Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 471435, 19 pages
doi:10.1155/2012/471435

Research Article

Critical Oscillation Constant for Difference Equations with Almost Periodic Coefficients

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Received 28 May 2012; Revised 10 September 2012; Accepted 13 September 2012

Academic Editor: Elena Braverman

We investigate a type of the Sturm-Liouville difference equations with almost periodic coefficients. We prove that there exists a constant, which is the borderline between the oscillation and the nonoscillation of these equations. We compute this oscillation constant explicitly. If the almost periodic coefficients are replaced by constants, our result reduces to the well-known result about the discrete Euler equation.

1. Introduction

In this paper, we analyse the second-order Sturm-Liouville equation

$$\Delta(r_k \Delta y_k) + q_k y_{k+1} = 0, \quad r_k > 0, \tag{$*$}$$

whose oscillation properties are widely studied over the last few decades. We begin with a short literature overview concerning the (non)oscillation of ($*$) and of some direct generalizations (including half-linear equations and dynamic equations on time scales).

Basic necessary and sufficient conditions for ($*$) in order to be oscillatory are derived in [1–3]. In [4] (see also [5]), the concept of a phase is established to obtain other oscillation criteria. For the matrix difference equations of the form of ($*$), we refer to [6]. Several oscillation criteria for slightly more general equations are presented in [7, 8]. The oscillation theory for the corresponding higher-order two-term Sturm-Liouville difference equations can be found in [9–11] (for differential case, see [12]).

Fundamental aspects of Sturmian theory (and some oscillation criteria) for second-order Sturm-Liouville equations on arbitrary time scales are formulated in [13]. Oscillation criteria for second-order difference equations can be obtained from oscillation criteria for more general dynamic equations. The oscillation properties of second-order linear dynamic equations, which have the Sturm-Liouville difference equations as special cases, are considered, for example, in [14].

As an illustration, we mention a particular Sturm-Liouville equation of which the complete oscillation classification is done as a consequence of general results on time scales. Using the comparison theorem for second-order linear dynamic equations, it is shown in [15] that the difference equation

$$\Delta^2 y_k + b\frac{(-1)^k}{k^c} y_{k+1} = 0 \tag{1.1}$$

is oscillatory for any $c < 1$ and $b \neq 0$. Further, it is obtained in [16] (based on results of [17, 18]) that the equation

$$\Delta^2 y_k + \left[\frac{a}{k^2} + b\frac{(-1)^k}{k}\right] y_{k+1} = 0 \tag{1.2}$$

is oscillatory if and only if $4a > 1 - b^2$. Finally in [16], applying the Willett-Wong-type theorems for second-order linear dynamic equations, there is given the full oscillation analysis of $(*)$ for

$$r_k = 1, \qquad q_k = \frac{a}{k^{c+1}} + b\frac{(-1)^k}{k^c} \tag{1.3}$$

with regard to arbitrary $a, b, c \in \mathbb{R}$.

The importance of the oscillation results about second-order equations lies among others in the fact that such results can be used to study the oscillation and nonoscillation properties of solutions of different equations. For example (see [19] and also [20]), all solutions of the delay equation $\Delta y_k + p_k y_{k-n} = 0$ oscillate if and only if all solutions of a certain type of $(*)$ with $r_k = 1$ oscillate.

The main aim of this paper is to present a sharp oscillation constant for the Euler-type difference equation

$$\Delta(r_k \Delta y_k) + \frac{\gamma s_k}{(k+1)k} y_{k+1} = 0, \tag{$**$}$$

where $\gamma \in \mathbb{R}$, $\inf\{r_k\} > 0$, and $\{r_k\}, \{s_k\}$ are positive almost periodic sequences. More precisely, we show that $(**)$ is the so-called conditionally oscillatory; that is, we prove that there exists a positive constant K (the oscillation constant) such that $(**)$ is oscillatory for $\gamma > K$ and non-oscillatory for $\gamma < K$.

Our research is motivated by the continuous case. It is a famous result due to Kneser [21] that the differential Euler equation

$$y''(t) + \frac{\gamma}{t^2} y(t) = 0 \tag{1.4}$$

is conditionally oscillatory with the oscillation constant $K = 1/4$. It is known (see [22]) that the equation

$$\left[r(t)y'(t)\right]' + \frac{\gamma s(t)}{t^2}\, y(t) = 0, \tag{1.5}$$

where r, s are positive periodic continuous functions, is conditionally oscillatory as well. We also refer to [23] and [24–29] which generalize [23] (for the discrete case, see [30]). Since the Euler difference equation

$$\Delta^2 y_k + \frac{\gamma}{(k+1)k} y_{k+1} = 0 \tag{1.6}$$

is conditionally oscillatory with the oscillation constant $K = 1/4$ (see [31]), it is natural to analyse the conditional oscillation of (∗∗). Note that the announced result is more general than the results known in the continuous case, because (∗∗) has almost periodic coefficients. The conditional oscillation of discrete equations with constant coefficients can be generalized in other ways. Point out [32], where an oscillation constant is characterized. The constant from [32] coincides with our oscillation constant if the considered coefficients are asymptotically constant.

Solutions of the second-order Sturm-Liouville difference equations with periodic coefficients are studied in [33] (see also [34, 35]). In [36], the half-linear differential equations of the second order with the Besicovitch almost periodic coefficients are considered and an oscillation theorem for these equations is obtained.

In the last years, many results dealing with the conditional oscillation of second-order equations and two-term equations of even order appeared. The two-term difference equation of even order

$$(-1)^{n+1}\Delta^n\left(\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\Delta^n y_k\right) + q_k y_{k+n} = 0, \tag{1.7}$$

where Γ denotes the gamma function, is studied in [9, 10]. Results concerning the half-linear difference equation

$$\Delta\left[r_k \Phi(\Delta y_k)\right] + q_k \Phi(y_{k+1}) = 0, \tag{1.8}$$

where

$$r_k > 0, \qquad \Phi(y) = |y|^{\alpha-1}\operatorname{sgn} y, \quad \alpha > 1, \tag{1.9}$$

can be found in [37] for $r_k = 1$, $q_k = \gamma(k+1)^{-\alpha}$ and also in [38, 39] (for dynamic half-linear equations on time scales, see [40–42]).

The paper is organized as follows. In Section 2, we mention only necessary preliminaries and an auxiliary result. Our main result is proved in Section 3, where the particular case concerning the equation with periodic coefficients is formulated as well. The paper is finished by concluding remarks and simple examples.

2. Preliminaries

We begin this section recalling some elements of the oscillation theory of the Sturm-Liouville difference equation

$$\Delta\left(r_k \Delta y_k\right) + q_k y_{k+1} = 0, \quad r_k > 0,\ k \in \mathbb{N}. \tag{2.1}$$

For more details, we can refer to books [43, 44] and references cited therein.

We recall that an interval $(a, a+1]$, $a \in \mathbb{N}$, contains the generalized zero of a solution $\{y_k\}$ of (2.1) if $y_a \neq 0$ and $y_a y_{a+1} \leq 0$. Equation (2.1) is said to be conjugate on $\{a, \ldots, a+n\}$, $n \in \mathbb{N}$, if there exists a solution which has at least two generalized zeros on $(a, \ldots, a+n+1]$ or if the solution $\{\widetilde{y}_k\}$ satisfying $\widetilde{y}_a = 0$ has at least one generalized zero on $(a, \ldots, a+n+1]$. Otherwise, (2.1) is said to be disconjugate on $\{a, \ldots, a+n\}$. Since Sturmian theory is valid for difference equations, all solutions of (2.1) have either a finite or an infinite number of generalized zeros on $\mathbb{N}$. Hence, we can categorize these equations as oscillatory and non-oscillatory.

Definition 2.1. Equation (2.1) is called non-oscillatory provided a solution of (2.1) is disconjugate at infinity, that is, there exists $N \in \mathbb{N}$ such that (2.1) is disconjugate on any set $[N, N+m] \cap \mathbb{N}$, $m \in \mathbb{N}$. Otherwise, we say that (2.1) is oscillatory.

Since we study a special case of (2.1), when the coefficients are almost periodic, we also mention the basics of the theory of almost periodic sequences. Here, we refer to each one of books [45, 46].

Definition 2.2. A real sequence $\{f_k\}_{k\in\mathbb{Z}}$ is called almost periodic if, for any $\varepsilon > 0$, there exists a positive integer $p(\varepsilon)$ such that any set consisting of $p(\varepsilon)$ consecutive integers contains at least one integer l with the property that

$$\left|f_{k+l} - f_k\right| < \varepsilon, \quad k \in \mathbb{Z}. \tag{2.2}$$

We say that a sequence $\{g_k\}_{k=1}^{\infty}$ is almost periodic if there exists an almost periodic sequence $\{f_k\}_{k\in\mathbb{Z}}$ for which $f_k = g_k$, $k \in \mathbb{N}$.

The above definition of almost periodicity is based on the Bohr concept. Now we formulate a necessary and sufficient condition for a sequence to be almost periodic. The following theorem is often used as an equivalent definition (the Bochner one) of almost periodicity for $k \in \mathbb{Z}$.

Theorem 2.3. *A sequence $\{f_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is almost periodic if and only if any sequence of the form $\{f_{k+h(n)}\}$, where $h(n) \in \mathbb{Z}$, $n \in \mathbb{N}$, has a uniformly convergent subsequence with respect to k.*

Proof. See [45, Theorem 1.26]. □

Corollary 2.4. *Let $\{f_k\}$ be almost periodic. The sequence $\{1/f_k\}$ is almost periodic if and only if*

$$\inf\left\{\left|f_k\right|;\ k \in \mathbb{N}\right\} > 0. \tag{2.3}$$

Proof. The corollary follows from [45, Theorem 1.27] and [47, Theorem 1.9] (or directly from Theorem 2.3). It suffices to use that (2.3) implies $\inf\{|f_k|\} > 0$ for any almost periodic sequence $\{f_k\}$ if $k \in \mathbb{Z}$. □

Note that there exist nonzero almost periodic sequences $\{f_k\}$ for which (2.3) is not satisfied (see, e.g., [48, Theorem 3]).

Theorem 2.5. *If $\{f_k\}$ is an almost periodic sequence, then the limit*

$$M(\{f_k\}) := \lim_{n\to\infty} \frac{f_k + f_{k+1} + \cdots + f_{k+n}}{n+1} \tag{2.4}$$

exists uniformly with respect to k.

Proof. See [45, Theorem 1.28]. □

Definition 2.6. Let $\{f_k\}$ be almost periodic. The number $M(\{f_k\})$ introduced in (2.4) is called the mean value of $\{f_k\}$.

Remark 2.7. For any positive almost periodic sequence $\{f_k\}$, we have $M(\{f_k\}) > 0$. Indeed, if we put $\varepsilon = f_1/2$ and find a corresponding $p(\varepsilon)$ in Definition 2.2, then we obtain

$$M(\{f_k\}) \geq \frac{f_1}{2p(\varepsilon)} > 0. \tag{2.5}$$

In the proof of our main result, we use an adapted Riccati technique. The classical Riccati technique deals with the so-called Riccati difference equation, which we obtain from (2.1) using the substitution $w_k = r_k(\Delta y_k / y_k)$, that is, we obtain the equation

$$\Delta w_k + q_k + \frac{w_k^2}{w_k + r_k} = 0, \quad k \in \mathbb{N}. \tag{2.6}$$

Putting $\zeta_k = -k w_k$, we adapt (2.6) to our purposes. A direct calculation leads to the equation

$$\Delta \zeta_k = \frac{1}{k}\left[k(k+1)q_k + \zeta_k + \frac{(k+1)\zeta_k^2}{k r_k - \zeta_k}\right], \quad k \in \mathbb{N}. \tag{2.7}$$

We also mention two lemmas which we use to prove the main result.

Lemma 2.8. *Let the equation*

$$\Delta(r_k \Delta y_k) + q_k y_{k+1} = 0, \quad k \in \mathbb{N}, \tag{2.8}$$

where $\sup\{r_k;\ k \in \mathbb{N}\} < \infty$ and $r_k, q_k > 0$, $k \in \mathbb{N}$, be non-oscillatory. For any solution $\{\zeta_k\}$ of the associated equation (2.7), there exists $k_0 \in \mathbb{N}$ such that, if $\zeta_{k_0+m} < 0$ for some $m \in \mathbb{N}$, then $\zeta_{k+m} < 0$, $k \geq k_0$.

Proof. Let $\{y_k\}$ be a solution of the non-oscillatory equation (2.8) for which $y_k y_{k+1} > 0$, $k \geq k_0$. From [43, Lemma 6.6.1] it follows that the sequence $\{w_k\}_{k=k_0}^{\infty}$, where $w_k = r_k(\Delta y_k / y_k)$, is decreasing. Further, [43, Theorem 6.6.2] implies that $\lim_{k\to\infty} w_k = 0$. Thus, the sequence $\{w_k\}_{k=k_0}^{\infty}$ is positive, that is, $\zeta_k = -k w_k < 0$, $k \geq k_0$. □

Lemma 2.9. *If there exists a solution $\{\zeta_k\}$ of the associated equation* (2.7) *satisfying $\zeta_k < 0$ for all $k \geq k_0$, then* (2.8) *is non-oscillatory.*

Proof. The statement of the lemma follows from [44, Theorem 6.16]. □

3. Oscillation Constant

This section is devoted to the main result of our paper. After its proof, within the concluding remarks, we formulate as a corollary the result which deals with periodic equations. This corollary is the discrete counterpart of the main result of [49].

Theorem 3.1. *Let the equation*

$$\Delta(r_k \Delta x_k) + \frac{\gamma s_k}{(k+1)k} x_{k+1} = 0, \quad k \in \mathbb{N}, \tag{3.1}$$

where $\gamma \in \mathbb{R}$ and $\{r_k\}$ and $\{s_k\}$ are positive almost periodic sequences satisfying

$$\inf\{r_k;\ k \in \mathbb{N}\} > 0, \tag{3.2}$$

be arbitrarily given. Let

$$K := \left[4M\left(\left\{r_k^{-1}\right\}\right)M(\{s_k\})\right]^{-1}. \tag{3.3}$$

Then, (3.1) *is oscillatory for $\gamma > K$ and non-oscillatory for $\gamma < K$.*

Proof. At first, let us prepare several estimates which we will use to prove the theorem. Henceforth, for given $\gamma \neq K$, we will consider $\alpha \in \mathbb{N}$ and $\vartheta > 0$ such that

$$\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i > 2\vartheta, \quad k \in \mathbb{N}, \tag{3.4}$$

$$8\left|\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i} - \frac{1}{\alpha}\sum_{i=l}^{l+\alpha-1}\frac{1}{r_i}\right| < \left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^2 |\gamma - K|\vartheta, \quad k, l \in \mathbb{N}, \tag{3.5}$$

$$2\left|K - \left(\frac{4}{\alpha^2}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i}\sum_{i=l}^{l+\alpha-1} s_i\right)^{-1}\right| < |\gamma - K|, \quad k, l \in \mathbb{N}. \tag{3.6}$$

The fact that such numbers α, ϑ exist follows from Theorem 2.5 and Remark 2.7 (consider also Corollary 2.4 with (3.2)). We put

$$r^{-} := \inf\{r_k;\ k \in \mathbb{N}\}, \qquad r^{+} := \sup\{r_k;\ k \in \mathbb{N}\}, \qquad s^{+} := \sup\{s_k;\ k \in \mathbb{N}\}. \tag{3.7}$$

The adapted Riccati equation associated to (3.1) has the form (see (2.7))

$$\Delta\zeta_k = \frac{1}{k}\left[\gamma s_k + \zeta_k + \frac{(k+1)\zeta_k^2}{kr_k - \zeta_k}\right]. \tag{3.8}$$

Since one can express

$$\zeta_k + \frac{(k+1)\zeta_k^2}{kr_k - \zeta_k} = \frac{k\zeta_k(r_k + \zeta_k)}{kr_k + |\ \zeta_k\ |} \quad \text{if } \zeta_k < 0 \text{ for some } k \in \mathbb{N}, \tag{3.9}$$

it is valid that

$$\Delta\zeta_k > \frac{\gamma s_k}{k} > 0 \quad \text{if } \zeta_k < -r^{+} \text{ for some } k \in \mathbb{N}, \tag{3.10}$$

and that ($C > 0$ is arbitrarily given)

$$\begin{aligned} |\Delta\zeta_k| &\le \frac{1}{k}\left[\gamma s^{+} + \frac{k|\zeta_k|(r_k + |\zeta_k|)}{kr_k + |\zeta_k|}\right] \\ &< \frac{\gamma s^{+} r^{-} + C(r^{+} + C)}{kr^{-}} \quad \text{if } \zeta_k \in (-C, 0) \text{ for some } k \in \mathbb{N}. \end{aligned} \tag{3.11}$$

Particularly, if $\zeta_k < 0$ for all sufficiently large k, then there exists $\delta < 0$ such that

$$\zeta_k > \delta \quad \text{for considered } k. \tag{3.12}$$

Indeed, it follows directly from (3.10) and (3.11).

Similarly, applying (3.10) and (3.11), it is seen that there exists $\overline{k} = \overline{k}(\delta, \alpha) \in \mathbb{N}$ for any $\delta < 0$ and $\alpha \in \mathbb{N}$ with the property that the solution $\{\zeta_k\}_{k=\overline{k}+m}^{\infty}$ of the Cauchy problem

$$\Delta\zeta_k = \frac{1}{k}\left[\gamma s_k + \zeta_k + \frac{(k+1)\zeta_k^2}{kr_k - \zeta_k}\right], \qquad \zeta_{\overline{k}+m} = \zeta_0 \in (2\delta, \delta], \tag{3.13}$$

where $m \in \mathbb{N}$, satisfies

$$\{\zeta_k\}_{k=\overline{k}+m}^{\infty} \subseteq (\min\{2\zeta_0, -2r^{+}\}, \infty), \qquad \{\zeta_k\}_{k=\overline{k}+m}^{\overline{k}+m+\alpha-1} \subseteq (\min\{2\zeta_0, -2r^{+}\}, 0), \tag{3.14}$$

and hence there exists $\Theta > 0$ (consider again (3.11)) for which

$$\left|\zeta_{\bar{k}+m} - \zeta_{\bar{k}+m+j}\right| < \frac{\Theta}{\bar{k}}, \quad j \in \{1,\dots,\alpha-1\},\ m \in \mathbb{N}. \tag{3.15}$$

Now we can proceed to the oscillatory part of the theorem. By contradiction, we suppose that $\gamma > K$ and that (3.1) is non-oscillatory. According to Lemma 2.8, any solution $\{\zeta_k\}_{k=k_0}^{\infty}$ of the associated adapted Riccati equation (3.8) for which $\zeta_{k_0} < 0$ satisfies $\zeta_k < 0$, $k \geq k_0$, if k_0 is enough large. Further, (3.12) gives the existence of $\delta < 0$ with the property that $\zeta_k \in (\delta, 0)$ for all $k \geq k_0$. Using (3.11), we obtain

$$|\Delta\zeta_k| < \frac{\gamma s^+ r^- - \delta(r^+ - \delta)}{k r^-}, \quad k \geq k_0. \tag{3.16}$$

Our goal is to achieve a contradiction with $\zeta_k \in (\delta, 0)$ by estimating the arithmetic mean of α subsequent values of ζ_k. We denote

$$\xi_k := \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1} \zeta_i \in (\delta, 0), \quad k \geq k_0, \tag{3.17}$$

and compute (for $\Delta\xi_k > 0$)

$$\begin{aligned}
\Delta\xi_k &= \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\Delta\zeta_i = \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{i}\left[\gamma s_i + \zeta_i + \frac{(i+1)\zeta_i^2}{ir_i - \zeta_i}\right] \\
&\geq \frac{1}{k+\alpha-1}\left[\frac{\gamma}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i + \xi_k + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i}\right] \\
&= \frac{1}{k+\alpha-1}\left\{\frac{\gamma}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i - \frac{A_k^2}{2} + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\left[\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i} - \frac{\zeta_i^2}{r_i}\right] + \xi_k + \frac{A_k^2}{2} + \frac{B_k^2}{2} + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{\zeta_i^2}{r_i} - \frac{B_k^2}{2}\right\}
\end{aligned} \tag{3.18}$$

or (for $\Delta\xi_k < 0$)

$$\begin{aligned}
\Delta\xi_k = \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\Delta\zeta_i \geq \frac{1}{k}\Bigg\{&\frac{\gamma}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i - \frac{A_k^2}{2} + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\left[\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i} - \frac{\zeta_i^2}{r_i}\right] \\
&+ \xi_k + \frac{A_k^2}{2} + \frac{B_k^2}{2} + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{\zeta_i^2}{r_i} - \frac{B_k^2}{2}\Bigg\},
\end{aligned} \tag{3.19}$$

where

$$A_k := \left(\frac{2}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i}\right)^{-1/2}, \qquad B_k := |\xi_k|\left(\frac{2}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i}\right)^{1/2}, \quad k \geq k_0. \tag{3.20}$$

Note that we can choose $k_0 \geq \alpha$. For reader's convenience, we will estimate $\Delta\xi_k$ stepwise.

Step 1. We show that there exist $k_1 \geq k_0$ and $\Gamma > 0$ such that

$$\frac{\gamma}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i - \frac{A_k^2}{2} + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\left[\frac{(i+1)\zeta_i^2}{ir_i-\zeta_i} - \frac{\zeta_i^2}{r_i}\right] \geq \Gamma, \quad k \geq k_1. \tag{3.21}$$

Applying $\zeta_k \in (\delta, 0)$ for $k \geq k_0$, we have (see (3.4) and (3.6))

$$\begin{aligned}
&\frac{\gamma}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i - \frac{A_k^2}{2} + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\left[\frac{(i+1)\zeta_i^2}{ir_i-\zeta_i} - \frac{\zeta_i^2}{r_i}\right] \\
&\quad = \frac{\gamma}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i - \frac{1}{2}\left(\frac{2}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i}\right)^{-1} + \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\left[\frac{\zeta_i^2 r_i + \zeta_i^3}{r_i(ir_i-\zeta_i)}\right] \\
&\quad \geq \frac{\gamma}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i - \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i}\right)^{-1} - \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\delta^2\frac{r^+-\delta}{r^-(ir^-)} \\
&\quad \geq \left(\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i\right)\left[\gamma - \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i}\right)^{-1}\left(\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1} s_i\right)^{-1}\right] \\
&\qquad - \frac{1}{\alpha k}\sum_{i=k}^{k+\alpha-1}\frac{\delta^2(r^+-\delta)}{(r^-)^2} > \vartheta(\gamma - K) - \frac{1}{k}\left(\frac{\delta^2(r^+-\delta)}{(r^-)^2}\right), \quad k \geq k_0.
\end{aligned} \tag{3.22}$$

Thus, there exist $\Gamma > 0$ and k_1 with the property that (3.21) is satisfied for all $k \geq k_1$.

Step 2. It holds (see (3.17) and (3.20))

$$\xi_k + \frac{A_k^2}{2} + \frac{B_k^2}{2} = \frac{A_k^2}{2} - |\ \xi_k\ | + \frac{B_k^2}{2} = \frac{1}{2}(A_k - B_k)^2 \geq 0, \quad k \geq k_0. \tag{3.23}$$

Step 3. We prove that there exists $k_2 \geq k_1$ satisfying

$$\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{\zeta_i^2}{r_i} - \frac{B_k^2}{2} \geq -\frac{\Gamma}{2}, \quad k \geq k_2, \tag{3.24}$$

where Γ is taken from Step 1. Considering (3.16), we obtain

$$|\zeta_m - \zeta_n| \leq \sum_{i=k}^{k+\alpha-1}|\Delta\zeta_i| < \sum_{i=k}^{k+\alpha-1}\frac{\gamma s^+ r^- - \delta(r^+-\delta)}{ir^-} \leq \frac{1}{k}\sum_{i=k}^{k+\alpha-1}\frac{\gamma s^+ r^- - \delta(r^+-\delta)}{r^-} \tag{3.25}$$

for each $m, n \in \{k, \dots, k+\alpha-1\}$, $k \geq k_0$. Thus, it is true

$$|\zeta_m - \zeta_n| < \frac{D}{k}, \quad m, n \in \{k, \dots, k+\alpha-1\},\ k \geq k_0, \tag{3.26}$$

where

$$D := \alpha \frac{\gamma s^+ r^- - \delta(r^+ - \delta)}{r^-}. \tag{3.27}$$

Now we can calculate (see (3.17))

$$\begin{aligned}
\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{\zeta_i^2}{r_i} - \frac{B_k^2}{2} &= \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{\zeta_i^2}{r_i} - \frac{1}{2}\left(\frac{2}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i}\right)\xi_k^2 \\
&= \frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{\zeta_i^2 - \xi_k^2}{r_i} = -\frac{1}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{(|\xi_k| + |\zeta_i|)(|\xi_k| - |\zeta_i|)}{r_i} \\
&> \frac{2\delta}{\alpha}\sum_{i=k}^{k+\alpha-1}\frac{|\xi_k - \zeta_i|}{r_i} > \frac{2\delta D}{\alpha k}\sum_{i=k}^{k+\alpha-1}\frac{1}{r_i} \geq \frac{2\delta D}{k r^-}, \quad k \geq k_0.
\end{aligned} \tag{3.28}$$

Let us discuss the inequality before the final one in more detail. If we denote

$$\begin{aligned}
\widehat{\zeta}_-^k &:= \max\left\{\zeta_j;\ \zeta_j \leq \xi_k,\ j \in \{k, \dots, k+\alpha-1\}\right\}, \\
\widehat{\zeta}_+^k &:= \min\left\{\zeta_j;\ \zeta_j \geq \xi_k,\ j \in \{k, \dots, k+\alpha-1\}\right\},
\end{aligned} \tag{3.29}$$

then we easily get (applying (3.26))

$$|\zeta_i - \xi_k| \leq \max\left\{\left|\zeta_i - \widehat{\zeta}_-^k\right|, \left|\zeta_i - \widehat{\zeta}_+^k\right|\right\} < \frac{D}{k}, \quad i \in \{k, \dots, k+\alpha-1\},\ k \geq k_0. \tag{3.30}$$

Of course, (3.28) implies the existence of $k_2 \geq k_1$ such that (3.24) is satisfied.

Using the previous steps, it is possible to prove the following result. If l tends to infinity, then so do ξ_l. Combining (3.21), (3.23), and (3.24), we obtain

$$\Delta\xi_k \geq \frac{1}{k+\alpha-1}\left(\Gamma + 0 - \frac{\Gamma}{2}\right) = \frac{\Gamma}{2(k+\alpha-1)}, \quad k \geq k_2. \tag{3.31}$$

We use the estimate (3.18) because $\Delta\xi_k > 0$. Summing inequality (3.31) from k_2 to an integer $(l-1) \geq k_2$, we have

$$\xi_l \geq \xi_{k_2} + \frac{\Gamma}{2}\sum_{i=k_2}^{l-1}\frac{1}{i+\alpha-1}. \tag{3.32}$$

This estimate implies that

$$\liminf_{l\to\infty} \xi_l \geq \xi_{k_2} + \frac{\Gamma}{2}\sum_{i=k_2}^{\infty}\frac{1}{i+\alpha-1} = \infty. \tag{3.33}$$

Particularly, $\xi_k > 0$ for sufficiently large k which means that $\zeta_k > 0$ for infinitely many k. This contradiction gives that (3.1) is oscillatory for $\gamma > K$.

To prove the non-oscillatory part of Theorem 3.1, we will consider the initial value problem

$$\Delta\zeta_k = \frac{1}{k}\left[\gamma s_k + \zeta_k + \frac{(k+1)\zeta_k^2}{kr_k - \zeta_k}\right], \qquad \zeta_{\tilde{k}} = -\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} \tag{3.34}$$

for some integer

$$\tilde{k} > \overline{k}\left(-\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}, \alpha\right), \tag{3.35}$$

where $\overline{k}$ satisfies (3.14) and (3.15). Let $\gamma < K$. Analogously as in the first part of the proof, we put

$$\xi_{\tilde{k}} := \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\zeta_i < 0 \tag{3.36}$$

and we express (for $\Delta\xi_{\tilde{k}} > 0$)

$$\begin{aligned}
\Delta\xi_{\tilde{k}} &= \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\Delta\zeta_i = \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{1}{i}\left[\gamma s_i + \zeta_i + \frac{(i+1)\zeta_i^2}{ir_i - \zeta_i}\right] \\
&\le \frac{1}{\tilde{k}}\left[\frac{\gamma}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}s_i + \xi_{\tilde{k}} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i}\right] \\
&= \frac{1}{\tilde{k}}\left[\frac{\gamma}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}s_i - \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} + \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}\right. \\
&\qquad \left. + \xi_{\tilde{k}} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{\zeta_i^2}{r_i} - \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{\zeta_i^2}{r_i} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i}\right]
\end{aligned} \tag{3.37}$$

or (for $\Delta\xi_{\tilde{k}} < 0$)

$$\begin{aligned}
\Delta\xi_{\tilde{k}} = \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\Delta\zeta_i \le \frac{1}{\tilde{k}+\alpha-1}&\left[\frac{\gamma}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}s_i - \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} + \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}\right. \\
&\left. + \xi_{\tilde{k}} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{\zeta_i^2}{r_i} - \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{\zeta_i^2}{r_i} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i}\right].
\end{aligned} \tag{3.38}$$

Again, we estimate $\Delta\xi_{\tilde{k}}$ stepwise. Using (3.14) and

$$-r^{+} < -\frac{r^{+}}{2} \leq -\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}, \tag{3.39}$$

we have

$$\zeta_k > -2r^{+}, \quad k \geq \tilde{k}, \quad \zeta_{\tilde{k}}, \ldots, \zeta_{\tilde{k}+\alpha-1} < 0. \tag{3.40}$$

Similarly to the first part of the proof, we can show that

$$\begin{aligned}
&\frac{\gamma}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1} s_i - \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i} - \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{\zeta_i^2}{r_i} \\
&\quad = \left(\frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1} s_i\right)\left[\gamma - \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}\left(\frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1} s_i\right)^{-1}\right] + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{r_i\zeta_i^2 + \zeta_i^3}{ir_i^2 - \zeta_i r_i} \\
&\quad \leq \left(\frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1} s_i\right)\frac{\gamma - K}{2} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{r^{+}(2r^{+})^2 + (2r^{+})^3}{i(r^{-})^2} \\
&\quad \leq \left(\frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1} s_i\right)\frac{\gamma - K}{2} + \frac{12(r^{+})^3}{\tilde{k}(r^{-})^2}.
\end{aligned} \tag{3.41}$$

Thus (consider (3.4)), there exist $\hat{k} \in \mathbb{N}$ and

$$\hat{\Gamma} > \frac{K-\gamma}{2}\,\vartheta \tag{3.42}$$

such that

$$\frac{\gamma}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1} s_i - \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{(i+1)\zeta_i^2}{ir_i - \zeta_i} - \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{\zeta_i^2}{r_i} \leq -\hat{\Gamma} \tag{3.43}$$

for $\tilde{k} \geq \hat{k}$. Henceforth, let $\tilde{k} \geq \hat{k}$.

Now we want to estimate

$$\Upsilon := \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} + \xi_{\tilde{k}} + \frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{\zeta_i^2}{r_i}. \tag{3.44}$$

Firstly, consider that, for $\zeta_i = \zeta_{\tilde{k}}$, $i \in \{\tilde{k}, \ldots, \tilde{k} + \alpha - 1\}$, it is valid

$$\Upsilon = \frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} - \left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} + \left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-2}\frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{1}{r_i}, \tag{3.45}$$

and hence (see (3.5) and (3.42))

$$|\,\Upsilon - 0\,| = \left|\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-2}\left[\frac{1}{\alpha}\sum_{i=\tilde{k}}^{\tilde{k}+\alpha-1}\frac{1}{r_i} - \frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right]\right| < \frac{1}{8}(K-\gamma)\vartheta < \frac{\widehat{\Gamma}}{4} \tag{3.46}$$

because

$$\frac{1}{4}\left(\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} - \left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} + \left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-2}\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i} = 0. \tag{3.47}$$

We repeat that (see (3.15))

$$\left|\zeta_{\tilde{k}} - \zeta_{\tilde{k}+j}\right| < \frac{\Theta}{\tilde{k}}, \quad j \in \{1, \ldots, \alpha - 1\}, \tag{3.48}$$

which gives

$$\left|\zeta_{\tilde{k}} - \xi_{\tilde{k}}\right| < \frac{\Theta}{\tilde{k}}. \tag{3.49}$$

Considering (3.40), (3.46), (3.48), and (3.49) together for general Υ, we have

$$\begin{aligned}|\Upsilon| &\leq \frac{\widehat{\Gamma}}{4} + \left|\xi_{\tilde{k}} - \zeta_{\tilde{k}}\right| + \frac{1}{\alpha}\sum_{i=0}^{\alpha-1}\frac{\left|\zeta_{\tilde{k}+i}^2 - \zeta_{\tilde{k}}^2\right|}{r_{\tilde{k}+i}} \\ &= \frac{\widehat{\Gamma}}{4} + \left|\xi_{\tilde{k}} - \zeta_{\tilde{k}}\right| + \frac{1}{\alpha}\sum_{i=0}^{\alpha-1}\frac{\left|\zeta_{\tilde{k}+i} + \zeta_{\tilde{k}}\right| \cdot \left|\zeta_{\tilde{k}+i} - \zeta_{\tilde{k}}\right|}{r_{\tilde{k}+i}} < \frac{\widehat{\Gamma}}{4} + \left(1 + \frac{4r^+}{r^-}\right)\frac{\Theta}{\tilde{k}}.\end{aligned} \tag{3.50}$$

Since it suffices to consider very large $\tilde{k}$, we can assume that $|\Upsilon| < 2\widehat{\Gamma}/3$.

Altogether, we obtain

$$\Delta\xi_{\tilde{k}} \leq \frac{1}{\tilde{k} + \alpha - 1}\left(-\widehat{\Gamma} + \frac{2\widehat{\Gamma}}{3}\right) < 0. \tag{3.51}$$

The resulting inequality (3.51) implies

$$\Delta\xi_{\widetilde{k}} = \frac{1}{\alpha}\sum_{i=\widetilde{k}}^{\widetilde{k}+\alpha-1}\Delta\zeta_i = \frac{\zeta_{\widetilde{k}+\alpha}-\zeta_{\widetilde{k}}}{\alpha} < 0, \quad \text{i.e.,}\ \zeta_{\widetilde{k}+\alpha} < \zeta_{\widetilde{k}}. \tag{3.52}$$

Partially, if

$$\zeta_k = -\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1} \quad \text{for some } k \geq \widetilde{k}, \tag{3.53}$$

then from (3.14) it follows

$$\zeta_k, \zeta_{k+1}, \dots, \zeta_{k+\alpha} < 0, \quad \zeta_{k+\alpha} < \zeta_k. \tag{3.54}$$

In fact (see the below given), this result remains true also if

$$\zeta_k \in \left(-\eta - \left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}, -\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}\right) \tag{3.55}$$

for a number $\eta > 0$ which depends only on γ and K. Considering (3.15) for large $\overline{k}$, it is seen that the solution of the Cauchy problem (3.34) satisfies

$$\zeta_{\widetilde{k}}, \zeta_{\widetilde{k}+\alpha}, \dots, \zeta_{\widetilde{k}+n\alpha}, \dots < -\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}, \tag{3.56}$$

and hence (see (3.54))

$$\zeta_{\widetilde{k}+n} < 0, \quad n \in \mathbb{N}. \tag{3.57}$$

Therefore, (3.57) and Lemma 2.9 say that (3.1) is non-oscillatory for $\gamma < K$.

It means that, to complete the proof, it suffices to find $\eta > 0$ which guaranties the above-mentioned generalization, that is, we need to prove (3.51) for (3.55) with $k = \widetilde{k}$. The concrete initial value was not used in the proof of (3.43). Thus, $\widehat{\Gamma}$ depends only on γ and K. Let

$$\eta := \min\left\{\left(\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{r_i}\right)^{-1}, \left(1+\frac{3r^+}{r^-}\right)^{-1}\frac{\widehat{\Gamma}}{4}\right\}. \tag{3.58}$$

In the estimate of Υ, since (3.14) and (3.15) remain true, we have (consider also (3.39), (3.48), and (3.49))

$$\zeta_{\tilde{k}+j} \in (-2r^{+}, 0),$$
$$\left| \zeta_{\tilde{k}+j} + \left(\frac{2}{\alpha} \sum_{i=1}^{\alpha} \frac{1}{r_i} \right)^{-1} \right| < \frac{\Theta}{\tilde{k}} + \eta, \qquad \left| \xi_{\tilde{k}} + \left(\frac{2}{\alpha} \sum_{i=1}^{\alpha} \frac{1}{r_i} \right)^{-1} \right| < \frac{\Theta}{\tilde{k}} + \eta, \tag{3.59}$$

where $j \in \{0, 1, \ldots, \alpha - 1\}$, and (see again (3.39))

$$\begin{aligned}
|\Upsilon| &\leq \frac{\widehat{\Gamma}}{4} + \left| \xi_{\tilde{k}} + \left(\frac{2}{\alpha} \sum_{i=1}^{\alpha} \frac{1}{r_i} \right)^{-1} \right| + \frac{1}{\alpha} \sum_{j=0}^{\alpha-1} \left[\frac{1}{r_{\tilde{k}+j}} \left| \zeta_{\tilde{k}+j}^{2} - \left(\frac{2}{\alpha} \sum_{i=1}^{\alpha} \frac{1}{r_i} \right)^{-2} \right| \right] \\
&< \frac{\widehat{\Gamma}}{4} + \frac{\Theta}{\tilde{k}} + \eta + \frac{1}{\alpha} \sum_{j=0}^{\alpha-1} \left[\frac{1}{r_{\tilde{k}+j}} \left| \zeta_{\tilde{k}+j} - \left(\frac{2}{\alpha} \sum_{i=1}^{\alpha} \frac{1}{r_i} \right)^{-1} \right| \cdot \left| \zeta_{\tilde{k}+j} + \left(\frac{2}{\alpha} \sum_{i=1}^{\alpha} \frac{1}{r_i} \right)^{-1} \right| \right] \\
&< \frac{\widehat{\Gamma}}{4} + \frac{\Theta}{\tilde{k}} + \eta + \frac{1}{r^{-}} \left(\frac{\Theta}{\tilde{k}} + \eta \right) (2r^{+} + r^{+}) \\
&= \frac{\widehat{\Gamma}}{4} + \left[1 + \frac{3r^{+}}{r^{-}} \right] \left(\frac{\Theta}{\tilde{k}} + \eta \right) \leq \left[1 + \frac{3r^{+}}{r^{-}} \right] \frac{\Theta}{\tilde{k}} + \frac{\widehat{\Gamma}}{2},
\end{aligned} \tag{3.60}$$

which confirms $|\Upsilon| < 2\widehat{\Gamma}/3$ and then the validity of (3.51). □

Remark 3.2. Let us point out that the constant K arises from the calculations in Step 1.

Remark 3.3. If $r_k = s_k = 1$, $k \in \mathbb{N}$, then $K = 1/4$ (see (3.3)); that is, Theorem 3.1 reduces to the result about the discrete Euler equation.

Example 3.4. For arbitrarily given continuous function $f : [-1, 1] \to \mathbb{R}^{+}$ and $a > 1$, $b, c \in \mathbb{R}$, let us consider

$$\Delta \left(\frac{\Delta x_k}{a + \sin(bk)\cos(ck)} \right) + \frac{\gamma f(\sin k)}{k(k+1)} x_{k+1} = 0, \quad k \in \mathbb{N}. \tag{3.61}$$

The almost periodicity of $\{r_k\} = \{[a + \sin(bk)\cos(ck)]^{-1}\}_{k\in\mathbb{N}}$ and $\{s_k\} = \{f(\sin k)\}_{k\in\mathbb{N}}$ follows from Corollary 2.4 and from, for example, [45, Theorem 1.27] and [47, Theorem 1.9]. It is seen that

$$M\left(\left\{r_k^{-1}\right\}\right) = a, \qquad M(\{s_k\}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\sin x)\,dx. \tag{3.62}$$

Thus, (3.61) is oscillatory if

$$\gamma > K = \frac{\pi}{2a \int_{-\pi}^{\pi} f(\sin x)\,dx}, \tag{3.63}$$

and non-oscillatory if $\gamma < K$.

Analogously, under the additional condition $b \neq 0$, the oscillation constant for the equation

$$\Delta([a+\sin(bk)\cos(bk)]\Delta x_k) + \frac{\gamma f(\cos k)}{k(k+1)} x_{k+1} = 0, \quad k \in \mathbb{N}, \tag{3.64}$$

is

$$K = \frac{\pi^2}{b\int_{-\pi}^{\pi} f(\cos x)dx \int_{-\pi/b}^{\pi/b} dy/\left(a+\sin(by)\cos(by)\right)} = \frac{\pi\sqrt{4a^2-1}}{4\int_{-\pi}^{\pi} f(\cos x)dx}. \tag{3.65}$$

Evidently, any periodic sequence is almost periodic. Thus, we also obtain this new result.

Corollary 3.5. *The equation*

$$\Delta(r_k \Delta x_k) + \frac{\gamma s_k}{(k+1)k} x_{k+1} = 0, \quad k \in \mathbb{N}, \tag{3.66}$$

where $\gamma \in \mathbb{R}$ and $\{r_k\}$ and $\{s_k\}$ are positive sequences with period $n \in \mathbb{N}$, is oscillatory if

$$\gamma > K := \left[\left(\frac{2}{n}\sum_{i=1}^{n}\frac{1}{r_i}\right)\left(\frac{2}{n}\sum_{i=1}^{n}s_i\right)\right]^{-1}, \tag{3.67}$$

and non-oscillatory if $\gamma < K$.

Remark 3.6. The border case given by $\gamma = K$ remains open. Nevertheless, based on the corresponding continuous case (see [23]) and other cases which generalize the discrete equation with constant coefficients (see, e.g., [10] with references cited therein), we conjecture that (3.66) (with periodic coefficients) is non-oscillatory even for $\gamma = K$.

Example 3.7. Let an odd integer $m \geq 3$ be given. We can use Corollary 3.5 for the equation

$$\Delta\left(\left|\sin\frac{(2k-1)\pi}{12}\right|\Delta x_k\right) + \frac{\gamma}{k(k+1)}\left(1+\cos\frac{2k\pi}{m}\right)x_{k+1} = 0, \quad k \in \mathbb{N}. \tag{3.68}$$

Since we can choose $n = 6m$, we obtain

$$\begin{gathered} M\left(\left\{r_k^{-1}\right\}\right) = \frac{1}{6m}\sum_{i=1}^{6m}\frac{1}{r_i} = \frac{1}{6}\sum_{i=1}^{6}\left|\sin\frac{(2i-1)\pi}{12}\right|^{-1} = \frac{\sqrt{2}}{3}\left(1+2\sqrt{3}\right), \\ M(\{s_k\}) = \frac{1}{6m}\sum_{i=1}^{6m}s_i = \frac{1}{m}\sum_{i=1}^{m}\left(1+\cos\frac{2i\pi}{m}\right) = 1. \end{gathered} \tag{3.69}$$

Hence, the oscillation constant for (3.68) is

$$K = \frac{3}{4\sqrt{2}\left(1+2\sqrt{3}\right)} \approx \frac{2}{17}. \tag{3.70}$$

We add that we can use Theorem 3.1 also in the case when one of the sequences $\{r_k\}$ and $\{s_k\}$ in (3.1) changes its sign. If the sequence $\{r_k\}$ in (3.71) changes its sign, then we have to generalize the definition of the generalized zeros as follows. An interval $(a, a+1]$, $a \in \mathbb{N}$, contains the generalized zero of a solution $\{x_k\}$ of (3.71) if $x_a \neq 0$ and $r_a x_a x_{a+1} \leq 0$.

Corollary 3.8. *Let the equation*

$$\Delta(r_k \Delta x_k) + \frac{\gamma s_k}{(k+1)k} x_{k+1} = 0, \quad k \in \mathbb{N}, \tag{3.71}$$

where $\gamma \in \mathbb{R}$ and $\{r_k\}$ and $\{s_k\}$ are nonzero almost periodic sequences, be given.

(i) *If* $\inf\{r_k;\ k \in \mathbb{N}\} > 0$ *and* $\gamma < \left[4M(\{r_k^{-1}\})M(\{|s_k|\})\right]^{-1}$, *then* (3.71) *is non-oscillatory.*

(ii) *If* $\inf\{|r_k|;\ k \in \mathbb{N}\} > 0$, $\{s_k\}$ *is positive and* $\gamma > \left[4M(\{|r_k|^{-1}\})M(\{s_k\})\right]^{-1}$, *then* (3.71) *is oscillatory.*

Proof. Since the almost periodicity of $\{f_k\}$ implies the almost periodicity of $\{|f_k|\}$, it suffices to apply the discrete Sturm comparison theorem and Theorem 3.1. □

At the end we remark that it is possible to find several definitions of almost periodicity for $k \in \mathbb{N}$ in the literature. For example, concerning almost periodic sequences with indices $k \in \mathbb{N}$, we refer to [50]. There is proved that, for any precompact sequence $\{x_k\}_{k\in\mathbb{N}}$, there exists a permutation P of the set of positive integers such that the sequence $\{x_{P(k)}\}_{k\in\mathbb{N}}$ is almost periodic. In fact, the so-called asymptotically almost periodic sequences are considered in [50] (based on the Bochner concept), where a bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ is called asymptotically almost periodic if the set of sequences $\{x_{k+p}\}_{k\in\mathbb{N}}$, $p \in \mathbb{N}$, is precompact in the space of all bounded sequences. We add that a sequence $\{x_k\}_{k\in\mathbb{N}}$ is asymptotically almost periodic if and only if it is the sum of an almost periodic sequence and a sequence which approaches zero as $k \to \infty$. One finds that this representation is unique. See, for example, [51, 52].

We consider difference equations with almost periodic coefficients given by the limitation of almost periodic sequences on $\mathbb{Z}$ because this approach is the standard one. But we conjecture that the main result can be similarly proved for almost periodic coefficients defined in other ways (e.g., for the above-mentioned asymptotically almost periodic sequences).

Acknowledgments

The authors thank the anonymous referees for their suggestions and references which improved the final version of the paper. This research is supported by the Czech Science Foundation under Grant P201/10/1032.

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