size in time space and $t_k = k\Delta t$ $(k = 0, 1, \ldots, n_T; T = n_T \Delta t)$. For simplicity, we denote $v^k(x) := v(x, t_k)$ by v^k and

$$v_t^k = \frac{v^{k+1} - v^k}{\Delta t}, \qquad v^{k+1/2} = \frac{v^{k+1} + v^k}{2}. \tag{3.6}$$

The fully discrete JDPG method for (3.1) is to find $u_N^k \in W_N$ such that

$$\begin{gathered} \left(u_{N_t}^k, v\right)_{w^{\alpha,\beta}} + \left(\partial_x^3 u_N^{k+1/2}, v\right)_{w^{\alpha,\beta}} = \left(f^{k+1/2}, v\right)_{w^{\alpha,\beta}}, \quad 0 \leq k \leq n_T - 1, \\ \left(u_N^0, v\right)_{w^{\alpha,\beta}} = (u_0, v)_{w^{\alpha,\beta}} \quad \forall v \in W_N^*. \end{gathered} \tag{3.7}$$

We give a brief description for the implementation of the scheme (3.7). At each time level, we need to solve the following equation: find $u_N^{k+1} \in W_N$,

$$\left(\frac{u_N^{k+1} - u_N^k}{\Delta t}, v\right)_{w^{\alpha,\beta}} + \left(\frac{\partial_x^3 (u_N^{k+1} + u_N^k)}{2}, v\right)_{w^{\alpha,\beta}} = \left(f^{k+1/2}, v\right)_{w^{\alpha,\beta}}, \quad \forall v \in W_N^*, \tag{3.8}$$

consequently

$$\left(u_N^{k+1}, v\right)_{w^{\alpha,\beta}} + \frac{\Delta t}{2}\left(\partial_x^3 u_N^{k+1}, v\right)_{w^{\alpha,\beta}} = \left(g^k, v\right)_{w^{\alpha,\beta}}, \quad \forall v \in W_N^*. \tag{3.9}$$

Now, we derive an algorithm for solving (3.9). For this purpose, let us denote

$$\begin{gathered} g_j^k = \left(g^k, \psi_j(x)\right)_{w^{\alpha,\beta}}, \qquad \hat{\mathbf{g}}^k = \left(g_0^k, g_1^k, \ldots, g_{N-3}^k\right)^T, \\ u_N^k(x) = \sum_{i=0}^{N-3} \hat{u}_i^k \phi_i(x), \qquad \hat{\mathbf{u}}^k = \left[\hat{u}_0^k, \hat{u}_1^k, \ldots, \hat{u}_{N-3}^k\right]^T, \\ a_{ji} = \left(\phi_i^{(3)}(x), \psi_j(x)\right)_{w^{\alpha,\beta}}, \qquad d_{ji} = \left(\phi_i(x), \psi_j(x)\right)_{w^{\alpha,\beta}}, \\ A = (a_{ji}), \quad D = (b_{ji}), \qquad 0 \leq j, i \leq N-3. \end{gathered} \tag{3.10}$$

Then scheme (3.9) takes the form:

$$\sum_{i=0}^{N-3}\left[\left(\phi_i(x), \psi_j(x)\right)_{w^{\alpha,\beta}} + \frac{\Delta t}{2}\left(\phi_i^{(3)}(x), \psi_j(x)\right)_{w^{\alpha,\beta}}\right]\hat{u}_i^{k+1} = \left(g^k, \psi_j(x)\right)_{w^{\alpha,\beta}}, \quad 0 \leq j \leq N-3. \tag{3.11}$$

The scheme (3.11) with the aid of (3.10) can be written in a matrix form as follows:

$$\left(D + \frac{\Delta t}{2}A\right)\hat{\mathbf{u}}^{k+1} = \hat{\mathbf{g}}^{k}. \tag{3.12}$$

The nonzero elements of the matrices A and D, using the properties of Jacobi polynomials and (2.6) are given explicitly as follows:

$$\begin{aligned}
&a_{kk} = 1,\\
&a_{kj} = \eta_k\eta_j\left[O_3(j,k,\alpha,\beta)h_k + O_3(j,k+1,\alpha,\beta)\rho_k h_{k+1}\right.\\
&\qquad\qquad + O_3(j,k+2,\alpha,\beta)\varrho_k h_{k+2}\\
&\qquad\qquad \left.+O_3(j,k+3,\alpha,\beta)\sigma_k h_{k+3}\right], \quad j = k+n, n \geq 1,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&d_{k+3,k} = \eta_k\eta_{k+3}\zeta_k h_{k+3}, \qquad d_{k,k+3} = \eta_k\eta_{k+3}\sigma_k h_{k+3},\\
&d_{k+2,k} = \eta_k\eta_{k+2}\left[\varepsilon_k h_{k+2} + \zeta_k\rho_{k+2}h_{k+3}\right],\\
&d_{k+1,k} = \eta_k\eta_{k+1}\left[\epsilon_k h_{k+1} + \varepsilon_k\rho_{k+1}h_{k+2} + \zeta_k\varrho_{k+1}h_{k+3}\right],\\
&d_{kk} = \eta_k^2\left[h_k + \epsilon_k\rho_k h_{k+1} + \varepsilon_k\varrho_k h_{k+2} + \zeta_k\sigma_k h_{k+3}\right],\\
&d_{k,k+1} = \eta_k\eta_{k+1}\left[\rho_k h_{k+1} + \epsilon_{k+1}\varrho_k h_{k+2} + \varepsilon_{k+1}\sigma_k h_{k+3}\right],\\
&d_{k,k+2} = \eta_k\eta_{k+2}\left[\varrho_k h_{k+2} + \epsilon_{k+2}\sigma_k h_{k+3}\right],
\end{aligned} \tag{3.14}$$

where,

$$\begin{aligned}
O_i(j,k,\alpha,\beta) = {}& C_i(j,k,\alpha,\beta) + \epsilon_j C_i(j+1,k,\alpha,\beta)\\
&+ \varepsilon_j C_i(j+2,k,\alpha,\beta) + \zeta_j C_i(j+3,k,\alpha,\beta).
\end{aligned} \tag{3.15}$$

The condition number of a matrix measures the sensitivity of the solution of a system of linear equations to errors in the data. It gives an indication of the accuracy of the results from matrix inversion and the linear equation solution. We can define the 2-norm condition number of a real $n \times n$ nonsingular matrix E as

$$\mathrm{Cond}(E) = \|E\|\left\|E^{-1}\right\|, \tag{3.16}$$

from which one can easily deduce that $\mathrm{Cond}(E) \geq 1$. This quantity gives an idea of the distribution of the eigenvalues of E in the complex plane. When $\mathrm{Cond}(E)$ is large, we generally expect scattered eigenvalues with considerable variation in magnitude. On the other hand, when $\mathrm{Cond}(E)$ is close to 1, the module of the eigenvalues are gathered together in a small interval.

However, at each time level, the Jacobi dual-Petrov-Galerkin approximations presented in this paper lead to systems with small condition numbers and are numerically stable. The system resulted from (3.11) is $E\hat{\mathbf{u}}^{k+1} = \hat{\mathbf{g}}^{k}$, where $E = D + (\Delta t/2)A$. In case of $\alpha = \beta = 0$, E

Table 1: Cond(E) for some different values of α, β.

N	α	β	$\Delta t = 10^{-1}$	$\Delta t = 10^{-2}$
8			3.35	23.89
16			3.36	25.10
32	0	0	3.36	25.12
64			3.36	25.12
128			3.36	25.12
8			3.49	22.80
16			3.53	25.35
32	$-1/4$	$-1/4$	3.54	25.36
64			3.55	25.36
128			3.55	25.36
8			3.80	22.21
16			3.99	25.81
32	$-1/2$	$-1/2$	4.31	25.83
64			4.83	25.84
128			5.53	25.84

is a symmetric banded matrix with lower bandwidth 3 and upper bandwidth 3, and for other values of $\alpha, \beta > -1$, E is lower banded matrix with lower bandwidth 3, so we note that the matrix E is well conditioned for all values of α, β. Hence the propagation of roundoff errors should not be very significant. The numerical examples presented in Section 5 confirm that our algorithms are numerically stable.

For the case of $\Delta t = 10^{-1}$ in system (3.12), one can easily show that the condition number of the matrix E is lower than the case of $\Delta t = 10^{-2}$, for all values of N and $\alpha, \beta > -1$. Table 1 illustrates the condition numbers of the matrix E in (3.12) for some different values of $\alpha, \beta, \Delta t$, and N.

A system of equations is considered to be well conditioned if a small change in the coefficient matrix or a small change in the right-hand side results in a small change in the solution vector. If the condition number is close to one, the matrix is well conditioned which means its inverse can be computed with good accuracy. Regarding system (3.12), an algebraic preconditioning yields a well-conditioned system, which is an improvement with respect to the well-known condition number $O(N^6)$ of collocation methods for third-order differential equations, and often exhibit unstable modes if the collocation points are not properly chosen (see, eg., [38, 39]).

4. Application to the KDV Equation

In this section, as an example of application to nonlinear equations, consider the following Korteweg-de Vries- (KdV-) type equation:

$$\partial_t u + \partial_x G(u) + \partial_x^3 u = 0, \quad x \in I, \ t \in (0, T], \tag{4.1}$$

We study the initial- and boundary-value problem of this equation in the space-time domain $[-1,1]\times[0,T]$ with the following initial and boundary values:

$$\begin{aligned} u(-1,t) &= q_1(t), \\ u(1,t) &= q_2(t), \\ \partial_x u(1,t) &= q_3(t), \quad t\in(0,T], \\ u(x,0) &= u_0(x), \quad x\in[-1,1]. \end{aligned} \tag{4.2}$$

As pointed out in [42, 44], the nonhomogeneous boundary conditions (4.2) can be converted into homogeneous boundary conditions. In fact, converting (4.1)–(4.2) into a problem with homogeneous boundary conditions introduces nonconstant coefficients and other terms into the equation which can be combined to the nonlinear term and the right-hand side of the new problem. Therefore, we shall assume, without loss of generality, that

$$\begin{aligned} \partial_t u + \partial_x F(u) + \partial_x^3 u &= f(x,t), \quad x\in I, t\in(0,T], \\ u(-1,t) = u(1,t) &= \partial_x u(1,t) = 0, \quad t\in(0,T], \\ u(x,0) &= u_0(x), \quad x\in I. \end{aligned} \tag{4.3}$$

We formulate the scheme basically in the above JDPG with general indexes $\alpha,\beta>-1$, but the nonlinear term in the Jacobi collocation method with other two general indexes $\theta,\vartheta>-1$.

The semidiscrete Jacobi dual-Petrov Galerkin and Jacobi collocation (JDPG-JC) scheme for (4.3) is to find $u_N\in W_N$ such that

$$\begin{aligned} &\left(\partial_t u_N(t),\upsilon\right)_{w^{\alpha,\beta}} + \left(\partial_x I_N^{P^{(\theta,\vartheta)}} F(u_N(t)),\upsilon\right)_{w^{\alpha,\beta}} + \left(\partial_x^3 u_N(t),\upsilon\right)_{w^{\alpha,\beta}} = \left(f(t),\upsilon\right)_{w^{\alpha,\beta}}, \quad 0<t<T, \\ &u_N(0) = I_N^{P^{(\theta,\vartheta)}} u_0 \quad \forall \upsilon \in W_N^*. \end{aligned} \tag{4.4}$$

For a given Δt, we set $t_k = k\Delta t$ and let u_N^1 be an appropriate approximation of $u(\cdot,t_1)$, for instance, we can compute u_N^1 using one step of a semi-implicit first-order scheme. Then, the second-order Crank-Nicholson-leap-frog scheme in time with the JDPG-JC approximation in space reads:

For $k=1,2,\ldots,n_T-1$, find $u_N^{k+1}\in W_N$ such that

$$\begin{aligned} &\frac{1}{2\Delta t}\left(u_N^{k+1}-u_N^{k-1},\upsilon\right)_{w^{\alpha,\beta}} + \left(\partial_x I_N^{P^{(\theta,\vartheta)}} F\left(u_N^k(t)\right),\upsilon\right)_{w^{\alpha,\beta}} + \frac{1}{2}\left(\partial_x^3\left(u_N^{k+1}+u_N^{k-1}\right),\upsilon\right)_{w^{\alpha,\beta}} \\ &\quad = \left(f^{k+1/2},\upsilon\right)_{w^{\alpha,\beta}}, \end{aligned} \tag{4.5}$$

consequently

$$\left(u_N^{k+1}, v\right)_{w^{\alpha,\beta}} + \Delta t\left(\partial_x^3 u_N^{k+1}, v\right)_{w^{\alpha,\beta}} = \left(g^k, v\right)_{w^{\alpha,\beta}}, \quad \forall v \in W_N^*, \tag{4.6}$$

where,

$$g^k = 2\Delta t f^{k+1/2} + u_N^{k-1} - \Delta t\partial_x^3 u_N^{k-1} - 2\Delta t\partial_x I_N^{P^{(\theta,\vartheta)}} F\left(u_N^k(t)\right). \tag{4.7}$$

It is clear that at each time step, (4.6) leads to

$$\sum_{i=0}^{N-3}\left[\left(\phi_i(x), \psi_j(x)\right)_{w^{\alpha,\beta}} + \Delta t\,\left(\phi_i^{(3)}(x), \psi_j(x)\right)_{w^{\alpha,\beta}}\right]\widehat{u}_i^{k+1} = \left(g^k, \phi_j(x)\right)_{w^{\alpha,\beta}}, \quad 0 \le j \le N-3. \tag{4.8}$$

The scheme (4.8) is equivalent to the following matrix equation

$$(D + \Delta t A)\widehat{\mathbf{u}}^{k+1} = \widehat{\mathbf{g}}^k, \qquad \widehat{\mathbf{u}}^k = \left[\widehat{u}_0^k, \widehat{u}_1^k, \ldots, \widehat{u}_{N-3}^k\right]^T, \tag{4.9}$$

where the nonzero elements of the matrices A and D are given explicitly in (3.13) and (3.14), respectively.

5. Numerical Results

We report in this section some numerical results obtained with the algorithms presented in the previous sections. We compute the linear model problem by using the JDPG scheme to check the accuracy. Then, we solve the KdV equation and the MKdV equation with the nonperiodic boundary condition by the JDPG-JC scheme. We use single solitary wave propagation and three solitary waves interaction to test the good accuracy of our method.

Example 5.1. Consider the linear model problem

$$\partial_t u + \partial_x^3 u = f(x,t), \quad x \in I,\ t \in (0,1], \tag{5.1}$$

with exact solution

$$u(x,t) = \sin(ax)^2 \sin(bx + ct), \quad -1 < x < 1, \tag{5.2}$$

and the right term is

$$\begin{aligned} f(x,t) = {} & \left[\left(c - b^3\right)\sin(ax)^2 + 6a^2 b\cos(2ax)\right]\cos(bx + c) \\ & - a\left(4a^2 + 3b^2\right)\sin(2ax)\sin(bx + ct). \end{aligned} \tag{5.3}$$

Table 2: Maximum pointwise error of $u - u_N$ using JDPG method for $\alpha = \beta = 0$.

N	α	β	JDPG method			
			$\Delta t = 10^{-1}$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-3}$	Ratio
16			$6.43 \cdot 10^{-1}$	$6.30 \cdot 10^{-1}$	$6.25 \cdot 10^{-1}$	
24	0	0	$7.63 \cdot 10^{-3}$	$2.84 \cdot 10^{-3}$	$2.81 \cdot 10^{-3}$	222.42
32			$5.79 \cdot 10^{-3}$	$4.31 \cdot 10^{-5}$	$9.41 \cdot 10^{-7}$	2986.18

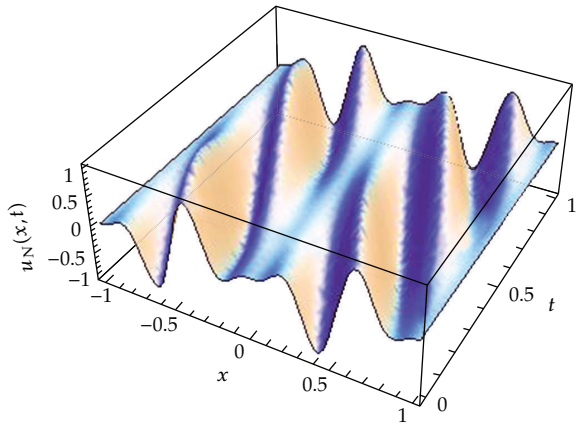

Figure 1: Space-time graph of the spectral solution $u_N(x)$ up to $t = 1s$ with $\Delta t = 10^{-3}$ and $N = 48$ for Example 5.1.

We extract the required initial and boundary functions from the exact solution. To see the accuracy of proposed JDPG method, we let Δt decrease from 10^{-1} to 10^{-3} and N increase from 16 to 32. The parameters $a = \pi$ (to meet the boundary conditions) and $b = c = 12$. The maximum pointwise errors of $u - u_N$, the proposed JDPG method, are given for various choices of α, β, Δt, and N in Tables 2, 3, and 4. We should note that for large values of N, the Legendre polynomial case ($\alpha = \beta = 0$) is always more accurate than the other choices of α and β. The graph of space-time of the approximate solution up to $t = 1$, and $\Delta t = 10^{-3}$, $\alpha = \beta = -1/2$, and $N = 48$ in Figure 1.

The seventh column in Tables 2, 3, and 4 represents the ratio between the errors in two consecutive calculations in the case of $\Delta t = 10^{-3}$, and it can be seen that the spatial error decays rapidly with a rate that is apparently increasing.

Example 5.2. In this example (see [21]), we consider the third-order nonlinear KdV equation, represented by

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0, \quad x \in [30, 80], t \in (0, T], \tag{5.4}$$

with a single solitary wave solution. The initial condition is given by

$$u(x, 0) = \frac{r}{2}\operatorname{sech}^2\left(\frac{\sqrt{r}}{2}x - 10\right), \quad r = 0.14, \tag{5.5}$$

Table 3: Maximum pointwise error of $u - u_N$ using JDPG method for $\alpha = \beta = 0.5$.

N	α	β	JDPG method			
			$\Delta t = 10^{-1}$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-3}$	Ratio
16			$5.78 \cdot 10^{-1}$	$5.75 \cdot 10^{-1}$	$5.62 \cdot 10^{-1}$	
24	1/2	1/2	$6.52 \cdot 10^{-3}$	$5.70 \cdot 10^{-3}$	$5.59 \cdot 10^{-3}$	100.537
32			$5.79 \cdot 10^{-3}$	$4.27 \cdot 10^{-5}$	$1.52 \cdot 10^{-6}$	3677.63

Table 4: Maximum pointwise error of $u - u_N$ using JDPG method for $\alpha = \beta = 1$.

N	α	β	JDPG method			
			$\Delta t = 10^{-1}$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-3}$	Ratio
16			$9.95 \cdot 10^{-1}$	$9.89 \cdot 10^{-1}$	$9.63 \cdot 10^{-1}$	
24	1	1	$2.13 \cdot 10^{-2}$	$1.57 \cdot 10^{-2}$	$8.31 \cdot 10^{-3}$	115.884
32			$5.79 \cdot 10^{-3}$	$4.15 \cdot 10^{-5}$	$8.21 \cdot 10^{-6}$	1012.18

Table 5: $L^2_{w^{\alpha,\beta}}$ and $L^\infty_{w^{\alpha,\beta}}$ errors using JDPG-JC method for $N = 8, 16, 32$, and 48 with $\Delta t = 10^{-1}$ and $t = 1$.

N	α	β	θ	ϑ	$L^2_{w^{\alpha,\beta}}$-error	$L^\infty_{w^{\alpha,\beta}}$-error	α	β	θ	ϑ	$L^2_{w^{\alpha,\beta}}$-error	$L^\infty_{w^{\alpha,\beta}}$-error
8					$2.00 \cdot 10^{-2}$	$1.05 \cdot 10^{-2}$					$1.74 \cdot 10^{-2}$	$5.80 \cdot 10^{-3}$
16	−1/2	−1/2	0	0	$2.36 \cdot 10^{-3}$	$7.06 \cdot 10^{-4}$	0	0	−1/2	−1/2	$1.79 \cdot 10^{-3}$	$2.07 \cdot 10^{-4}$
32					$2.16 \cdot 10^{-5}$	$2.40 \cdot 10^{-7}$					$1.85 \cdot 10^{-5}$	$1.84 \cdot 10^{-7}$
48					$4.67 \cdot 10^{-6}$	$2.28 \cdot 10^{-6}$					$2.83 \cdot 10^{-6}$	$5.40 \cdot 10^{-7}$
8					$1.74 \cdot 10^{-2}$	$5.80 \cdot 10^{-3}$					$1.53 \cdot 10^{-2}$	$3.56 \cdot 10^{-3}$
16	0	0	1/2	1/2	$1.80 \cdot 10^{-3}$	$2.08 \cdot 10^{-4}$	1/2	1/2	−1/2	−1/2	$1.38 \cdot 10^{-3}$	$3.34 \cdot 10^{-5}$
32					$1.85 \cdot 10^{-5}$	$1.79 \cdot 10^{-7}$					$1.61 \cdot 10^{-5}$	$8.19 \cdot 10^{-8}$
48					$2.54 \cdot 10^{-6}$	$1.42 \cdot 10^{-7}$					$2.07 \cdot 10^{-6}$	$7.03 \cdot 10^{-8}$

and the exact solution is

$$u(x,t) = \frac{r}{2}\operatorname{sech}^2\left(\frac{\sqrt{r}}{2}(x - rt) - 10\right), \quad r = 0.14. \tag{5.6}$$

We extract the boundary functions from the exact solution. The $L^\infty_{w^{\alpha,\beta}}$, $L^2_{w^{\alpha,\beta}}$, and $H^1_{w^{\alpha,\beta}}$ and errors are obtained in Tables 5 and 6 for various choices of t, Δt, α, β, θ, ϑ, and N.

The graph of analytical and spectral solutions for $t = 1$, $\Delta t = 10^{-1}$, $\alpha = \beta = 0$, $\theta = \vartheta = 1/4$, and $N = 48$ is given in Figure 2. We also draw the absolute error graph for $t = 1$, $\Delta t = 10^{-1}$, $\alpha = \beta = 0$, $\theta = \vartheta = 1/4$, and $N = 48$ and space-time graph of the approximate solution up to $t = 4$, and $\Delta t = 10^{-3}$, $\alpha = \beta = 0$, $\theta = \vartheta = 1/4$, and $N = 48$ in Figures 3 and 4, respectively.

Example 5.3. Here, we compute by the JDPG-JC method the interaction of three solitons of the KdV equation

$$\partial_t u + u \;\; \partial_x u + \partial_x^3 u = 0. \tag{5.7}$$

The initial condition is given by

$$u(x,t) = \sum_{i=1}^{3} 12\kappa_i^2 \operatorname{sech}^2\left(\kappa_i\left(x - 4\kappa_i^2 t - \tau_i\right)\right), \quad -90 < x < 90, \; t = 0. \tag{5.8}$$

Table 6: $L^2_{w^{\alpha,\beta}}$, $L^\infty_{w^{\alpha,\beta}}$, and $H^1_{w^{\alpha,\beta}}$ errors using JDPG-JC method for Example 5.2.

N	α	β	θ	ϑ	$t = 0.5$ with $\Delta t = 5 \times 10^{-3}$			$t = 1$ with $\Delta t = 1 \times 10^{-2}$		
					$L^2_{w^{\alpha,\beta}}$-error	$L^\infty_{w^{\alpha,\beta}}$-error	$H^1_{w^{\alpha,\beta}}$-error	$L^2_{w^{\alpha,\beta}}$-error	$L^\infty_{w^{\alpha,\beta}}$-error	$H^1_{w^{\alpha,\beta}}$-error
8					$1.52 \cdot 10^{-2}$	$3.36 \cdot 10^{-3}$	$1.64 \cdot 10^{-2}$	$1.52 \cdot 10^{-2}$	$3.58 \cdot 10^{-3}$	$1.64 \cdot 10^{-2}$
16	$-1/2$	$-1/2$	$1/2$	$1/2$	$1.29 \cdot 10^{-3}$	$4.47 \cdot 10^{-6}$	$1.89 \cdot 10^{-3}$	$1.35 \cdot 10^{-3}$	$3.16 \cdot 10^{-5}$	$1.85 \cdot 10^{-3}$
32					$1.45 \cdot 10^{-5}$	$6.53 \cdot 10^{-8}$	$2.50 \cdot 10^{-5}$	$1.62 \cdot 10^{-5}$	$1.97 \cdot 10^{-7}$	$2.75 \cdot 10^{-5}$

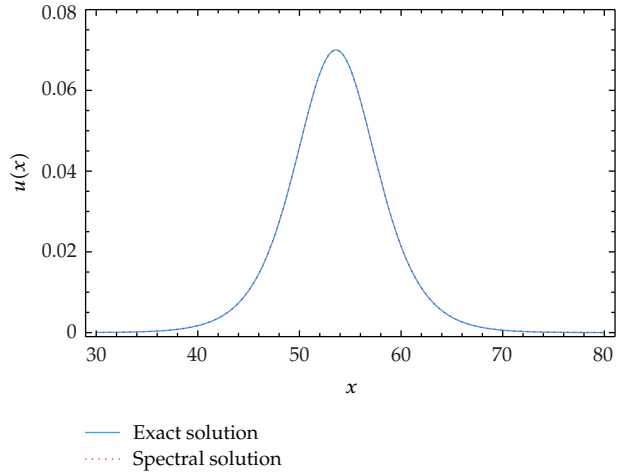

Figure 2: Analytical and spectral solutions in $t = 1s$, $\Delta t = 10^{-1}$, and $N = 48$ for Example 5.2.

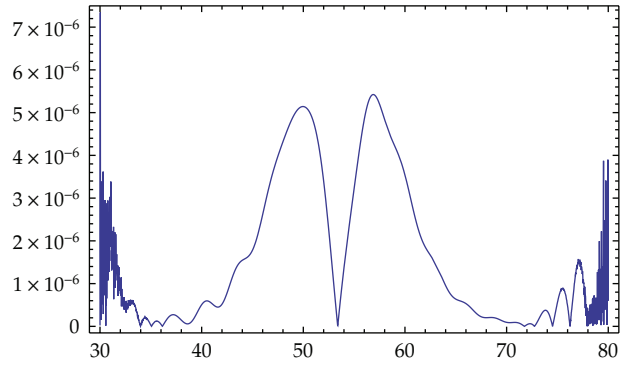

Figure 3: Absolute value of $u(x) - u_N(x)$ at $t = 1s$, $\Delta t = 10^{-1}$, and $N = 48$ for Example 5.2.

The parameters are taken as follows:

$$\begin{aligned} \kappa_1 &= 0.3, \quad \kappa_2 = 0.25, \quad \kappa_3 = 0.2, \\ \tau_1 &= -60, \quad \tau_2 = -44, \quad \tau_3 = -26. \end{aligned} \tag{5.9}$$

The boundary functions can be extracted from the exact solution.

The $L^\infty_{w^{\alpha,\beta}}$, $L^2_{w^{\alpha,\beta}}$, and $H^1_{w^{\alpha,\beta}}$ and errors are obtained in Table 7 at $t = 1$ for $\Delta t = 0.1$, $N = 40$ with various choices of α, β, θ, and ϑ.

Table 7: $L^2_{w^{\alpha,\beta}}$, $L^\infty_{w^{\alpha,\beta}}$, and $H^1_{w^{\alpha,\beta}}$ errors using JDPG-JC method for $t = 1$, $\Delta t = 0.1$, and $N = 40$.

N	α	β	θ	ϑ	$L^2_{w^{\alpha,\beta}}$-error	$L^\infty_{w^{\alpha,\beta}}$-error	$H^1_{w^{\alpha,\beta}}$-error
	0	0	1/2	1/2	$5.95 \cdot 10^{-2}$	$4.76 \cdot 10^{-3}$	$7.37 \cdot 10^{-2}$
40	0	0	1	1	$5.95 \cdot 10^{-2}$	$4.73 \cdot 10^{-3}$	$7.34 \cdot 10^{-2}$
	1/2	1/2	−1/2	−1/2	$5.23 \cdot 10^{-2}$	$3.60 \cdot 10^{-3}$	$6.15 \cdot 10^{-2}$

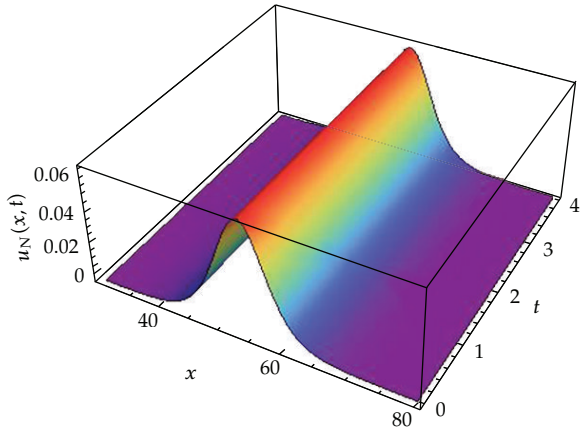

Figure 4: Space-time graph of the spectral solution $u_N(x)$ up to $t = 4s$ with $\Delta t = 10^{-3}$, $\alpha = \beta = 0$, $\theta = \vartheta = 1/4$ and $N = 48$ for Example 5.2.

6. Conclusion

In this paper, we have presented some efficient direct solvers for the linear third-order differential equations by using Jacobi dual Petrov-Galerkin approximation with general parameters α, $\beta > -1$. The modified Galerkin method is suitable for the unbalanced third-order differential equations. Moreover, we developed a new approach implementing Jacobi dual-Petrov Galerkin-Jacobi collocation method in space combined with Crank-Nicholson-leap-frog method in time such that at each time step only a sparse banded linear algebraic system needs to be solved. The numerical results given in the previous section demonstrate the good accuracy of these algorithms.

Acknowledgments

The authors would like to thank the editor and the reviewers for their constructive comments and suggestions to improve the quality of the paper. This study was supported by the Deanship of Scientific Research of Northern Border University.

References

[1] J. L. Bona, V. A. Dougalis, O. A. Karakashian, and W. R. McKinney, "Conservative, high-order numerical schemes for the generalized Korteweg-de Vries equation," *Philosophical Transactions of the Royal Society of London A*, vol. 351, no. 1695, pp. 107–164, 1995.

[2] G. F. Carey and Y. Shen, "Approximations of the KdV equation by least squares finite elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 93, no. 1, pp. 1–11, 1991.
[3] R. C. Cascaval, "Variable coefficient KdV equations and waves in elastic tubes," in *Evolution Equations Lecture Notes in Pure and Applied Mathematics*, vol. 234 of *Lecture Notes in Pure and Appl. Math.*, pp. 57–69, Dekker, New York, NY, USA, 2003.
[4] N. J. Zabusky and M. D. Kruskal, "Interaction of "solitons" in a collisionless plasma and the recurrence of initial states," *Physical Review Letters*, vol. 15, no. 6, pp. 240–243, 1965.
[5] D. G. Crighton, "Applications of KdV," *Acta Applicandae Mathematicae*, vol. 39, no. 1-3, pp. 39–67, 1995, KdV '95 (Amsterdam, 1995).
[6] G. C. Das and J. Sarma, "A new mathematical approach for finding the solitary waves in dusty plasma," *Physics of Plasmas*, vol. 6, no. 11, pp. 4394–4397, 1999.
[7] A. R. Osborne, "The inverse scattering transform: tools for the nonlinear Fourier analysis and filtering of ocean surface waves," *Chaos, Solitons and Fractals*, vol. 5, no. 12, pp. 2623–2637, 1995, Solitons in science and engineering: theory and applications.
[8] L. Ostrovsky and Y. A. Stepanyants, "Do internal solitons exist in the ocean?" *Reviews of Geophysics*, vol. 27, pp. 293–310, 1989.
[9] A. Ludu and J. P. Draayer, "Nonlinear modes of liquid drops as solitary waves," *Physical Review Letters*, vol. 80, no. 10, pp. 2125–2128, 1998.
[10] L. Reatto, "What is a roton?" *International Journal of Modern Physics B*, vol. 13, no. 5-6, pp. 607–616, 1999.
[11] S. K. Turitsyn, A. B. Aceves, C. K. R. T. Jones, and V. Zharnitsky, "Average dynamics of the optical soliton in communication lines with dispersion management: analytical results," *Physical Review E*, vol. 58, no. 1, pp. R48–R51, 1998.
[12] M. W. Coffey, "Nonlinear dynamics of vortices in ultraclean type-II superconductors: integrable wave equations in cylindrical geometry," *Physical Review B*, vol. 54, no. 2, pp. 1279–1285, 1996.
[13] C. Q. Dai and J. F. Zhang, "Application of he's EXP-function method to the stochastic mKdV equation," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 5, pp. 675–680, 2009.
[14] J.-H. He and X.-H. Wu, "Exp-function method for nonlinear wave equations," *Chaos, Solitons and Fractals*, vol. 30, no. 3, pp. 700–708, 2006.
[15] A.-M. Wazwaz, "Two reliable methods for solving variants of the KdV equation with compact and noncompact structures," *Chaos, Solitons and Fractals*, vol. 28, no. 2, pp. 454–462, 2006.
[16] H. Triki, T. R. Taha, and A.-M. Wazwaz, "Solitary wave solutions for a generalized KdV-mKdV equation with variable coefficients," *Mathematics and Computers in Simulation*, vol. 80, no. 9, pp. 1867–1873, 2010.
[17] D. Pavoni, "Single and multidomain Chebyshev collocation methods for the Korteweg-de Vries equation," *Calcolo*, vol. 25, no. 4, pp. 311–346, 1988.
[18] W. Heinrichs, "Spectral approximation of third-order problems," *Journal of Scientific Computing*, vol. 14, no. 3, pp. 275–289, 1999.
[19] H. Ma and W. Sun, "A Legendre-Petrov-Galerkin and Chebyshev collocation method for third-order differential equations," *SIAM Journal on Numerical Analysis*, vol. 38, no. 5, pp. 1425–1438, 2000.
[20] J. O. Skogestad and H. Kalisch, "A boundary value problem for the KdV equation: comparison of finite-difference and Chebyshev methods," *Mathematics and Computers in Simulation*, vol. 80, no. 1, pp. 151–163, 2009.
[21] M. Dehghan and A. Shokri, "A numerical method for KdV equation using collocation and radial basis functions," *Nonlinear Dynamics*, vol. 50, no. 1-2, pp. 111–120, 2007.
[22] A. H. Bhrawy and W. M. Abd-Elhameed, "New algorithm for the numerical solutions of nonlinear third-order differential equations using Jacobi-Gauss collocation method," *Mathematical Problems in Engineering*, vol. 2011, Article ID 837218, 14 pages, 2011.
[23] A. H. Bhrawy and A. S. Alofi, "A Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 62–70, 2012.
[24] A. H. Bhrawy, A. S. Alofi, and S. I. El-Soubhy, "Spectral shifted Jacobi tau and collocation methods for solving fifth-order boundary value problems," *Abstract and Applied Analysis*, vol. 2011, Article ID 823273, 14 pages, 2011.
[25] E. H. Doha and A. H. Bhrawy, "Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials," *Applied Numerical Mathematics*, vol. 58, no. 8, pp. 1224–1244, 2008.

[26] E. H. Doha, W. M. Abd-Elhameed, and A. H. Bhrawy, "Efficient spectral ultraspherical-Galerkin algorithms for the direct solution of 2nth-order linear differential equations," *Applied Mathematical Modelling*, vol. 33, no. 4, pp. 1982–1996, 2009.
[27] K. Aghigh, M. Masjed-Jamei, and M. Dehghan, "A survey on third and fourth kind of Chebyshev polynomials and their applications," *Applied Mathematics and Computation*, vol. 199, no. 1, pp. 2–12, 2008.
[28] M. Fernandino, C. A. Dorao, and H. A. Jakobsen, "Jacobi galerkin spectral method for cylindrical and spherical geometries," *Chemical Engineering Science*, vol. 62, no. 23, pp. 6777–6783, 2007.
[29] A. H. Bhrawy, A. S. Alofi, and S. I. El-Soubhy, "An extension of the Legendre-Galerkin method for solving sixth-order differential equations with variable polynomial coefficients," *Mathematical Problems in Engineering*, vol. 2012, Article ID 896575, 13 pages, 2012.
[30] E. H. Doha and A. H. Bhrawy, "An efficient direct solver for multidimensional elliptic robin boundary value problems using a legendre Spectral-Galerkin method," *Computers and Mathematics with Applications*, vol. 64, no. 4, pp. 558–571, 2012.
[31] E. H. Doha and W. M. Abd-Elhameed, "Efficient solutions of multidimensional sixth-order boundary value problems using symmetric generalized Jacobi-Galerkin method," *Abstract and Applied Analysis*, vol. 2012, Article ID 749370, 19 pages, 2012.
[32] E. H. Doha, A. H. Bhrawy, and R. M. Hafez, "On shifted Jacobi spectral method for high-order multi-point boundary value problems," *Communications in Nonlinear Scienceand Numerical Simulation*, vol. 17, pp. 3802–3810, 2012.
[33] E. H. Doha, A. H. Bhrawy, and R. M. Hafez, "A Jacobi dual-Petrov-Galerkin method for solving some odd-order ordinary differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 947230, 21 pages, 2011.
[34] Y. Maday and A. Quarteroni, "Error analysis for spectral approximation of the Korteweg-de Vries equation," *RAIRO Modélisation Mathématique et Analyse Numérique*, vol. 22, no. 3, pp. 499–529, 1988.
[35] H. Kalisch, "Rapid convergence of a Galerkin projection of the KdV equation," *Comptes Rendus Mathématique*, vol. 341, no. 7, pp. 457–460, 2005.
[36] M. Bjørkavåg and H. Kalisch, "Exponential convergence of a spectral projection of the KdV equation," *Physics Letters A*, vol. 365, no. 4, pp. 278–283, 2007.
[37] H. Kalisch and X. Raynaud, "On the rate of convergence of a collocation projection of the KdV equation," *M2AN. Mathematical Modelling and Numerical Analysis*, vol. 41, no. 1, pp. 95–110, 2007.
[38] W. J. Merryfield and B. Shizgal, "Properties of collocation third-derivative operators," *Journal of Computational Physics*, vol. 105, no. 1, pp. 182–185, 1993.
[39] W. Z. Huang and D. M. Sloan, "The pseudospectral method for third-order differential equations," *SIAM Journal on Numerical Analysis*, vol. 29, no. 6, pp. 1626–1647, 1992.
[40] W. S. Don and D. Gottlieb, "The Chebyshev-Legendre method: implementing Legendre methods on Chebyshev points," *SIAM Journal on Numerical Analysis*, vol. 31, no. 6, pp. 1519–1534, 1994.
[41] H. Ma, "Chebyshev-Legendre super spectral viscosity method for nonlinear conservation laws," *SIAM Journal on Numerical Analysis*, vol. 35, no. 3, pp. 893–908, 1998.
[42] E. H. Doha, A. H. Bhrawy, and R. M. Hafez, "A Jacobi-Jacobi dual-Petrov-Galerkin method for third- and fifth-order differential equations," *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1820–1832, 2011.
[43] J.-M. Yuan and J. Wu, "A dual-Petrov-Galerkin method for two integrable fifth-order KdV type equations," *Discrete and Continuous Dynamical Systems A*, vol. 26, no. 4, pp. 1525–1536, 2010.
[44] J. Shen, "A new dual-Petrov-Galerkin method for third and higher odd-order differential equations: application to the KdV equation," *SIAM Journal on Numerical Analysis*, vol. 41, no. 5, pp. 1595–1619, 2003.
[45] J.-M. Yuan, J. Shen, and J. Wu, "A dual-Petrov-Galerkin method for the Kawahara-type equations," *Journal of Scientific Computing*, vol. 34, no. 1, pp. 48–63, 2008.
[46] B.-y. Guo and L.-l. Wang, "Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces," *Journal of Approximation Theory*, vol. 128, no. 1, pp. 1–41, 2004.
[47] E. H. Doha, "On the construction of recurrence relations for the expansion and connection coefficients in series of Jacobi polynomials," *Journal of Physics*, vol. 37, no. 3, pp. 657–675, 2004.