Research Article

Existence Results for Fractional Differential Inclusions with Multivalued Term Depending on Lower-Order Derivative

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This paper is concerned with a class of fractional differential inclusions whose multivalued term depends on lower-order fractional derivative with fractional (non)separated boundary conditions. The cases of convex-valued and non-convex-valued right-hand sides are considered. Some existence results are obtained by using standard fixed point theorems. A possible generalization for the inclusion problem with integral boundary conditions is also discussed. Examples are given to illustrate the results.

1. Introduction

Recently, the subject of fractional differential equations has emerged as an important area of investigation. Indeed, we can find numerous applications of fractional-order derivatives in the mathematical modeling of physical and biological phenomena in various fields of science and engineering [1–3]. A variety of results on initial and boundary value problems of fractional differential equations and inclusions can easily be found in the literature on this topic. For some recent results, we can refer to, for instance, [4–20] (equations) [21–27] (inclusions) and the references therein.

Ahmad and Ntouyas [22] considered a boundary value problem of fractional differential inclusions with fractional separated boundary conditions given by

$$^{c}D^{q}x(t) \in F(t, x(t)), \quad t \in [0,1], \ 1 < q \le 2,$$

$$\alpha_1 x(0) + \beta_1 \left(^{c}D^{p}x(0)\right) = \gamma_1, \qquad \alpha_2 x(1) + \beta_2 \left(^{c}D^{p}x(1)\right) = \gamma_2, \quad 0 < p < 1, \tag{1.1}$$

where $^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,1] \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a multivalued map, and $\alpha_i, \beta_i, \gamma_i$ $(i = 1,2)$ are real constants, with $\alpha_1 \neq 0$.

In Cernea [24], the following multipoint boundary value problem for a fractional-order differential inclusion was studied

$$D^\alpha x(t) \in F(t, x(t), x'(t)) \quad \text{a.e. } t \in [0,1], \ 2 < \alpha \leq 3,$$

$$x(0) = x'(0) = 0, \qquad x(1) - \sum_{i=1}^{m} a_i x(\xi_i) = \lambda, \tag{1.2}$$

where D^α is the standard Riemann-Liouville fractional derivative, $m \geq 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\sum_{i=1}^{m} a_i \xi_i^{\alpha-1} < 1$, $\lambda > 0$, $a_i > 0$, $i = 1, 2, \ldots, m$, and $F : [0,1] \times \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a multivalued map.

In Khan et al. [11], the authors studied the existence and uniqueness results of nonlinear fractional differential equation of the type

$$^cD^q x(t) = f(t, x(t), {}^cD^\sigma x(t)), \quad t \in [0,T],$$

$$\alpha x(0) - \beta x'(0) = \int_0^T g(s,x)ds, \qquad \gamma x(T) + \delta x'(T) = \int_0^T h(s,x)ds, \tag{1.3}$$

where $0 < \sigma < 1$, $1 < q < 2$, $\alpha, \delta > 0$, $\beta, \gamma \geq 0$ (or $\alpha, \delta \geq 0$, $\beta, \gamma > 0$) and $^cD^q, {}^cD^\sigma$ are the Caputo fractional derivatives. The results in [11, 22, 24] are obtained by using appropriate standard fixed point theorems.

Motivated by the papers cited above, in this paper, we consider the existence results for a new class of fractional differential inclusions of the form

$$^cD^\alpha x(t) \in F(t, x(t), {}^cD^\beta x(t)), \quad \text{a.e. } t \in [0,T], \tag{1.4}$$

where $^cD^\alpha$ denotes the Caputo fractional derivative of order α, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a multivalued map, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, and $T > 0$. We study (1.4) subject to two families of boundary conditions:

(1) separated boundary conditions

$$a_1 x(0) + b_1({}^cD^\gamma x(0)) = c_1, \qquad a_2 x(T) + b_2({}^cD^\gamma x(T)) = c_2, \tag{1.5}$$

(2) Nonseparated boundary conditions

$$a_1 x(0) + b_1 x(T) = c_1, \qquad a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2, \tag{1.6}$$

where a_i, b_i, c_i, $i = 1,2$ are real constants and $0 < \gamma < 1$.

The results of this paper can easily to be generalized to the boundary value problems of fractional differential inclusions (1.4) with the following integral boundary conditions:

$$a_1 x(0) + b_1 \left({}^c D^\gamma x(0)\right) = c_1 \int_0^T g(s, x(s)) ds,$$

$$a_2 x(T) + b_2 \left({}^c D^\gamma x(T)\right) = c_2 \int_0^T h(s, x(s)) ds, \tag{1.7}$$

$$a_1 x(0) + b_1 x(T) = c_1 \int_0^T g(s, x(s)) ds,$$

$$a_2 \left({}^c D^\gamma x(0)\right) + b_2 \left({}^c D^\gamma x(T)\right) = c_2 \int_0^T h(s, x(s)) ds, \tag{1.8}$$

where $g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given functions.

We remark that when the third variable of the multifunction F in (1.4) vanishes, the problem (1.4), (1.5) reduces to the case considered in [22]. When $a_1 = b_1 = 1$, $a_2 = b_2 = 1$, and $c_1 = c_2 = 0$, the problem (1.4), (1.6) reduces to an antiperiodic fractional boundary value problem (the case of $F = f$ a given continuous function was studied in [4, 15]). Our results generalize some results from the literature cited above and constitute a contribution to this emerging field of research.

The rest of the paper is organized as follows: in Section 2 we present the notations and definitions and give some preliminary results that we need in the sequel, Section 3 is dedicated to the existence results of the fractional differential inclusion (1.4) with boundary conditions (1.5) and (1.6), in Section 4 we indicate a possible generalization for the inclusion problem (1.4) with integral boundary conditions (1.7) and (1.8), and two illustrative examples are given in Section 5.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $(X, \| \cdot \|)$ be a normed space. We use the notations $P(X) = \{Y \subseteq X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$, $P_{cp,c}(X) = \{Y \in P(X) : Y \text{ compact, convex}\}$, and so on.

Let $A, B \in P_{cl}(X)$; the Pompeiu-Hausdorff distance of A, B is defined as

$$h(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}. \tag{2.1}$$

A multivalued map $F : X \to P(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$. F is called upper semicontinuous on X, if for every $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and for every open set O of X containing $F(x_0)$, there exists an open neighborhood U_0 of x_0 such that $F(U_0) \subseteq O$. Equivalently, F is upper semicontinuous if the set $\{x \in X : F(x) \subseteq O\}$ is open for any open set O of X. F is called lower semicontinuous

if the set $\{x \in X : F(x) \cap O \neq \emptyset\}$ is open for each open set O in X. If a multivalued map F is completely continuous with nonempty compact values, then F is upper semicontinuous if and only if F has a closed graph, that is, if $x_n \to x_*$ and $y_n \to y_*$, then $y_n \in F(x_n)$ implies $y_* \in F(x_*)$ [28].

A multivalued map $F : [0,T] \to P_{cl}(X)$ is said to be measurable if, for every $x \in X$, the function $t \to d(x, F(t)) = \inf\{d(x,y) : y \in F(t)\}$ is a measurable function.

Definition 2.1. A multivalued map $F : X \to P_{cl}(X)$ is called

(1) γ-Lipschitz if there exists $\gamma > 0$ such that

$$h(F(x), F(y)) \leq \gamma d(x,y), \quad \text{for each } x, y \in X, \tag{2.2}$$

(2) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Definition 2.2. A multivalued map $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is said to be Carathéodory if

(1) $t \to F(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$;

(2) $(x,y) \to F(t,x,y)$ is upper semicontinuous for a.e. $t \in [0,T]$.

Further, a Carathéodory function F is said to be L^1- Carathéodory if

(3) for each $l > 0$, there exists $\varphi_l \in L^1([0,T], \mathbb{R}^+)$ such that

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \leq \varphi_l(t) \tag{2.3}$$

for all $|x| \leq l$, $|y| \leq l$ and a.e. $t \in [0,T]$.

Lemma 2.3 (see [29]). *Let X be a Banach space. Let $G : [0,T] \times X \to P_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and Γ a linear continuous map from $L^1([0,T], X)$ to $C([0,T], X)$, then the operator*

$$\Gamma \circ S_G : C([0,T], X) \longrightarrow P_{cp,c}(C([0,T], X)), \qquad y \longmapsto (\Gamma \circ S_G)(y) = \Gamma(S_{G,y}) \tag{2.4}$$

is a closed graph operator in $C([0,T], X) \times C([0,T], X)$.

Here $S_{G,y} = \{v \in L^1([0,T], X) : v(t) \in G(t, y(t)) \text{ for a.e. } t \in [0,T]\}$.

Definition 2.4 (see [30]). The Riemann-Liouville fractional integral of order q for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad q > 0, \tag{2.5}$$

provided the integral exists.

Definition 2.5 (see [30]). For at least n-times differentiable function f, the Caputo derivative of order q is defined as

$$^cD^qf(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \quad n-1 < q < n, \ n = [q] + 1, \tag{2.6}$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.6 (see [20]). *Let $\alpha > 0$; then the differential equation*

$$^cD^\alpha h(t) = 0 \tag{2.7}$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ and

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}; \tag{2.8}$$

here $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1$.

The following lemma obtained in [6] is useful in the rest of the paper.

Lemma 2.7 (see [6]). *For a given $y \in C([0,T], \mathbb{R})$, the unique solution of the fractional separated boundary value problem*

$$^cD^\alpha x(t) = y(t), \quad t \in [0,T], \ 1 < \alpha \leq 2,$$
$$a_1 x(0) + b_1(^cD^\gamma x(0)) = c_1, \qquad a_2 x(T) + b_2(^cD^\gamma x(T)) = c_2, \quad 0 < \gamma < 1, \tag{2.9}$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds$$
$$- \frac{t}{v_1} \left(a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \right) + v_2 t + \frac{c_1}{a_1}, \tag{2.10}$$

where

$$v_1 = \frac{a_2 T\Gamma(2-\gamma) + b_2 T^{1-\gamma}}{\Gamma(2-\gamma)}, \qquad v_2 = \frac{a_1 c_2 - a_2 c_1}{a_1 v_1}. \tag{2.11}$$

We notice that the solution (2.10) of the problem (2.9) does not depend on the parameter b_1, that is to say, the parameter b_1 is of arbitrary nature for this problem. And by (2.10), we should assume that $a_1 \neq 0$ and $a_2 T^\gamma \Gamma(2-\gamma) \neq -b_2$.

Lemma 2.8. *For any $y \in C([0,T], \mathbb{R})$, the unique solution of the fractional nonseparated boundary value problem*

$$
{}^cD^\alpha x(t) = y(t), \quad t \in [0,T], \ 1 < \alpha \le 2,
$$
$$
a_1 x(0) + b_1 x(T) = c_1, \qquad a_2\left({}^cD^\gamma x(0)\right) + b_2\left({}^cD^\gamma x(T)\right) = c_2, \quad 0 < \gamma < 1,
$$
(2.12)

is given by

$$
\begin{aligned}
x(t) = {}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds + \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \\
& - \frac{b_1}{a_1 + b_1}\left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \right) \\
& - \frac{1}{a_1 + b_1}\left(\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1 \right).
\end{aligned}
$$
(2.13)

Proof. For $1 < \alpha \le 2$, by Lemma 2.6, we know that the general solution of the equation ${}^cD^\alpha x(t) = y(t)$ can be written as

$$
x(t) = I^\alpha y(t) - k_1 - k_2 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 t,
$$
(2.14)

where $k_1, k_2 \in \mathbb{R}$ are arbitrary constants. Since ${}^cD^\gamma k = 0$ (k is a constant), ${}^cD^\gamma t = t^{1-\gamma}/\Gamma(2-\gamma)$, ${}^cD^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t)$ (see [30]), from (2.14), we have

$$
{}^cD^\gamma x(t) = I^{\alpha-\gamma} y(t) - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)} = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)}.
$$
(2.15)

Using the boundary conditions, we obtain

$$
a_1(-k_1) + b_1\left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 T \right) = c_1,
$$
$$
a_2 \times 0 + b_2\left(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 T^{1-\gamma}}{\Gamma(2-\gamma)} \right) = c_2.
$$
(2.16)

Therefore, we have

$$
\begin{aligned}
k_1 &= \frac{1}{a_1 + b_1}\left(\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1\right) \\
&\quad + \frac{b_1}{a_1 + b_1}\left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - T^\gamma \Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds\right), \\
k_2 &= \frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\left(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{c_2}{b_2}\right).
\end{aligned}
\tag{2.17}
$$

Substituting the values of k_1, k_2 in (2.14), we obtain (2.13). This completes the proof. $\qquad\square$

From the proof of the above lemma, we notice that the solution (2.13) of the problem (2.12) does not depend on the parameter a_2, that is to say, the parameter a_2 is of arbitrary nature for this problem. In this situation, we need to assume that $a_1 + b_1 \neq 0$ and $b_2 \neq 0$.

Let us define what we mean by a solution of the problem (1.4), (1.5) and the problem (1.4), (1.6).

Definition 2.9. A function $x \in AC^1([0,T],\mathbb{R})$ is a solution of the problem (1.4), (1.5) if it satisfies the boundary conditions (1.5) and there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t),{}^cD^\beta x(t))$ a.e. on $t \in [0,T]$ and

$$
\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
&\quad - \frac{t}{v_1}\left(a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s)ds\right) + v_2 t + \frac{c_1}{a_1}.
\end{aligned}
\tag{2.18}
$$

Definition 2.10. A function $x \in AC^1([0,T],\mathbb{R})$ is a solution of the problem (1.4), (1.6) if it satisfies the boundary conditions (1.6) and there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t),{}^cD^\beta x(t))$ a.e. on $t \in [0,T]$ and

$$
\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s)ds + \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \\
&\quad - \frac{b_1}{a_1+b_1}\left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds - T^\gamma \Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s)ds\right) \\
&\quad - \frac{1}{a_1+b_1}\left(\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1\right).
\end{aligned}
\tag{2.19}
$$

Let $C([0,T],\mathbb{R})$ be the space of all continuous functions defined on $[0,T]$. Define the space $\mathcal{X} = \{x : x \text{ and } {}^cD^\beta x \in C([0,T],\mathbb{R})\}$ $(0 < \beta \leq 1)$ endowed with the norm $\|x\| = \max_{t\in[0,T]}|x(t)| + \max_{t\in[0,T]}|{}^cD^\beta x(t)|$. We know that $(\mathcal{X}, \|\cdot\|)$ is a Banach space (see [14]).

We end this section with two fixed point theorems, which will be used in the sequel.

Theorem 2.11 (nonlinear alternative of Leray-Schauder type [31]). *Let X be a Banach space, C a closed convex subset of X, U an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,c}(C)$ is an upper semicontinuous compact map. Then either (1) F has a fixed point in $\overline{U}$, or (2) there is a $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x \in \lambda F(x)$.*

Theorem 2.12 (Covitz and Nadler Jr. [32]). *Let (X, d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then F has a fixed point.*

3. Existence Results

In this section, we will give some existence results for the problems (1.4), (1.5) and (1.4), (1.6).

For each $x \in \mathcal{X}$, define the set of selections of F by

$$S_{F,x} = \left\{ v \in L^1([0, T], \mathbb{R}) : v(t) \in F\left(t, x(t), {}^c D^\beta x(t)\right) \text{ for a.e. } t \in [0, T] \right\}. \tag{3.1}$$

In view of Lemmas 2.7 and 2.8, we define operators $N, M : \mathcal{X} \to P(\mathcal{X})$ as

$$N(x) = \{ h \in \mathcal{X} : h(t) = (Su)(t), \ u \in S_{F,x} \}, \tag{3.2}$$

$$M(x) = \{ h \in \mathcal{X} : h(t) = (Ku)(t), \ u \in S_{F,x} \} \tag{3.3}$$

with

$$
\begin{aligned}
(Su)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds \\
&\quad - \frac{t}{v_1} \left(a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} u(s) ds \right) + v_2 t + \frac{c_1}{a_1}, \\
(Ku)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} u(s) ds + \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \\
&\quad - \frac{b_1}{a_1+b_1} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds - T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} u(s) ds \right) \\
&\quad - \frac{1}{a_1+b_1} \left(\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1 \right).
\end{aligned}
\tag{3.4}
$$

It is clear that if $x \in \mathcal{X}$ is a fixed point of the operator N (the operator M), then x is a solution of the problem (1.4), (1.5) (the problem (1.4), (1.6)).

Now we are in a position to present our main results. The methods used to prove the existence results are standard; however, their exposition in the framework of problems (1.4), (1.5) and (1.4), (1.6) is new.

3.1. Convex Case

We consider first the case when F is convex valued.

(H1): (1) $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map; (2) there exist $m \in L^\infty([0,T], \mathbb{R}^+)$ and $\varphi, \psi : [0, \infty) \to (0, \infty)$ continuous, nondecreasing such that

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \le m(t)(\varphi(|x|) + \psi(|y|)) \tag{3.5}$$

for $x, y \in \mathbb{R}$ and a.e. $t \in [0,T]$.

Theorem 3.1. *Assume that (H1) is satisfied and there exists $L > 0$ such that*

$$\frac{L}{P + (\varphi(L) + \psi(L))\|m\|_{L^\infty}Q} > 1, \tag{3.6}$$

where

$$P = |v_2|T + \frac{|c_1|}{|a_1|} + \frac{|v_2|T^{1-\beta}}{\Gamma(2-\beta)}, \tag{3.7}$$

$$Q = \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|v_1|}\left(1 + \frac{T^{-\beta}}{\Gamma(2-\beta)}\right)\left(\frac{|a_2|T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2|T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right).$$

Then the problem (1.4), (1.5) has at least one solution on $[0,T]$.

Proof. Consider the operator $N : \mathcal{X} \to P(\mathcal{X})$ defined by (3.2). From (H1), we have for each $x \in \mathcal{X}$, the set $S_{F,x}$ is nonempty [29]. For $x \in \mathcal{X}$, let $u \in S_{F,x}$ and $h = Su$, that is, $h \in N(x)$, we have

$$^cD^\beta h(t) = \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}u(s)ds - \frac{kt^{1-\beta}}{\Gamma(2-\beta)}, \tag{3.8}$$

where k is a constant given by

$$k = \frac{1}{v_1}\left(a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}u(s)ds\right) - v_2. \tag{3.9}$$

Hence we know that the operator $N : \mathcal{X} \to P(\mathcal{X})$ is well defined.

We put $Su = S_1u + S_2u$ where

$$(S_1u)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds, \qquad (S_2u)(t) = -k_u t + \frac{c_1}{a_1}. \tag{3.10}$$

Here k_u means that the constant k defined by (3.9) is related to u.

We will show that N satisfies the requirements of the nonlinear alternative of Leray-Schauder type. The proof will be given in five steps.

Step 1 ($N(x)$ is convex valued). Since F is convex valued, we know that $S_{F,x}$ is convex and therefore it is obvious that $N(x)$ is convex for each $x \in \mathcal{X}$.

Step 2 (N maps bounded sets into bounded sets in $\mathcal{X}$). Let B_r be a bounded subset of $\mathcal{X}$ such that for any $x \in B_r$, $\|x\| \le r$, $r > 0$. We prove that there exists a constant $l > 0$ such that for each $x \in B_r$, one has $\|h\| \le l$ for each $h \in N(x)$. Let $x \in B_r$ and $h \in N(x)$, then there exists $u \in S_{F,x}$ such that

$$h(t) = (Su)(t) \quad \text{for } t \in [0, T]. \tag{3.11}$$

By simple calculations, we have

$$|(S_1 u)(t)| \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds \le (\varphi(r) + \psi(r)) \|m\|_{L^\infty} \frac{T^\alpha}{\Gamma(\alpha+1)},$$

$$|(S_2 u)(t)| \le \frac{(\varphi(r) + \psi(r)) \|m\|_{L^\infty} T}{|v_1|} \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) + |v_2| T + \frac{|c_1|}{|a_1|},$$

$$\left| {}^c D^\beta h(t) \right| \le \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |u(s)| ds + \frac{|k_u| T^{1-\beta}}{\Gamma(2-\beta)} \tag{3.12}$$

$$\le (\varphi(r) + \psi(r)) \|m\|_{L^\infty} \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{|v_2| T^{1-\beta}}{\Gamma(2-\beta)}$$

$$+ \frac{(\varphi(r) + \psi(r)) \|m\|_{L^\infty} T^{1-\beta}}{|v_1| \Gamma(2-\beta)} \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right).$$

Hence we obtain

$$\|h\| \le |v_2| T + \frac{|c_1|}{|a_1|} + \frac{|v_2| T^{1-\beta}}{\Gamma(2-\beta)} + (\varphi(r) + \psi(r)) \|m\|_{L^\infty}$$

$$\times \left(\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \left(\frac{T}{|v_1|} + \frac{T^{1-\beta}}{|v_1| \Gamma(2-\beta)} \right) \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) \right)$$

$$\le P + (\varphi(r) + \psi(r)) \|m\|_{L^\infty} Q = l \quad \text{(a constant)}.$$

$$\tag{3.13}$$

Step 3 (N maps bounded sets into equicontinuous sets in $\mathcal{X}$). Let B_r be a bounded set of $\mathcal{X}$ as in Step 2. Let $0 \le t_1 < t_2 \le T$ and $x \in B_r$. For each $h \in N(x)$, then there is $u \in S_{F,x}$ such that $h(t) = (Su)(t)$. Since

$$
\begin{aligned}
\left|(S_1 u)(t_2) - (S_1 u)(t_1)\right| &= \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds \right| \\
&\leq \frac{(\varphi(r) + \psi(r)) \|m\|_{L^\infty}}{\Gamma(\alpha+1)} \left((t_2-t_1)^\alpha + \left| t_2^\alpha - (t_2-t_1)^\alpha - t_1^\alpha \right| \right) \\
&\leq \frac{(\varphi(r) + \psi(r)) \|m\|_{L^\infty} (t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
\left|(S_2 u)(t_2) - (S_2 u)(t_1)\right| &= \left| -k_u t_2 + \frac{c_1}{a_1} + k_u t_1 - \frac{c_1}{a_1} \right| \\
&\leq \left(\frac{(\varphi(r) + \psi(r)) \|m\|_{L^\infty}}{|v_1|} \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) + |v_2| \right) (t_2 - t_1),
\end{aligned}
$$

$$
\begin{aligned}
\left| {}^c D^\beta h(t_2) - {}^c D^\beta h(t_1) \right| &= \left| I^{\alpha-\beta} u(t_2) - \frac{k_u t_2^{1-\beta}}{\Gamma(2-\beta)} - I^{\alpha-\beta} u(t_1) + \frac{k_u t_1^{1-\beta}}{\Gamma(2-\beta)} \right| \\
&\leq \left(\frac{(\varphi(r) + \psi(r)) \|m\|_{L^\infty}}{|v_1| \Gamma(2-\beta)} \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) \right. \\
&\quad \left. + \frac{|v_2|}{\Gamma(2-\beta)} \right) \left(t_2^{1-\beta} - t_1^{1-\beta} \right) + \frac{(\varphi(r) + \psi(r)) \|m\|_{L^\infty} \left(t_2^{\alpha-\beta} - t_1^{\alpha-\beta} \right)}{\Gamma(\alpha-\beta+1)},
\end{aligned}
$$

$$(3.14)$$

we obtain that (since $\alpha > 1$, $\alpha - \beta > 0$ and $1 - \beta \geq 0$)

$$
|h(t_2) - h(t_1)| \longrightarrow 0, \qquad \left| {}^c D^\beta h(t_2) - {}^c D^\beta h(t_1) \right| \longrightarrow 0 \quad \text{as } t_2 \longrightarrow t_1 \tag{3.15}
$$

and the limits are independent of $x \in B_r$ and $h \in N(x)$.

Step 4 (N has a closed graph). Let $x_n \to x_*$, $h_n \in N(x_n)$, and $h_n \to h_*$; we need to show $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies that there exists $u_n \in S_{F,x_n}$ such that $h_n(t) = (Su_n)(t)$ for $t \in [0,T]$. Let us consider the continuous linear operator $\Gamma : L^1([0,T], \mathbb{R}) \to \mathcal{X}$ given by

$$
(\Gamma u)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds - \frac{t}{v_1} \left(a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} u(s) ds \right) \tag{3.16}
$$

and denote $w(t) = v_2 t + c_1/a_1$. Then $h_n(t) - w(t) = (\Gamma u_n)(t)$ and

$$
\begin{aligned}
\|h_n - h_*\| &= \max_{t \in [0,T]} |h_n(t) - w(t) - (h_*(t) - w(t))| \\
&\quad + \max_{t \in [0,T]} \left| {}^c D^\beta (h_n - w)(t) - {}^c D^\beta (h_* - w)(t) \right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.17}
$$

We apply Lemma 2.3 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $h_n - w \in \Gamma \circ S_F(x_n)$. Since $x_n \to x_*$, $h_n - w \to h_* - w$, it follows the existence of $u_* \in S_{F,x_*}$ such that $h_* - w = \Gamma(u_*)$. This means that $h_* \in N(x_*)$.

Step 5 (a priori bounds on solutions). Let $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Then there exists $u \in S_{F,x}$ such that $x(t) = \lambda(Su)(t)$ for $t \in [0,T]$. With the same arguments as in Step 2 of our proof, for each $t \in [0,T]$, we obtain

$$|x(t)| + \left| {}^cD^\beta x(t) \right| \le P + (\varphi(\|x\|) + \psi(\|x\|))\|m\|_{L^\infty}Q. \tag{3.18}$$

Thus

$$\|x\| \le P + (\varphi(\|x\|) + \psi(\|x\|))\|m\|_{L^\infty}Q. \tag{3.19}$$

Now we set

$$U = \{x \in \mathcal{X} : \|x\| < L\}. \tag{3.20}$$

Clearly, U is an open subset of $\mathcal{X}$ and $0 \in U$. As a consequence of Steps 1–4, together with the Arzela-Ascoli theorem, we can conclude that $N : \overline{U} \to P_{cp,c}(\mathcal{X})$ is upper semicontinuous and completely continuous. From the choice of the U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Therefore, by the nonlinear alternative of Leary-Schauder type (Theorem 2.11), we deduce that N has a fixed point $x \in \overline{U}$, which is a solution of the problem (1.4), (1.5). This completes the proof.   □

Theorem 3.2. *Assume that (H1) is satisfied and there exists $L_1 > 0$ such that*

$$\frac{L_1}{P_1 + (\varphi(L_1) + \psi(L_1))\|m\|_{L^\infty}Q_1} > 1, \tag{3.21}$$

where

$$P_1 = \frac{T^\gamma \Gamma(2-\gamma)|c_2|}{|b_2|}\left(1 + \frac{T^{-\beta}}{\Gamma(2-\beta)}\right) + \left|\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\right|,$$

$$Q_1 = T^\alpha\left(1 + \frac{|b_1|}{|a_1+b_1|}\right)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\right) \tag{3.22}$$

$$+ T^{\alpha-\beta}\left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(2-\beta)\Gamma(\alpha-\gamma+1)}\right).$$

Then the problem (1.4), (1.6) has at least one solution on $[0,T]$.

Proof. To obtain the result, the main aim is to study the properties of the operator M defined in (3.3). The proof of them is similar to those of Theorem 3.1, so we omit the details. Here

we just give some estimations, which are needed in the following theorems. Let $x \in \mathcal{X}$ and $h \in M(x)$; then there exists $u \in S_{F,x}$ such that

$$h(t) = (Ku)(t), \quad \text{for } t \in [0, T]. \tag{3.23}$$

We put $Ku = K_1 u + K_2 u$ and

$$(K_1 u)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds, \qquad (K_2 u)(t) = -k_2^u t - k_1^u; \tag{3.24}$$

here k_1^u and k_2^u are constants given by

$$
\begin{aligned}
k_1^u &= \frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1 + b_1) b_2} - \frac{c_1}{a_1 + b_1} \\
&\quad + \frac{b_1}{a_1 + b_1} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds - T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} u(s) ds \right), \\
k_2^u &= \frac{\Gamma(2-\gamma)}{T^{1-\gamma}} \left(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} u(s) ds - \frac{c_2}{b_2} \right).
\end{aligned}
\tag{3.25}
$$

By simple calculations, we have

$$
\left| (K_1 u)(t) \right| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds \leq \left(\varphi(\|x\|) + \psi(\|x\|) \right) \|m\|_{L^\infty} \frac{T^\alpha}{\Gamma(\alpha+1)},
$$

$$
\left| (K_2 u)(t) \right| \leq T |k_2^u| + |k_1^u|,
$$

$$
T |k_2^u| \leq T^\gamma \Gamma(2-\gamma) \left(\left(\varphi(\|x\|) + \psi(\|x\|) \right) \|m\|_{L^\infty} \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{|c_2|}{|b_2|} \right),
$$

$$
|k_1^u| \leq \frac{|b_1|}{|a_1 + b_1|} \left(\left(\varphi(\|x\|) + \psi(\|x\|) \right) \|m\|_{L^\infty} \frac{T^\alpha}{\Gamma(\alpha+1)} + \left(\varphi(\|x\|) + \psi(\|x\|) \right) \|m\|_{L^\infty} \frac{\Gamma(2-\gamma) T^\alpha}{\Gamma(\alpha-\gamma+1)} \right)
$$

$$
+ \left| \frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1 + b_1) b_2} - \frac{c_1}{a_1 + b_1} \right|,
$$

$$
\left| {}^c D^\beta h(t) \right| \leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |u(s)| ds + \frac{|k_2^u| T^{1-\beta}}{\Gamma(2-\beta)}
$$

$$
\leq \left(\varphi(\|x\|) + \psi(\|x\|) \right) \|m\|_{L^\infty} \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(2-\gamma) T^{\gamma-\beta}}{\Gamma(2-\beta)}
$$

$$
\times \left(\left(\varphi(\|x\|) + \psi(\|x\|) \right) \|m\|_{L^\infty} \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{|c_2|}{|b_2|} \right).
$$

$$\tag{3.26}$$

Hence we obtain

$$\|h\| \leq \frac{T^\gamma \Gamma(2-\gamma)|c_2|}{|b_2|}\left(1 + \frac{T^{-\beta}}{\Gamma(2-\beta)}\right) + \left|\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\right|$$

$$+ (\varphi(\|x\|) + \psi(\|x\|))\|m\|_{L^\infty} T^\alpha \left(1 + \frac{|b_1|}{|a_1+b_1|}\right)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\right) \qquad (3.27)$$

$$+ (\varphi(\|x\|) + \psi(\|x\|))\|m\|_{L^\infty} T^{\alpha-\beta}\left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(2-\beta)\Gamma(\alpha-\gamma+1)}\right).$$

This is the end of the proof.   □

3.2. Nonconvex Case

Now we study the case when F is not necessarily convex valued.

A subset A of $L^1([0,T], \mathbb{R})$ is decomposable if for all $u, v \in A$ and $J \subseteq [0,T]$ Lebesgue measurable, then $u\chi_J + v\chi_{[0,T]-J} \in A$, where χ stands for the characteristic function.

(H2): $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multivalued map such that (1) $(t, x, y) \to F(t, x, y)$ is $\Sigma \otimes \mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R}$ measurable; (2) $(x, y) \to F(t, x, y)$ is lower semicontinuous for a.e. $t \in [0,T]$.

Theorem 3.3. *Let (H1)(2), (H2), and relation* (3.6) *hold; then the problem* (1.4), (1.5) *has at least one solution on* $[0,T]$.

Proof. From (H1)(2), (H2), and [33, Lemma 4.1], the map

$$\mathcal{F} : \mathcal{X} \to P\left(L^1([0,T], \mathbb{R})\right), \qquad x \longrightarrow \mathcal{F}(x) = S_{F,x} \qquad (3.28)$$

is lower semicontinuous and has nonempty closed and decomposable values. Then from a selection theorem due to Bressan and Colombo [34], there exists a continuous function $f : \mathcal{X} \to L^1([0,T], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in \mathcal{X}$. That is to say, we have $f(x)(t) \in F(t, x(t), {}^cD^\beta x(t))$ for a.e. $t \in [0,T]$. Now consider the problem

$$^cD^\alpha x(t) = f(x)(t), \quad t \in [0,T] \qquad (3.29)$$

with the boundary conditions (1.5). Note that if $x \in \mathcal{X}$ is a solution of the problem (3.29), then x is a solution to the problem (1.4), (1.5).

Problem (3.29) is then reformulated as a fixed point problem for the operator $N_1 : \mathcal{X} \to \mathcal{X}$ defined by

$$N_1(x)(t) = (Sf(x))(t). \qquad (3.30)$$

It can easily be shown that N_1 is continuous and completely continuous and satisfies all conditions of the Leray-Schauder nonlinear alternative for single-valued maps [31]. The

remaining part of the proof is similar to that of Theorem 3.1, so we omit it. This completes the proof. $\qquad\square$

Theorem 3.4. *Let (H1)(2), (H2), and relation (3.21) hold, then the problem (1.4), (1.6) has at least one solution on $[0, T]$.*

The proof of this theorem is similar to that of Theorem 3.3.

(H3): $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multivalued map such that (1) F is integrably bounded and the map $t \to F(t, x, y)$ is measurable for all $x, y \in \mathbb{R}$; (2) there exists $m \in L^\infty([0, T], \mathbb{R}^+)$ such that for a.e. $t \in [0, T]$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$h\big(F(t, x_1, y_1), F(t, x_2, y_2)\big) \leq m(t)\big(|x_1 - x_2| + |y_1 - y_2|\big). \tag{3.31}$$

Theorem 3.5. *Let (H3) hold, if, in addition,*

$$\|m\|_{L^\infty} \left[\frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{T^{\alpha - \beta}}{\Gamma(\alpha - \beta + 1)} + \left(\frac{T}{|v_1|} + \frac{T^{1-\beta}}{|v_1|\Gamma(2 - \beta)} \right) \left(\frac{|a_2|T^\alpha}{\Gamma(\alpha + 1)} + \frac{|b_2|T^{\alpha - \gamma}}{\Gamma(\alpha - \gamma + 1)} \right) \right] < 1, \tag{3.32}$$

then the problem (1.4), (1.5) has at least one solution on $[0, T]$.

Proof. From (H3), we have that the multivalued map $t \to F(t, x(t), {}^c D^\beta x(t))$ is measurable [28, Proposition 2.7.9] and closed valued for each $x \in \mathcal{X}$. Hence it has measurable selection [28, Theorem 2.2.1] and the set $S_{F,x}$ is nonempty. Let N be defined in (3.2). We will show that, under this situation, N satisfies the requirements of Theorem 2.12.

Step 1. For each $x \in \mathcal{X}$, $N(x) \in P_{cl}(\mathcal{X})$. Let $h_n \in N(x)$, $n \geq 1$ such that $h_n \to h$ in $\mathcal{X}$. Then $h \in \mathcal{X}$ and there exists $u_n \in S_{F,x}$, $n \geq 1$ such that

$$h_n(t) = (Su_n)(t), \quad t \in [0, T]. \tag{3.33}$$

By (H3), the sequence u_n is integrable bounded. Since F has compact values, we may pass to a subsequence if necessary to get that u_n converges to u in $L^1([0, T], \mathbb{R})$. Thus $u \in S_{F,x}$ and for each $t \in [0, T]$

$$h_n(t) \longrightarrow h(t) = (Su)(t). \tag{3.34}$$

This means that $h \in N(x)$ and $N(x)$ is closed.

Step 2. There exists $\rho < 1$ such that

$$h\big(N(x), N(y)\big) \leq \rho \|x - y\|, \quad \forall x, y \in \mathcal{X}. \tag{3.35}$$

Let $x, y \in \mathcal{X}$ and $h_1 \in N(y)$; then there exists $u_1 \in S_{F,y}$ such that

$$h_1(t) = (Su_1)(t), \quad t \in [0, T]. \tag{3.36}$$

From (H3)(2), we deduce

$$
\begin{aligned}
h\Big(F\big(t, x(t), {}^cD^\beta x(t)\big), F\big(t, y(t), {}^cD^\beta y(t)\big)\Big) \\
\leq m(t)\Big(\big|x(t) - y(t)\big| + \big|{}^cD^\beta x(t) - {}^cD^\beta y(t)\big|\Big).
\end{aligned}
\tag{3.37}
$$

Hence, for a.e. $t \in [0, T]$, there exists $v \in F(t, x(t), {}^cD^\beta x(t))$ such that

$$
|u_1(t) - v| \leq m(t)\Big(\big|x(t) - y(t)\big| + \big|{}^cD^\beta x(t) - {}^cD^\beta y(t)\big|\Big).
\tag{3.38}
$$

Consider the multivalued map $V : [0, T] \to P(\mathbb{R})$ given by

$$
V(t) = \Big\{u \in \mathbb{R} : |u_1(t) - u| \leq m(t)\Big(\big|x(t) - y(t)\big| + \big|{}^cD^\beta x(t) - {}^cD^\beta y(t)\big|\Big)\Big\}.
\tag{3.39}
$$

Since $u_1(t), \alpha(t) = m(t)(|x(t) - y(t)| + |{}^cD^\beta x(t) - {}^cD^\beta y(t)|)$ are measurable, [35, Theorem III.41] implies that V is measurable. It follows from (H3) that the map $t \to F(t, x(t), {}^cD^\beta x(t))$ is measurable. Hence by (3.38) and [28, Proposition 2.1.43], the multivalued map $t \to V(t) \cap F(t, x(t), {}^cD^\beta x(t))$ is measurable and nonempty closed valued. Therefore, we can find $u_2(t) \in F(t, x(t), {}^cD^\beta x(t))$ such that for a.e. $t \in [0, T]$,

$$
|u_1(t) - u_2(t)| \leq m(t)\Big(\big|x(t) - y(t)\big| + \big|{}^cD^\beta x(t) - {}^cD^\beta y(t)\big|\Big).
\tag{3.40}
$$

Let $h_2(t) = (Su_2)(t)$, that is, $h_2 \in N(x)$. Since

$$
\begin{aligned}
|(S_1 u_1)(t) - (S_1 u_2)(t)| &= \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(u_1(s) - u_2(s))ds\right| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\Big(\big|x(s) - y(s)\big| + \big|{}^cD^\beta x(s) - {}^cD^\beta y(s)\big|\Big)ds \\
&\leq \frac{\|m\|_{L^\infty} T^\alpha}{\Gamma(\alpha+1)}\|x - y\|, \\
|(S_2 u_1)(t) - (S_2 u_2)(t)| &= |t(k_{u_1} - k_{u_2})| \\
&\leq T\left|\frac{a_2}{v_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}(u_1(s) - u_2(s))ds\right. \\
&\quad \left. + \frac{b_2}{v_1}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}(u_1(s) - u_2(s))ds\right| \\
&\leq \frac{\|m\|_{L^\infty} T}{|v_1|}\left(\frac{|a_2|T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2|T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right)\|x - y\|,
\end{aligned}
$$

$$\left| {}^cD^\beta h_1(t) - {}^cD^\beta h_2(t) \right| = \left| I^{\alpha-\beta} u_1(t) - \frac{k_{u_1} t^{1-\beta}}{\Gamma(2-\beta)} - I^{\alpha-\beta} u_2(t) + \frac{k_{u_2} t^{1-\beta}}{\Gamma(2-\beta)} \right|$$

$$\le \frac{\|m\|_{L^\infty} T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|x-y\|$$

$$+ \frac{\|m\|_{L^\infty} T^{1-\beta}}{|v_1| \Gamma(2-\beta)} \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) \|x-y\|,$$

$$(3.41)$$

we obtain

$$\|h_1 - h_2\| \le \|m\|_{L^\infty} \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right.$$

$$\left. + \left(\frac{T}{|v_1|} + \frac{T^{1-\beta}}{|v_1| \Gamma(2-\beta)} \right) \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) \right] \|x-y\|.$$

$$(3.42)$$

Denote

$$\rho = \|m\|_{L^\infty} \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \left(\frac{T}{|v_1|} + \frac{T^{1-\beta}}{|v_1| \Gamma(2-\beta)} \right) \left(\frac{|a_2| T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_2| T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \right) \right].$$

$$(3.43)$$

By using an analogous relation obtained by interchanging the roles of x and y, we get

$$h(N(x), N(y)) \le \rho \|x-y\|.$$

$$(3.44)$$

Therefore, from condition (3.32), Theorem 2.12 implies that N has a fixed point, which is a solution of the problem (1.4), (1.5). This completes the proof. □

Theorem 3.6. *Let (H3) hold, if, in addition,*

$$\|m\|_{L^\infty} T^\alpha \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)} \right)$$

$$+ \|m\|_{L^\infty} T^{\alpha-\beta} \left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(2-\beta)\Gamma(\alpha-\gamma+1)} \right) < 1,$$

$$(3.45)$$

then the problem (1.4), (1.6) *has at least one solution on* $[0, T]$.

Using the arguments employed in the proof of **Theorem 3.5**, we can prove this theorem similarly. Hence the details are omitted here.

4. Integral Boundary Conditions

In this section, the existence results of the problems (1.4), (1.5) and (1.4), (1.6) obtained in the previous section will be extended to the ones of the problems of fractional differential inclusions (1.4) subject to the integral boundary conditions (1.7) and (1.8).

Lemma 4.1. *For any y, ξ, $\chi \in C([0,T],\mathbb{R})$, the unique solution of the fractional separated integral boundary value problem,*

$$^cD^\alpha x(t) = y(t), \quad t \in [0,T], \ 1 < \alpha \leq 2,$$

$$a_1 x(0) + b_1\left(^cD^\gamma x(0)\right) = c_1 \int_0^T \xi(s)ds,$$

$$a_2 x(T) + b_2\left(^cD^\gamma x(T)\right) = c_2 \int_0^T \chi(s)ds, \quad 0 < \gamma < 1,$$

$$(4.1)$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds$$

$$- \frac{t}{v_1}\left(a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds\right)$$

$$+ \frac{c_2 t}{v_1}\int_0^T \chi(s)ds + \frac{c_1(v_1 - a_2 t)}{a_1 v_1}\int_0^T \xi(s)ds.$$

$$(4.2)$$

Lemma 4.2. *For any $y,\xi,\chi \in C([0,T],\mathbb{R})$, the unique solution of the fractional nonseparated integral boundary value problem,*

$$^cD^\alpha x(t) = y(t), \quad t \in [0,T], \ 1 < \alpha \leq 2,$$

$$a_1 x(0) + b_1 x(T) = c_1 \int_0^T \xi(s)ds,$$

$$a_2\left(^cD^\gamma x(0)\right) + b_2\left(^cD^\gamma x(T)\right) = c_2 \int_0^T \chi(s)ds, \quad 0 < \gamma < 1,$$

$$(4.3)$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s) ds + \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \int_0^T \chi(s) ds$$

$$- \frac{b_1}{a_1+b_1} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s) ds \right)$$

$$- \frac{b_1 T^\gamma \Gamma(2-\gamma)c_2}{b_2(a_1+b_1)} \int_0^T \chi(s) ds + \frac{c_1}{a_1+b_1} \int_0^T \xi(s) ds.$$

$$(4.4)$$

To obtain the existence results of the problems (1.4), (1.7) and (1.4), (1.8), in view of Lemmas 4.1 and 4.2, we define two operators $\Pi, \Omega : \mathcal{X} \to P(\mathcal{X})$ as

$$\Pi(x) = \{h \in \mathcal{X} : h = Hv, \; v \in S_{F,x}\}, \tag{4.5}$$

$$\Omega(x) = \{h \in \mathcal{X} : h = Zv, \; v \in S_{F,x}\} \tag{4.6}$$

with

$$(Hv)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds$$

$$- \frac{t}{v_1} \left(a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s) ds \right)$$

$$+ \frac{c_2 t}{v_1} \int_0^T h(s, x(s)) ds + \frac{c_1}{a_1} \int_0^T g(s, x(s)) ds - \frac{c_1 a_2 t}{a_1 v_1} \int_0^T g(s, x(s)) ds,$$

$$(Zv)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s) ds \tag{4.7}$$

$$+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} \int_0^T h(s, x(s)) ds$$

$$- \frac{b_1}{a_1+b_1} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds - T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s) ds \right)$$

$$- \frac{b_1 T^\gamma \Gamma(2-\gamma)c_2}{b_2(a_1+b_1)} \int_0^T h(s, x(s)) ds + \frac{c_1}{a_1+b_1} \int_0^T g(s, x(s)) ds.$$

Observe that if $x \in \mathcal{X}$ is a fixed point of the operator Π (the operator Ω), that is, $x \in \Pi(x)$ ($x \in \Omega(x)$), then x is a solution of the problem (1.4), (1.7) (the problem (1.4), (1.8)).

From the definitions of the operators N, Π (see (3.2), (4.5)), we know that the difference between them is very apparent, that is, c_1, c_2 in (3.2) were replaced by $c_1 \int_0^T g(s, x(s)) ds$ and $c_2 \int_0^T h(s, x(s)) ds$ in (4.5). This fact is also true for the operators M, Ω (see (3.3), (4.6)).

In the following, we state some existence results for the problems (1.4), (1.7) and (1.4), (1.8). We omit the proofs as these are similar to the ones given in Section 3.

(A1): The functions $g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous. There exist functions $m_2, m_3 \in L^1([0, T], \mathbb{R}^+)$ and $\varphi_2, \varphi_3 : [0, \infty) \to (0, \infty)$ continuous, nondecreasing such that

$$\left| g(t, x) \right| \le m_2(t) \varphi_2(|x|), \qquad |h(t, x)| \le m_3(t) \varphi_3(|x|) \tag{4.8}$$

for all $x \in \mathbb{R}$ and a.e. $t \in [0, T]$.

Theorem 4.3. *Assume that (H1) and (A1) hold. If there exists a constant $I > 0$ such that*

$$\frac{I}{(\varphi(I) + \psi(I)) \|m\|_{L^\infty} Q + \varphi_3(I) \|m_3\|_{L^1} R + \varphi_2(I) \|m_2\|_{L^1} W} > 1, \tag{4.9}$$

here Q is defined by (3.7) and

$$R = \frac{|c_2| T}{|v_1|} \left(1 + \frac{T^{-\beta}}{\Gamma(2 - \beta)} \right), \qquad W = \frac{|c_1|}{|a_1|} \left(1 + \frac{|a_2| T}{|v_1|} \left(1 + \frac{T^{-\beta}}{\Gamma(2 - \beta)} \right) \right). \tag{4.10}$$

Then the boundary value problem (1.4), (1.7) has at least one solution on $[0, T]$.

Theorem 4.4. *Assume that (H1) and (A1) hold. If there exists a constant $I_1 > 0$ such that*

$$\frac{I_1}{(\varphi(I_1) + \psi(I_1)) \|m\|_{L^\infty} Q_1 + \varphi_3(I_1) \|m_3\|_{L^1} R_1 + \varphi_2(I_1) \|m_2\|_{L^1} W_1} > 1, \tag{4.11}$$

here Q_1 is defined by (3.22) and

$$R_1 = \frac{|c_2| T^\gamma \Gamma(2 - \gamma)}{|b_2|} \left(1 + \frac{|b_1|}{|a_1 + b_1|} + \frac{T^{-\beta}}{\Gamma(2 - \beta)} \right), \qquad W_1 = \frac{|c_1|}{|a_1 + b_1|}. \tag{4.12}$$

Then the boundary value problem (1.4), (1.8) has at least one solution on $[0, T]$.

Theorem 4.5. *Assume that (H1)(2), (H2), (A1) and condition (4.9) hold. Then the boundary value problem (1.4), (1.7) has at least one solution on $[0, T]$.*

Theorem 4.6. *Assume that (H1)(2), (H2), (A1) and condition (4.11) hold. Then the boundary value problem (1.4), (1.8) has at least one solution on $[0, T]$.*

(A2): *The functions $g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy*

$$\left| g(t, x) - g(t, y) \right| \le m_2(t) \left| x - y \right|, \qquad \left| h(t, x) - h(t, y) \right| \le m_3(t) \left| x - y \right| \tag{4.13}$$

for all $x, y \in \mathbb{R}$ and a.e. $t \in [0, T]$; here $m_2, m_3 \in L^1([0, T], \mathbb{R}^+)$.

Theorem 4.7. *Assume that (H3) and (A2) hold. If, in addition,*

$$\|m\|_{L^\infty}Q + \|m_3\|_{L^1}R + \|m_2\|_{L^1}W < 1, \tag{4.14}$$

here Q is defined by (3.7) and R, W are defined by (4.10), then the boundary value problem (1.4), (1.7) has at least one solution on $[0,T]$.

Theorem 4.8. *Assume that (H3) and (A2) hold. If, in addition,*

$$\|m\|_{L^\infty}Q_1 + \|m_3\|_{L^1}R_1 + \|m_2\|_{L^1}W_1 < 1, \tag{4.15}$$

here Q_1 is defined by (3.22) and R_1, W_1 are defined by (4.12), then the boundary value problem (1.4), (1.8) has at least one solution on $[0,T]$.

5. Examples

In this section, we give two simple examples to show the applicability of our results.

Example 5.1. Consider the following fractional boundary value problem:

$$^cD^{3/2}x(t) \in F\left(t, x(t), {}^cD^{3/4}x(t)\right), \quad t \in [0,1],$$

$$x(0) - \frac{1}{2}\left({}^cD^{1/2}x(0)\right) = 2.5, \tag{5.1}$$

$$2x(1) + \frac{1}{3}\left({}^cD^{1/2}x(1)\right) = -\frac{1}{3},$$

where $\alpha = 3/2$, $\beta = 3/4$, $\gamma = 1/2$, $a_1 = 1$, $b_1 = -1/2$, $c_1 = 2.5$, $a_2 = 2$, $b_2 = 1/3$, $c_2 = -1/3$, $T = 1$, and $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map given by

$$F(t,x,y) = \left\{ u \in \mathbb{R} : e^{-|x|} - \frac{|y|}{1+|y|} + \sin t \le u \le 5 + \frac{|x|}{1+x^2} + 6t^3 + \cos y \right\}. \tag{5.2}$$

In the context of this problem, we have

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \le 7 + 6t^3 \le 13, \quad \text{for } t \in [0,1], \ x, y \in \mathbb{R}. \tag{5.3}$$

It is clear that F is convex compact valued and is of Carathéodory type. Let $m(t) \equiv 1$ and $\varphi(|x|) \equiv 3$, $\psi(|y|) \equiv 10$; we get for $t \in [0,1]$, $x, y \in \mathbb{R}$

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \le m(t)(\varphi(|x|) + \psi(|y|)). \tag{5.4}$$

As for the condition (3.6), since $P + (\varphi(|x|) + \psi(|y|))\|m\|_{L^\infty}Q = P + 13Q$ (P, Q defined in (3.6)) is a constant, we can choose L large enough so that

$$\frac{L}{P + (\varphi(L) + \psi(L))\|m\|_{L^\infty}Q} > 1. \tag{5.5}$$

Thus, by the conclusion of Theorem 3.1, the boundary value problem (5.1) has at least one solution on $[0, 1]$.

Example 5.2. Consider the following fractional differential inclusion with integral boundary conditions:

$$^cD^{7/4}x(t) \in F\left(t, x(t), {}^cD^{1/2}x(t)\right), \quad t \in [0, 1],$$

$$3x(0) + \frac{1}{3}x(1) = \int_0^1 g(s, x(s))ds, \tag{5.6}$$

$$2\left({}^cD^{1/4}x(0)\right) + 3\left({}^cD^{1/4}x(1)\right) = \frac{1}{4}\int_0^1 h(s, x(s))ds,$$

where $\alpha = 7/4, \beta = 1/2, \gamma = 1/4, T = 1, a_1 = 3, b_1 = 1/3, c_1 = 1, a_2 = 2, b_2 = 3, c_2 = 1/4,$

$$F(t, x, y) = \left[-l_1(t) - \frac{\sin x}{(4+t)^2} - 2, -\frac{1}{10}\right] \bigcup \left[0, \frac{1}{16}\frac{|y|}{1+|y|} + l_2(t)\right],$$

$$g(t, x) = \frac{1}{(3+t)^2}\cos x, \qquad h(t, x) = x, \tag{5.7}$$

and $l_1, l_2 \in L^1([0, 1], \mathbb{R}^+)$.

From the data given above, we have for $t \in [0, 1], x, y \in \mathbb{R}$,

$$\sup\{|u| : u \in F(t, x, y)\} \leq 3 + \frac{1}{(4+t)^2} + l_1(t) + l_2(t),$$

$$h(F(t, x_1, y_1), F(t, x_2, y_2)) \leq \frac{1}{(4+t)^2}|x_1 - x_2| + \frac{1}{16}|y_1 - y_2|, \tag{5.8}$$

$$|g(t, x) - g(t, y)| \leq \frac{1}{(3+t)^2}|x - y|, \qquad |h(t, x) - h(t, y)| \leq |x - y|.$$

Then let $m_2(t) = 1/(3+t)^2, m_3(t) = 1$, and $m(t) = 1/16 + 1/(4+t)^2$; we have

$$h(F(t, x_1, y_1), F(t, x_2, y_2)) \leq m(t)(|x_1 - x_2| + |y_1 - y_2|),$$

$$\|m\|_{L^\infty}Q_1 + \|m_3\|_{L^1}R_1 + \|m_2\|_{L^1}W_1 \leq \frac{1}{8} \times 3.1071 + 1 \times 0.1707 + \frac{1}{9} \times \frac{3}{10} = 0.5924 < 1. \tag{5.9}$$

Here Q_1 is defined by (3.22) and R_1, W_1 are defined by (4.12). Hence all the assumptions of Theorem 4.8 are satisfied, and by the conclusion of it, the boundary value problem (5.6) has at least one solution on $[0, 1]$.

Acknowledgment

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