Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 450318, 13 pages
doi:10.1155/2012/450318

Research Article

Coefficient Conditions for Starlikeness of Nonnegative Order

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Received 9 October 2012; Accepted 25 November 2012

Academic Editor: Graziano Crasta

Sufficient conditions on a sequence $\{a_k\}$ of nonnegative numbers are obtained that ensures $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is starlike of nonnegative order in the unit disk. A result of Vietoris on trigonometric sums is extended in this pursuit. Conditions for close to convexity and convexity in the direction of the imaginary axis are also established. These results are applied to investigate the starlikeness of functions involving the Gaussian hypergeometric functions.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0 = f'(0) - 1$. Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is starlike with respect to the origin and convex if $f(\mathbb{D})$ is a convex domain. These classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively, are subsets of $\mathcal{S}$. The generalized classes $\mathcal{S}^*(\mu)$ and $\mathcal{C}(\mu)$ of starlike and convex functions of order μ, $\mu < 1$ are defined, respectively, by the analytic characterizations

$$f \in \mathcal{S}^*(\mu) \Longleftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \mu, \qquad f \in \mathcal{C}(\mu) \Longleftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \mu, \tag{1.1}$$

with $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$.

An extension of starlike functions is the class of close-to-convex functions $f \in \mathcal{A}$ of order μ defined analytically by

$$\operatorname{Re}\left(e^{i\eta}\frac{zf'(z)}{g(z)}\right) > \mu, \quad g \in \mathcal{S}^*, \tag{1.2}$$

for some real $\eta \in (-\pi/2, \pi/2)$. The family of close-to-convex functions of order μ with respect to $g \in \mathcal{S}^*$ is denoted by $\mathcal{K}_g(\mu)$, with $\mathcal{K}_g := \mathcal{K}_g(0)$. Exposition on the geometric properties of functions in these classes can be found in [1, 2].

A function f satisfying $(\operatorname{Im} z)\ (\operatorname{Im} f(z)) > 0$ in $\mathbb{D}$ is said to be typically real, and f is convex in the direction of the imaginary axis if every line parallel to the imaginary axis either intersects $f(\mathbb{D})$ in an interval or has an empty intersection. For $f \in \mathcal{A}$ with real coefficients, Robertson [3] proved that being convex in the direction of the imaginary axis is equivalent to zf' being typically real, which in turn is equivalent to $\operatorname{Re}(1-z^2)f'(z) > 0$. For $f \in \mathcal{A}$ satisfying f' is typically real and $\operatorname{Re} f'(z) > 0$ in $\mathbb{D}$, Ruscheweyh [4] proved that it is necessarily starlike. The latter result is extended in [5] to include starlike functions of a nonnegative order.

Lemma 1.1 (see [5]). *For $0 \le \alpha < 1$, let $f \in \mathcal{A}$ satisfy f' and $f' - \alpha f/z$ be typically real in $\mathbb{D}$. If $\operatorname{Re} f'(z) > \max\{0, \alpha \operatorname{Re}(f(z)/z)\}$ in $\mathbb{D}$, then $f \in \mathcal{S}^*(\alpha)$.*

Trigonometric series, in particular the cosine and sine series along with their partial sums, have found widely important applications in many works, for example, those of [4–9]. Vietoris [10] (also see [11]) showed that if $c_{2k} = c_{2k+1} = (1/2)_k/k!$, $k = 0, 1, \ldots$, then

$$\sum_{k=0}^{n} c_k \cos k\theta > 0, \qquad \sum_{k=1}^{n} c_k \sin k\theta > 0, \quad 0 < \theta < \pi, \tag{1.3}$$

for any positive integer n. Here the Pochhammer symbol $(a)_\lambda$ is defined by $(a)_0 = 1$, and $(a)_\lambda = (a+\lambda-1)(a)_{\lambda-1}$, $\lambda \in \mathbb{N}$. Using Abel's partial summation formula

$$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n-1} \left((b_k - b_{k+1}) \sum_{j=0}^{k} c_j \right) + b_n \sum_{k=0}^{n} c_k, \tag{1.4}$$

equation (1.3) yields the following classical result on the positivity of cosine and sine sums.

Theorem 1.2 (see [10]). *Let $\{a_k\}_{k=0}^{\infty}$ be a decreasing sequence of nonnegative real numbers satisfying $a_0 > 0$ and $(2k)a_{2k} \le (2k-1)a_{2k-1}$, $k \ge 1$. Then*

$$\sum_{k=0}^{n} a_k \cos k\theta > 0, \qquad \sum_{k=1}^{n} a_k \sin k\theta > 0, \quad 0 < \theta < \pi, \tag{1.5}$$

for any positive integer n.

Using Theorem 1.2, Ruscheweyh [4] obtained sufficient coefficient conditions for functions $f \in \mathcal{A}$ to be starlike which can readily be tested. This paper aims to extend Ruscheweyh's concise result. Specifically in the next section, sufficient conditions on a sequence $\{a_k\}$ of nonnegative numbers are obtained that ensures $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is starlike of order $(1-2\mu)/(1-\mu)$, $\mu \in (0, 1/2]$ in the unit disk. Coefficient conditions for $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ to be either close to convex or convex in the direction of the imaginary axis are also derived. The final section is devoted to finding conditions on the triplets (a, b, c) that will ensure a normalized Gaussian hypergeometric function $z_2F_1(a, b; c; z)$ is starlike of order $(1-2\mu)/(1-\mu)$, $\mu \in (0, 1/2]$.

The following extension of Theorem 1.2 will be required.

Theorem 1.3 (see [12]). *Let $c_{2k} = c_{2k+1} = (\mu)_k/k!$, $\mu \in (0,1)$. For any positive integer n and $0 < \theta < \pi$, then*

(i) $\sum_{k=0}^{n} c_k \cos k\theta > 0$ *if and only if* $0 < \mu \le \mu_0$,

(ii) $\sum_{k=1}^{2n+1} c_k \sin k\theta > 0$ *if and only if* $0 < \mu \le \mu_0$,

(iii) $\sum_{k=1}^{2n} c_k \sin k\theta > 0$ *if* $0 < \mu \le 1/2$.

Here $\mu_0 = 0.691556\cdots$ is the unique root in $(0,1)$ of

$$\int_0^{3\pi/2} \frac{\cos t}{t^{1-\mu}}\, dt = 0. \tag{1.6}$$

2. Main Results

For our purpose, it will be more expedient to allow the terms of the sequence in Theorem 1.3 to consist of nonnegative numbers. Thus as a prelude to the main results, Theorem 1.3 is first appropriately adapted to yield the following two preliminary results.

Lemma 2.1. *Let $\{b_k\}$ be a decreasing sequence of nonnegative numbers satisfying $b_0 > 0$ and $kb_{2k} \le (k+\mu-1)b_{2k-1}$, $k \ge 1$, $\mu \in (0,1)$. For any positive integer n and $0 < \theta < \pi$, then*

$$\sum_{k=0}^{n} b_k \cos k\theta > 0 \quad \text{iff } 0 < \mu \le \mu_0. \tag{2.1}$$

Proof. Let the sequence $\{c_k\}$ be given by $c_{2k} = c_{2k+1} = (\mu)_k/k!$. It is evident from Theorem 1.3(i) that

$$\sum_{k=0}^{n} c_k \cos k\theta > 0 \quad \text{iff } 0 < \mu \le \mu_0. \tag{2.2}$$

Using (1.4), rewrite $\sum_{k=0}^{n} b_k \cos k\theta$ in the form

$$\sum_{k=0}^{n} b_k \cos k\theta = \sum_{k=0}^{n-1} \left(\frac{b_k}{c_k} - \frac{b_{k+1}}{c_{k+1}} \right) \sum_{j=0}^{k} c_j \cos(j\theta) + \frac{b_n}{c_n} \sum_{j=0}^{n} c_j \cos(j\theta). \tag{2.3}$$

If $b_k > 0$ for $k = 1, \ldots, n$, then a computation gives

$$\frac{b_{2k-1}}{c_{2k-1}} - \frac{b_{2k}}{c_{2k}} = \frac{(k-1)!}{(\mu)_k} \left((k+\mu-1)b_{2k-1} - kb_{2k} \right) \ge 0. \tag{2.4}$$

Similarly,

$$\frac{b_{2k}}{c_{2k}} - \frac{b_{2k+1}}{c_{2k+1}} = \frac{(k)!}{(\mu)_k} (b_{2k} - b_{2k+1}) \ge 0. \tag{2.5}$$

Thus

$$\left(\frac{b_k}{c_k} - \frac{b_{k+1}}{c_{k+1}}\right) \geq 0 \tag{2.6}$$

for $k = 0, \ldots, n-1$. Together with (2.2), the latter implies that the expression on the right side of (2.3) is positive.

Now suppose there is an m, $1 \leq m \leq n$, so that $b_m = 0$ while $b_k > 0$ for $0 \leq k \leq m-1$. Then the conditions on $\{b_k\}$ would imply that $b_m = b_{m+1} = \cdots = b_n = 0$. If $b_1 = 0$, evidently $\sum_{k=0}^{n} b_k \cos k\theta = b_0 > 0$. Let $m \geq 2$. It is shown above in (2.6) that $(b_k/c_k - b_{k+1}/c_{k+1}) \geq 0$ for $k = 0, \ldots, m-2$. The conditions $b_k \geq b_{k+1}$ and $kb_{2k} \leq (k+\mu-1)b_{2k-1}$ imply that

$$\left(\frac{b_0}{c_0} - \frac{b_1}{c_1}\right)c_0 + \left(\frac{b_1}{c_1} - \frac{b_2}{c_2}\right)(c_0 + c_1\cos\theta) + \cdots + \frac{b_{m-1}}{c_{m-1}}\sum_{j=0}^{m-1} c_j \cos j\theta > 0, \tag{2.7}$$

which yields the desired result. □

The following result is readily obtained by using a similar argument used in Lemma 2.1.

Lemma 2.2. *Let $\{b_k\}$ be a decreasing sequence satisfying $b_1 > 0$ and $kb_{2k} \leq (k+\mu-1)b_{2k-1}$, $k \geq 1$, $\mu \in (0,1)$. If $0 < \mu \leq 1/2$, then*

$$\sum_{k=1}^{n} b_k \sin k\theta > 0 \tag{2.8}$$

for any positive integer n and $0 < \theta < \pi$.

The preceding lemmas will next be used to establish the following result on starlikeness.

Theorem 2.3. *Let $a_1 = 1$, $a_k \geq 0$ satisfy*

$$\big((1-\mu)k - 1 + 2\mu\big)a_k \geq \big((1-\mu)k + \mu\big)a_{k+1}, \tag{2.9}$$

$$(k+\mu-1)\big(2(1-\mu)k - 1 + 2\mu\big)a_{2k} \geq k\big(2(1-\mu)k + \mu\big)a_{2k+1}, \tag{2.10}$$

$k \geq 1$, $0 < \mu \leq 1/2$. Then $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is starlike of order $(1-2\mu)/(1-\mu)$. The result is sharp as illustrated by the function $f(z) = z + \mu z^2$.

Proof. Let $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$, $\alpha := (1-2\mu)/(1-\mu)$, and $b_k = (k+1-\alpha)a_{k+1}$, $k \geq 0$. Then

$$f_n'(z) - \alpha\frac{f_n(z)}{z} = \sum_{k=0}^{n-1} b_k z^k. \tag{2.11}$$

With $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ in (2.11), it follows that

$$\operatorname{Re}\left(f_n'(z) - \alpha\frac{f_n(z)}{z}\right) = \sum_{k=0}^{n-1} b_k \cos k\theta. \tag{2.12}$$

Now $b_0 > 0$ and $b_k \ge 0$, $k \ge 1$. Condition (2.9) shows that

$$\begin{aligned} b_{k-1} &= (k-\alpha)a_k = \frac{1}{1-\mu}\big((1-\mu)k - 1 + 2\mu\big)a_k \\ &\ge \frac{1}{1-\mu}\big((1-\mu)k + \mu\big)a_{k+1} \\ &= (k+1-\alpha)a_{k+1} = b_k, \end{aligned} \tag{2.13}$$

while inequality (2.10) yields

$$\begin{aligned} &\big(k+\mu-1\big)b_{2k-1} - kb_{2k} \\ &\quad = \big(k+\mu-1\big)(2k-\alpha)a_{2k} - k(2k+1-\alpha)a_{2k+1} \\ &\quad = \frac{1}{1-\mu}\big((k+\mu-1)\big(2(1-\mu)k - 1 + 2\mu\big)a_{2k} - k\big(2(1-\mu)k+\mu\big)a_{2k+1}\big) \ge 0. \end{aligned} \tag{2.14}$$

Evidently $\{b_k\}$ satisfies the hypothesis of Lemma 2.1, and therefore

$$\sum_{k=0}^{n-1} b_k \cos k\theta \ge 0, \quad 0 \le \theta \le \pi. \tag{2.15}$$

The minimum principle for harmonic functions implies that $\operatorname{Re}(f_n'(z) - \alpha f_n(z)/z)$ is either identically zero or positive. Since $\operatorname{Re}(f_n'(z) - \alpha f_n(z)/z) = 1 - \alpha$ at $z = 0$, it follows that $\operatorname{Re}(f_n'(z) - \alpha f_n(z)/z) > 0$ in $\mathbb{D}$.

Similarly, taking $z = e^{i\theta}$ in (2.11) results in

$$\operatorname{Im}\left(f_n'(z) - \alpha\frac{f_n(z)}{z}\right) = \sum_{k=1}^{n-1} b_k \sin k\theta, \tag{2.16}$$

for $z \in \mathbb{D} \cap \{z : \operatorname{Im} z > 0\}$. Now Lemma 2.2 implies that

$$\sum_{k=1}^{n-1} b_k \sin k\theta > 0 \quad \text{for } 0 < \theta < \pi. \tag{2.17}$$

Since the coefficients b_k are real, (2.17) shows that $\operatorname{Im}(f_n'(z) - \alpha f_n(z)/z) \ge 0$ on $\partial(\mathbb{D} \cap \{z : \operatorname{Im} z > 0\})$. Again by the minimum principle, $\operatorname{Im}(f_n'(z) - \alpha f_n(z)/z)$ is either identically zero or positive in $\mathbb{D} \cap \{z : \operatorname{Im} z > 0\}$. The former implies that $f_n(z) = z$, which is starlike. In the

latter case, the reflection principle yields $\operatorname{Im}(f_n'(z) - \alpha f_n(z)/z) < 0$ in $\mathbb{D} \cap \{z : \operatorname{Im} z < 0\}$. Thus $f_n' - \alpha f/z$ is typically real.

It remains to show that f_n' is typically real satisfying $\operatorname{Re} f_n'(z) > 0$. Let $c_k = (k+1)a_{k+1}$, and thus

$$f_n'(z) = \sum_{k=0}^{n-1} c_k z^k. \tag{2.18}$$

Inequality (2.9) yields

$$c_{k-1} - c_k = ka_k - (k+1)a_{k+1} \geq \frac{1-2\mu}{(1-\mu)k - 1 + 2\mu} a_{k+1} \geq 0, \tag{2.19}$$

while (2.10) implies

$$\begin{aligned}
&(k+\mu-1)c_{2k-1} - kc_{2k} \\
&\quad = (k+\mu-1)2ka_{2k} - k(2k+1)a_{2k+1} \\
&\quad \geq \frac{k}{2(1-\mu)k - 1 + 2\mu}\big(2k\big(2(1-\mu)k + \mu\big) - (2k+1)\big(2(1-\mu)k - 1 + 2\mu\big)\big)a_{2k+1} \\
&\quad = \frac{k}{2(1-\mu)k - 1 + 2\mu}\Big(4(1-\mu)k^2 + 2\mu k - 4(1-\mu)k^2 \\
&\qquad\qquad +2k(1-2\mu) - 2(1-\mu)k + (1-2\mu) + 2\mu k - 4(1-\mu)\Big)a_{2k+1} \\
&\quad = \frac{k(1-2\mu)}{2(1-\mu)k - 1 + 2\mu} a_{2k+1} \geq 0.
\end{aligned} \tag{2.20}$$

Thus $\{c_k\}$ also satisfies the hypothesis of Lemmas 2.1 and 2.2, and following the same arguments used earlier, f_n' is deduced to be typically real with $\operatorname{Re} f_n'(z) > 0$.

Lemma 1.1 now implies that f_n is starlike of order $(1-2\mu)/(1-\mu)$, $\mu \in (0, 1/2]$. Since the class of starlike functions of a fixed order is a compact family, it is evident that $f = \lim_{n\to\infty} f_n$ is also starlike of order $(1-2\mu)/(1-\mu)$.

Finally note that when $f(z) = z + \mu z^2$, then

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\frac{1+2\mu z}{1+\mu z} \longrightarrow \frac{1-2\mu}{1-\mu} \quad \text{as } z \longrightarrow -1. \tag{2.21}$$

Hence the order of starlikeness is sharp. □

For $\mu = 1/2$, Theorem 2.3 reduces to the following result of Ruscheweyh.

Corollary 2.4 (see [4]). *Let* $a_1 = 1$, $a_k \geq 0$ *satisfy*

$$ka_k \geq (k+1)a_{k+1}, \qquad (2k-1)a_{2k} \geq (2k+1)a_{2k+1}, \quad k \geq 1. \tag{2.22}$$

Then $f(z) = \sum_{k=1}^{\infty} a_k z^k$ *is starlike.*

Using Lemma 2.1 and the minimum principle for harmonic functions, the following sufficient condition for $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ to be close to convex of order $1 - \mu/\mu_0$ with respect to the starlike function $g(z) = z$ is obtained.

Theorem 2.5. *Let* $a_1 = 1$, $a_k \geq 0$ *satisfy*

$$0 \leq na_n \leq \cdots \leq (k+1)a_{k+1} \leq ka_k \leq \cdots \leq 3a_3 \leq 2a_2 \leq \frac{\mu}{\mu_0}, \quad \mu \in (0, \mu_0], \tag{2.23}$$

$$2(k+\mu-1)a_{2k} \geq (2k+1)a_{2k+1}, \quad 1 \leq k \leq \frac{n-1}{2}. \tag{2.24}$$

Then $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ *satisfies* $\operatorname{Re} f_n'(z) > 1 - \mu/\mu_0$.

Proof. Let $\beta := 1 - \mu/\mu_0$, $b_0 = 1$, and $b_k := [(k+1)/(1-\beta)]a_{k+1}$, $1 \leq k \leq n-1$. Then

$$\frac{f_n'(z) - \beta}{1-\beta} = \sum_{k=0}^{n-1} b_k z^k. \tag{2.25}$$

Letting $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, in (2.25), it follows that

$$\operatorname{Re} \frac{f_n'(z) - \beta}{1-\beta} = \sum_{k=0}^{n-1} b_k \cos k\theta. \tag{2.26}$$

Employing the same argument used in the proof of Theorem 2.3, it is sufficient to consider only the interval $0 \leq \theta \leq \pi$.

Now $a_k \geq 0$ implies $b_k \geq 0$, and inequality (2.23) shows that

$$b_k - b_{k+1} = \frac{(k+1)a_{k+1} - (k+2)a_{k+2}}{1-\beta} \geq 0, \quad 1 \leq k \leq n-2. \tag{2.27}$$

Also, $b_0 \geq b_1$ since $a_2 \leq (1-\beta)/2 = \mu/(2\mu_0)$.

Inequality (2.24) also yields

$$(k+\mu-1)b_{2k-1} - kb_{2k} = \frac{k\left((2k+2\mu-2)a_{2k} - (2k+1)a_{2k+1}\right)}{1-\beta} \geq 0. \tag{2.28}$$

Thus $\{b_k\}$ satisfies the hypothesis of Lemma 2.1. The minimum principle for harmonic functions yields $\operatorname{Re} f_n'(z) > \beta = 1 - \mu/\mu_0$. □

The next result gives a sufficient condition for f_n to be convex in the direction of the imaginary axis, which is equivalent to $f_n \in \mathcal{K}_g$ with $g(z) = z/(1-z^2)$.

Theorem 2.6. *Let $a_1 = 1$, $a_k \geq 0$ satisfy*

$$\begin{aligned} ka_k &\geq (k+1)a_{k+1}, \quad (k = 1, \ldots, n-1), \\ (k+\mu-1)(2k-1)a_{2k-1} &\geq 2k^2 a_{2k}, \quad \left(k = 1, \ldots, \frac{n}{2}\right), \end{aligned} \tag{2.29}$$

$\mu \in (0,1)$. Then $f_n(z) = \sum_{k=1}^{n} a_k z^k$ is convex in the direction of the imaginary axis whenever $\mu \in (0, 1/2]$.

Proof. Since the coefficients of f_n are real, f_n is convex in the direction of the imaginary axis if and only if zf_n' is typically real. Let $b_k = ka_k$. Then

$$zf_n'(z) = \sum_{k=1}^{n} b_k z^k. \tag{2.30}$$

Inequality (2.29) shows that the coefficients b_k satisfy the hypothesis of Lemma 2.2. Hence by taking $z = e^{i\theta}$, $\theta \in (0, \pi)$, in (2.30) and using Lemma 2.2, it follows that

$$\operatorname{Im} zf_n'(z) = \sum_{k=1}^{n} b_k \sin k\theta > 0 \tag{2.31}$$

for $z \in \mathbb{D} \cap \{z : \operatorname{Im} z > 0\}$. A similar argument used in the proof of Theorem 2.3 now leads to the conclusion that zf_n' is typically real for $0 < \mu \leq 1/2$. □

Corollary 2.7. *Let $a_1 = 1$, $a_k \geq 0$ satisfy*

$$ka_k \geq (k+1)a_{k+1}, \qquad (k+\mu-1)(2k-1)a_{2k-1} \geq 2k^2 a_{2k}, \tag{2.32}$$

$0 < \mu \leq 1/2$. Then $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is convex in the direction of the imaginary axis.

Proof. It is evident from Theorem 2.6 that f_n is convex in the direction of the imaginary axis for any positive integer n. The result now follows in light of the compactness of the class of functions convex in the direction of the imaginary axis. □

The choice of $\mu = 1/2$ in Corollary 2.7 reduces to a result of Acharya [6].

Corollary 2.8 (see [6, Theorem 2.3.5, page 33]). *Let $a_1 = 1$, $a_k \geq 0$ satisfy*

$$ka_k \geq (k+1)a_{k+1}, \qquad (2k-1)^2 a_{2k-1} \geq (2k)^2 a_{2k}, \quad k \geq 1. \tag{2.33}$$

Then $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is convex in the direction of the imaginary axis.

3. Starlikeness of the Gaussian Hypergeometric Functions

For complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by the series

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k. \tag{3.1} $$

When $a = -m$ or $b = -m$, ${}_2F_1(a,b;c;z)$ reduces to a hypergeometric polynomial of degree m. Properties on the hypergeometric functions are treated in [13]. The geometry of close to convexity, starlikeness, and convexity of $z\,{}_2F_1(a,b;c;z)$ has been studied in various works, for example, those of [6, 14–19]. Notwithstanding these works, the exact range of the triplets (a,b,c) for starlikeness as well as for the other geometric structures of normalized Gaussian hypergeometric functions remains a formidable challenge.

In this section, conditions on the triplets (a,b,c) are determined that will ensure the function $z\,{}_2F_1(a,b;c;z)$ is starlike of a certain order in $\mathbb{D}$. Several examples are presented to compare the range obtained with some of those earlier works. Sufficient conditions for starlikeness of the odd Gaussian hypergeometric functions $z\,{}_2F_1(a,b;c;z^2)$ are also obtained.

Theorem 3.1. *Let $\mu \in (0, 1/2]$ and $a, b \leq \mu - 1$ satisfy $(a)_k (b)_k \geq 0$ for $k \geq 2$. If $\mu c \geq ab$, then $z\,{}_2F_1(a,b;c;z)$ is starlike of order $(1-2\mu)/(1-\mu)$.*

Proof. The function $z\,{}_2F_1(a,b;c;z)$ can be expressed as

$$ z\,{}_2F_1(a,b;c;z) = \sum_{k=1}^{\infty} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} z^k = \sum_{k=1}^{\infty} a_k z^k, \tag{3.2} $$

where $a_1 = 1$ and

$$ a_{k+1} = \frac{(a+k-1)(b+k-1)}{k(c+k-1)} a_k, \quad k \geq 1. \tag{3.3} $$

The sequence $\{a_k\}$ is first shown to satisfy conditions (2.9) and (2.10) in Theorem 2.3. Now consider

$$ \begin{aligned} &\big((1-\mu)k - 1 + 2\mu\big)a_k - \big((1-\mu)k + \mu\big)a_{k+1} \\ &\quad = \big((1-\mu)k - 1 + 2\mu\big)a_k - \big((1-\mu)k + \mu\big)\frac{(a+k-1)(b+k-1)}{k(c+k-1)} a_k \\ &\quad = \frac{a_k}{k(c+k-1)}\big(k\big((1-\mu)k - 1 + 2\mu\big)(c+k-1) - \big((1-\mu)k + \mu\big)(a+k-1)(b+k-1)\big) \\ &\quad = \frac{a_k}{k(c+k-1)}\Big(A_1(k-1)^2 + A_2(k-1) + A_3\Big), \end{aligned} \tag{3.4} $$

with

$$\begin{aligned} A_1 &= (c-a-b)(1-\mu), \\ A_2 &= (-a-b+\mu+ab\mu)+(c-ab), \\ A_3 &= \mu c - ab. \end{aligned} \tag{3.5}$$

The conditions $a, b \leq \mu - 1 \leq -1/2$ and $\mu c \geq ab > 0$ show that $A_1 > 0$, $A_3 \geq 0$, and $A_2 > c - ab \geq (1/\mu - 1)ab > 0$. Thus

$$A_1(k-1)^2 + A_2(k-1) + A_3 \geq 0, \quad k \geq 1, \tag{3.6}$$

and inequality (2.9) holds.

To verify inequality (2.10), consider

$$\begin{aligned} &(k+\mu-1)(2(1-\mu)k-1+2\mu)a_{2k} - k(2(1-\mu)k+\mu)a_{2k+1} \\ &\quad = \frac{a_{2k}}{2k(c+2k-1)}\big(2k(2(1-\mu)k+\mu)(k+\mu-1)(c+2k-1) \\ &\qquad\qquad -k(2(1-\mu)k+\mu)(a+2k-1)(b+2k-1)\big) \\ &\quad = \frac{a_{2k}}{2(c+2k-1)}\Big(B_1(k-1)^2 + B_2(k-1) + B_3\Big), \end{aligned} \tag{3.7}$$

with

$$\begin{aligned} B_1 &= 4\Big(-2-(a+b)(1-\mu)+4\mu-2\mu^2\Big)+4c(1-\mu), \\ B_2 &= 2\Big(-4-(a+b)(3-2\mu)+7\mu-2\mu^2\Big)+2c\Big(1+2\mu-2\mu^2\Big)+2ab\mu, \\ B_3 &= (-2-(a+b)(2-\mu)+3\mu)+2(c\mu-ab)+ab\mu. \end{aligned} \tag{3.8}$$

Again from the conditions $a, b \leq \mu - 1 \leq -1/2$ and $\mu c \geq ab > 0$, computations show that

$$\begin{aligned} B_1 &> 4\Big(-2-2(\mu-1)(1-\mu)+4\mu-2\mu^2\Big) = 0, \\ B_2 &> 2\Big(-4-2(\mu-1)(3-2\mu)+7\mu-2\mu^2\Big) = 2\Big(2-3\mu+2\mu^2\Big) > 0, \\ B_3 &> -2-2(\mu-1)(2-\mu)+3\mu = 2-3\mu+2\mu^2 > 0. \end{aligned} \tag{3.9}$$

Hence inequality (2.10) also holds. The desired result now readily follows from Theorem 2.3. □

Choosing $a = b = \mu - n$ and $a = b = -\mu - n$, respectively, in Theorem 3.1 yields the following result.

Corollary 3.2. *Let $\mu \in (0, 1/2]$ and n be a fixed integer.*

(i) *If $\mu c \geq (\mu - n)^2$, then $z\,{}_2F_1(\mu - n, \mu - n; c; z)$ is starlike of order $(1 - 2\mu)/(1 - \mu)$. In particular, $f(z) = z\sqrt{1 - z} + z\sqrt{z}\ \arcsin(\sqrt{z})$ is starlike.*

(ii) *If $\mu c \geq (\mu + n)^2$, then $z\,{}_2F_1(-\mu - n, -\mu - n; c; z)$ is starlike of order $(1 - 2\mu)/(1 - \mu)$.*

Observe that the starlikeness of

$$z\,{}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; z\right) = z\sqrt{1 - z} + z\sqrt{z}\arcsin(\sqrt{z}) \tag{3.10}$$

follows from (i) by taking $n = 1$, $\mu = 1/2$, and $c = 1/2$.

For $\mu = 1/2$, Theorem 3.1 leads to the following result.

Corollary 3.3. *Let $a, b \leq -1/2$ satisfy $(a)_k(b)_k \geq 0$ for $k \geq 2$. If $c \geq 2ab$, then $z\,{}_2F_1(a, b; c; z)$ is starlike.*

Remark 3.4. Sufficient conditions for starlikeness (of order 0) for $z\,{}_2F_1(a, b; c; z)$ are also obtained in [6, Theorem 4.3.1, page 60]. The result in Theorem 3.1 however investigated conditions for $z\,{}_2F_1(a, b; c; z)$ to be starlike of a certain order.

Corollary 3.5. *Let $\mu \in (0, 1/2]$ and $a, b \leq \mu - 2$ satisfy $(a + 1)_k(b + 1)_k \geq 0$, $k \geq 2$. If $\mu(c + 1) \geq (a + 1)(b + 1)$, then the function $(c/ab)\ [{}_2F_1(a, b; c; z) - 1]$ is convex of order $(1 - 2\mu)/(1 - \mu)$.*

Proof. If f is starlike of a certain order, then Alexander's transformation $\int_0^z f(t)/t\,dt$ yields a function convex of the same order. Under the given hypothesis and using Theorem 3.1, it is clear that $z\,{}_2F_1(a + 1, b + 1; c + 1; z)$ is starlike of order $(1 - 2\mu)/(1 - \mu)$. The Gaussian hypergeometric function satisfies the identity

$$ab\,{}_2F_1(a + 1, b + 1; c + 1; z) = c\,{}_2F_1'(a, b; c; z). \tag{3.11}$$

Thus

$$\frac{c}{ab}[{}_2F_1(a, b; c; z) - 1] = \int_0^z \frac{t\,{}_2F_1(a + 1, b + 1; c + 1; t)}{t}dt \tag{3.12}$$

is convex of order $(1 - 2\mu)/(1 - \mu)$. □

We state the following recent result by Hästö et al. [14] related to the starlikeness of $z\,{}_2F_1(a, b; c; z)$.

Theorem 3.6 (see [14]). *Let a, b, and c be nonzero real numbers such that ${}_2F_1(a, b; c; z)$ has no zero in $\mathbb{D}$. Then $z\,{}_2F_1(a, b; c; z)$ is starlike of order $(1 - 2\mu)/(1 - \mu)$, $\mu \in (0, 1/2]$, if*

(1) $c \geq 1 + a + b - ab/\widetilde{\mu}$,

(2) $C + \widetilde{\mu} \geq 2A$,

(3) $(\widetilde{\mu} + \widetilde{\mu}^2)C + 2BD + D^2 \geq 0$,

where $\widetilde{\mu} = \mu/(1 - \mu)$, $A = \widetilde{\mu}^2 - \widetilde{\mu}(a + b) + ab$, $B = \widetilde{\mu}(a + b) - 2\widetilde{\mu}^2$, $C = \widetilde{\mu}\widetilde{c} + ab$, $D = \widetilde{\mu}\widetilde{c}$, and $\widetilde{c} = c - 1 - a - b$.

Next we provide some examples to show that Theorem 3.1 gives a better range of triplets (a,b,c) than those obtained in earlier works.

Example 3.7. If $a,b \in (-1,-2/3]$, then $z\,{}_2F_1(a,b;3ab;z)$ is starlike of order $1/2$. The latter fact follows from Theorem 3.1 by taking $\mu = 1/3$. This result cannot be obtained from Hästö et al. [14, Corollary 1.7] since

$$\max\{1+a+b-2ab, 1+2ab, 1+|a-b|\} = 1+2ab > 3ab. \tag{3.13}$$

Example 3.8. Let $\mu \in (0,1/2]$ and $-1-\sqrt{(1+\mu)/(1-\mu)} < a \le \mu - 1$. Then $z\,{}_2F_1(a,a;a^2/\mu;z)$ belongs to $\mathcal{S}^*((1-2\mu)/(1-\mu))$, which follows from Theorem 3.1 with $a=b$ and $c = a^2/\mu$. Comparing with Theorem 3.6, note that

$$\begin{aligned} C+\tilde{\mu}-2A &= (c-1-a-b)\tilde{\mu} + ab + \tilde{\mu} - 2\tilde{\mu}^2 + 2\tilde{\mu}(a+b) - 2ab \\ &= c\tilde{\mu} - 2a\tilde{\mu} + a^2 - 2\tilde{\mu}^2 + 4\tilde{\mu}a - 2a^2 \\ &= \frac{a^2}{\mu}\tilde{\mu} - a^2 + 2a\tilde{\mu} - 2\tilde{\mu}^2 \\ &= \tilde{\mu}\left(a^2 + 2a - 2\tilde{\mu}\right) \\ &= \tilde{\mu}\left((a+1)^2 - \frac{1+\mu}{1-\mu}\right) < 0, \end{aligned} \tag{3.14}$$

and so the second condition in Theorem 3.6 does not hold. Therefore, the range of the parameters in Theorem 3.6 does not include the range in this example.

The next result gives conditions on triplets (a,b,c) for which the odd Gaussian hypergeometric functions $z\,{}_2F_1(a,b;c;z^2)$ are starlike of order $(1-3\mu)/(1-\mu)$.

Theorem 3.9. *Let $a,b \le \mu - 1$, $\mu \in (0,1/3]$ satisfy $(a)_k(b)_k \ge 0$, $k \ge 2$. If $\mu c \ge ab$, then $z\,{}_2F_1(a,b;c;z^2)$ is starlike of order $(1-3\mu)/(1-\mu)$.*

Proof. Let $f(z) = z\,{}_2F_1(a,b;c;z)$ and $zg(z) = f(z^2)$. Then

$$\frac{zg'(z)}{g(z)} = 2\frac{z^2 f'(z^2)}{f(z^2)} - 1. \tag{3.15}$$

Theorem 3.1 shows that $f \in \mathcal{S}^*((1-2\mu)/(1-\mu))$, and therefore

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = 2\operatorname{Re}\frac{z^2 f'(z^2)}{f(z^2)} - 1 > \frac{1-3\mu}{1-\mu}, \tag{3.16}$$

that is, $g \in \mathcal{S}^*((1-3\mu)/(1-\mu))$. □

For $\mu = 1/3$, Theorem 3.9 reduces to the following result.

Corollary 3.10. *Let $a, b \leq -2/3$ satisfy $(a)_k(b)_k \geq 0$, $k \geq 2$. Then $z\,{}_2F_1(a, b; c; z^2)$ is in $\mathcal{S}^*$ for $c \geq 3ab$.*

Note that when $a, b \in (-1, -2/3]$, then $\max\{1 + a + b - 2ab, 1 + 2ab, 1 + |a - b|\} = 1 + 2ab$ and a result in [14, Corollary 1.9] yields that $z\,{}_2F_1(a, b; c; z^2)$ is starlike provided $c \geq 1 + 2ab$. However for the given range of a and b above, evidently $1 + 2ab > 3ab$, and hence Corollary 3.10 gives a better range for c.

Acknowledgment

The work presented here was supported in part by a Research University Grant from Universiti Sains Malaysia.

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