Research Article

Minimization of Vectors of Curvilinear Functionals on Second-Order Jet Bundle: Dual Program Theory

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A previous paper (2011), Pitea and Postolache, considered the problem of minimization of vectors of curvilinear functionals (well known as mechanical work), thought as multitime multiobjective variational problem, subject to PDE and/or PDI constraints. They have chosen the suitable framework offered by the second-order jet bundle, and initiated an optimization theory for this class of problems by introducing necessary conditions. As natural continuation of these results, the present work introduces a dual program theory, the general setting, and the theory which is new as a whole, containing our results.

1. Introduction

Fractional programming problems arise in wide areas of research in pure and applied sciences and in the new technology as well. We have in mind portfolio selection, stock cutting, game theory, and various decision problems in management science. The study of duality for this class of problems is an active area of research due to its wide scope of applications, see [1] by Chinchuluun and Pardalos, for an excellent paper in multiobjective optimization.

The duality theory for multiobjective optimization problems based on a vector-valued Lagrangian function were extensively investigated in [2] by Bitran, and [3] by Tanino and Sawaragi. Several authors were interested in this problem, see [4] by Egudo, [5] by Mond and Husain, [6] by Weir and Mond. Later, an interesting and rich duality theory for such problems was developed by Aghezzaf and Hachimi, [7], Preda [8, 9], and recently, duality problems were studied in [10] by Boţ and Grad, [1] by Chinchuluun and Pardalos, [11] by Pitea et al.

In the problems of our study, the objective function is of curvilinear integral type and could mould the mechanical work when we have in mind design problems in engineering.

Also, such kind of problems proved to be useful tools for minimizing the cost or maximizing the profit when we discuss economical problems.

Motivated by the study reported above, in a very recent work [12], Pitea and Postolache considered the problem of minimization of vectors of curvilinear functionals (well known as mechanical work), thought as multitime multiobjective variational problem, subject to PDE and/or PDI constraints. They have chosen the suitable framework offered by the second-order jet bundle and initiated an optimization theory for this class of problems by introducing necessary conditions. As natural continuation of these results, the present work introduces a dual program theory. It is organized as follows. Next, in Section 2, we introduce the general setting, while in Section 3 we prove our results. Finally, we conclude the paper.

2. General Setting

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively. The local coordinates on T and M will be written $t = (t^\alpha)$ and $x = (x^i)$, respectively. Let $J^2(T, M)$ be the second-order jet bundle associated to T and M, see [13].

Throughout this work, we use the customary relations between two vectors of the same dimension, [11]. With the product-order relation on $\mathbb{R}^p$, the hyperparallelepiped Ω_{t_0, t_1}, in $\mathbb{R}^p$, with the diagonal opposite points $t_0 = (t_0^1, \ldots, t_0^p)$ and $t_1 = (t_1^1, \ldots, t_1^p)$, can be written as interval $[t_0, t_1]$. Suppose γ_{t_0, t_1} is a piecewise C^2-class curve joining the points t_0 and t_1.

To simplify the notations, denote by

$$x_\gamma(t) = \frac{\partial x}{\partial t^\gamma}(t), \quad \gamma = \overline{1, p}, \qquad x_{\theta\sigma}(t) = \frac{\partial^2 x}{\partial t^\theta \partial t^\sigma}(t), \quad \theta, \sigma = \overline{1, p}, \tag{2.1}$$

the partial velocities and partial accelerations, respectively. Also, in our subsequent theory, we will set $\pi_x(t) = (t, x(t), x_\gamma(t), x_{\theta\sigma}(t))$.

On $J^2(T, M)$ with values on $\mathbb{R}^r$, consider the closed Lagrange 1-forms densities of C^∞-class:

$$f_\alpha = \left(f_\alpha^\ell \right), \qquad k_\alpha = \left(k_\alpha^\ell \right), \quad \ell = \overline{1, r}, \ \alpha = \overline{1, p}, \tag{2.2}$$

which determine the following path-independent functionals:

$$F^\ell(x(\cdot)) = \int_{\gamma_{t_0, t_1}} f_\alpha^\ell(\pi_x(t)) dt^\alpha, \qquad K^\ell(x(\cdot)) = \int_{\gamma_{t_0, t_1}} k_\alpha^\ell(\pi_x(t)) dt^\alpha. \tag{2.3}$$

The closeness conditions (complete integrability conditions) are

$$D_\beta f_\alpha^\ell = D_\alpha f_\beta^\ell, \qquad D_\beta k_\alpha^\ell = D_\alpha k_\beta^\ell, \quad \alpha, \beta = \overline{1, p}, \ \alpha \neq \beta, \ \ell = \overline{1, r}, \tag{2.4}$$

where D_β is the total derivative.

We accept that the Lagrange matrix densities:

$$g = \left(g_a^b \right) : J^2(T, M) \longrightarrow \mathbb{R}^{md}, \quad a = \overline{1, d}, \ b = \overline{1, m}, \ m < n, \tag{2.5}$$

of C^∞-class define the partial differential inequations (PDI) of evolution

$$g(\pi_x(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1}, \tag{2.6}$$

and the Lagrange matrix densities,

$$h = \left(h_a^b\right) : J^2(T, M) \longrightarrow \mathbb{R}^{qd}, \quad a = \overline{1, s}, \; b = \overline{1, q}, \; q < n, \tag{2.7}$$

define the partial differential equations (PDE), of evolution

$$h(\pi_x(t)) = 0, \quad t \in \Omega_{t_0,t_1}. \tag{2.8}$$

For each $\ell = \overline{1, r}$, suppose $K^\ell(x(\cdot)) > 0$, and consider

$$\frac{F(x(\cdot))}{K(x(\cdot))} = \left(\frac{F^1(x(\cdot))}{K^1(x(\cdot))}, \ldots, \frac{F^r(x(\cdot))}{K^r(x(\cdot))}\right). \tag{2.9}$$

With conditions (2.6) and (2.8), we denote by

$$\mathcal{F}(\Omega_{t_0,t_1}) = \left\{x \in C^\infty(\Omega_{t_0,t_1}, M) \mid x(t_0) = x_0, \; x(t_1) = x_1, g(\pi_x(t)) \leqq 0, h(\pi_x(t)) = 0, \; t \in \Omega_{t_0,t_1}\right\} \tag{2.10}$$

the *set of all feasible solutions* of problem:

$$(\text{MFP}) \begin{cases} \min \dfrac{F(x(\cdot))}{K(x(\cdot))}, \\ \text{subject to } x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}), \end{cases} \tag{2.11}$$

a PDI- and/or PDE-constrained minimum problem.

In [12], we introduced necessary efficiency conditions for problem (MFP). The aim of this work is to introduce and study two dual programs:

(1) the multitime multiobjective fractional variational problem (MFP) of minimizing a vector of quotients of path-independent curvilinear functionals;

(2) the multiobjective variational dual problem:

$$
(\text{MFD}) \begin{cases}
\displaystyle\max_{y(\cdot)} \left(\dfrac{F(y(\cdot))}{K(y(\cdot))} \right) \\[2ex]
\text{subject to} \\[1ex]
\Lambda_\ell^{10} \dfrac{\partial f_\alpha^\ell}{\partial y}(\pi_y(t)) - \Lambda_\ell^{20} \dfrac{\partial k_\alpha^\ell}{\partial y}(\pi_y(t)) + \left\langle \mu_\alpha(t), \dfrac{\partial g}{\partial y}(\pi_y(t)) \right\rangle \\[2ex]
\quad + \left\langle \nu_\alpha(t), \dfrac{\partial h}{\partial y}(\pi_y(t)) \right\rangle - D_\gamma \left(\Lambda_\ell^{10} \dfrac{\partial f_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) - \Lambda_\ell^{20} \dfrac{\partial k_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) \right. \\[2ex]
\qquad\qquad\qquad\qquad \left. + \left\langle \mu_\alpha(t), \dfrac{\partial g}{\partial y_\gamma}(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), \dfrac{\partial h}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right) \\[2ex]
\quad + D_{\theta\sigma}^2 \left(\left\langle \lambda, \dfrac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_y(t)) \right\rangle + \left\langle \mu_\alpha(t), \dfrac{\partial g}{\partial x_{\theta\sigma}}(\pi_y(t)) \right\rangle \right. \\[2ex]
\qquad\qquad \left. + \left\langle \nu_\alpha(t), \dfrac{\partial h}{\partial x_{\theta\sigma}}(\pi_y(t)) \right\rangle \right) = 0, \quad t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1,p} \\[2ex]
\langle \mu_\alpha(t), g(\pi_y(t)) \rangle + \langle \nu_\alpha(t), h(\pi_y(t)) \rangle \geqq 0, \quad \alpha = \overline{1,p}, \ t \in \Omega_{t_0,t_1} \\[2ex]
\Lambda^{10} \geq 0,
\end{cases}
$$

$$(2.12)$$

taking into account that the functions $x(t)$ and $y(t)$ have to satisfy the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$, or $x(t)|_{\partial\Omega_{t_0,t_1}} = \chi =$ given, respectively, $y(t_0) = x_0$, $y(t_1) = x_1$, or $y(t)|_{\partial\Omega_{t_0,t_1}} = \chi =$ given, the partial differential inequations of evolution (2.6), and the partial differential equations of evolution (2.8).

To develop the theory in our main section, we need the notion of efficient solution.

Definition 2.1. A feasible solution $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is called *efficient solution* for the program (MFP) if and only if for any feasible solution $x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$, one has the implication:

$$
\frac{F(x(\cdot))}{K(x(\cdot))} \geqq \frac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))} \implies \frac{F(x(\cdot))}{K(x(\cdot))} = \frac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))}.
\tag{2.13}
$$

3. Main Results

To state our results, we have to introduce an appropriate generalized convexity. For this purpose, consider ρ be a real number, $b : C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to [0, \infty)$ a functional, and $a = (a_\alpha)$, $\alpha = \overline{1,p}$, a closed 1-form. To a we associate the curvilinear integral

$$
A(x(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha(\pi_x(t)) dt^\alpha.
\tag{3.1}
$$

Definition 3.1. The functional A is called *[strictly]* (ρ, b)-*quasiinvex at the point* $x^\circ(\cdot)$ *if there exists a vector function* $\eta : J^2(\Omega_{t_0,t_1}, M) \times J^2(\Omega_{t_0,t_1}, M) \rightarrow \mathbb{R}^n$, *vanishing at the point* $(\pi_{x^\circ}(t), \pi_{x^\circ}(t))$, *and the function* θ *defined on* $C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M)$ *to* $\mathbb{R}^n$, *such that for any* $x(\cdot)[x(\cdot) \neq x^\circ(\cdot)]$, *the following implication holds:*

$$
(A(x(\cdot)) \leq A(x^\circ(\cdot)))
$$

$$
\Longrightarrow \Bigg(bx((\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[\Bigg\langle \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x}(\pi_{x^\circ}(t)) \Bigg\rangle
$$

$$
+ \Bigg\langle D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) \Bigg\rangle \tag{3.2}
$$

$$
+ \Bigg\langle D_{\theta\sigma}^2 \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) \Bigg\rangle \Bigg] dt^\alpha
$$

$$
[<] \leq -\rho b(x(\cdot), x^\circ(\cdot)) \| \theta(x(\cdot), x^\circ(\cdot)) \|^2 \Bigg).
$$

The notion of quasiinvexity is used, in appropriate forms, in recent works for studies of some multiobjective programming problems, for example, see [14] by Nahak and Mohapatra.

Let us denote by $\varpi(x^\circ(\cdot))$ the minimizing functional vector of problem (MFP) at the point $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ and by $(\delta(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ the maximizing functional vector of dual (MFD) at the point $(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot)) \in \Delta$, where Δ is the domain of problem (MFD).

Theorem 3.2 (Weak Duality). *Let* $x^\circ(\cdot)$ *be a feasible solution of the problem* (MFP) *and let* $y(\cdot)$ *be an efficient solution of problem* (MFP). *Assume that the following conditions are fulfilled:*

(a) $\Lambda_\ell^{10} > 0$, $\Lambda_\ell^{20} > 0$, $\Lambda_\ell^{10} F^\ell(y(\cdot)) - \Lambda_\ell^{20} K^\ell(y(\cdot)) = 0$, $\ell = \overline{1, r}$;

(b) *for any* $\ell = \overline{1, r}$, *the functional* $F^\ell(x(\cdot))$ *is* (ρ'^ℓ, b)-*quasiinvex at the point* $y(\cdot)$ *and the functional* $-K^\ell(x(\cdot))$ *is* (ρ''^ℓ, b)-*quasiinvex at the point* $y(\cdot)$ *with respect to* η *and* θ;

(c) *the functional,*

$$
\int_{\gamma_{t_0,t_1}} \Big[\langle \mu_\alpha(t), g(\pi_x(t)) \rangle + \langle \nu_\alpha(t), h(\pi_x(t)) \rangle \Big] dt^\alpha, \tag{3.3}
$$

is (ρ''', b)-*quasiinvex at the point* $y(\cdot)$ *with respect to* η *and* θ;

(d) *either the functional of* (b) *or the functional of* (c) *is strictly quasiinvex;*

(e) $\rho'^\ell \Lambda_\ell^{10} + \rho''^\ell \Lambda_\ell^{20} + \rho''' \geq 0$.

Then, the inequality $\varpi(x^\circ(\cdot)) \leq \delta(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ *is false.*

Proof. From condition (b), it follows that

$$
\left(F^\ell(x^\circ(\cdot)) F^\ell(y(\cdot)) \right)
$$

$$
\Longrightarrow \left(b(x^\circ(\cdot), y(\cdot)) \int_{\gamma_{t_0,t_1}} \left[\left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial f_\alpha^\ell}{\partial y}(\pi_y(t)) \right\rangle \right. \right.
$$

$$
+ \left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial f_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) \right\rangle \tag{3.4}
$$

$$
\left. \left. + \left\langle D_{\theta\sigma}^2 \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial f_\alpha^\ell}{\partial y_{\theta\sigma}}(\pi_y(t)) \right\rangle \right] dt^\alpha \right.
$$

$$
\left. \leqq -\rho'^\ell b(x^\circ(\cdot), y(\cdot)) \| \theta(x^\circ(\cdot), y(\cdot)) \|^2 \right), \quad \ell = \overline{1, r},
$$

$$
\left(-K^\ell(x^\circ(\cdot)) - K^\ell(y(\cdot)) \right)
$$

$$
\Longrightarrow \left(b(x^\circ(\cdot), y(\cdot)) \int_{\gamma_{t_0,t_1}} \left[\left\langle -\eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial k_\alpha^\ell}{\partial y}(\pi_y(t)) \right\rangle \right. \right.
$$

$$
- \left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial k_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) \right\rangle \tag{3.5}
$$

$$
\left. \left. - \left\langle D_{\theta\sigma}^2 \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial k_\alpha^\ell}{\partial y_{\theta\sigma}}(\pi_y(t)) \right\rangle \right] dt^\alpha \right.
$$

$$
\left. \leqq \rho''^\ell b(x^\circ(\cdot), y(\cdot)) \| \theta(x^\circ(\cdot), y(\cdot)) \|^2 \right), \quad \ell = \overline{1, r}.
$$

We multiply (3.4) by $\Lambda_\ell^{10} > 0$ and (3.5) by $\Lambda_\ell^{20} > 0$. We make the sum and we obtain the following implications:

$$
\left(\Lambda_\ell^{10} F^\ell(x^\circ(\cdot)) - \Lambda_\ell^{20} K^\ell(x^\circ(\cdot)) \leqq 0 \right)
$$

$$
\Longrightarrow \left(b(x^\circ(\cdot), y(\cdot)) \int_{\gamma_{t_0,t_1}} \left\{ \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \Lambda_\ell^{10} \frac{\partial f_\alpha^\ell}{\partial y}(\pi_y(t)) - \Lambda_\ell^{20} \frac{\partial k_\alpha^\ell}{\partial y}(\pi_y(t)) \right\rangle \right. \right.
$$

$$
+ \left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \Lambda_\ell^{10} \frac{\partial f_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) - \Lambda_\ell^{20} \frac{\partial k_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) \right\rangle
$$

$$
\left. \left. + \left\langle D_{\theta\sigma}^2 \eta(\pi_{x^\circ}(t), \pi_y(t)), \Lambda_\ell^{10} \frac{\partial f_\alpha^\ell}{\partial y_{\theta\sigma}}(\pi_y(t)) - \Lambda_\ell^{20} \frac{\partial k_\alpha^\ell}{\partial y_{\theta\sigma}}(\pi_y(t)) \right\rangle \right\} dt^\alpha \right.
$$

$$
\left. \leqq -b(x^\circ(\cdot), y(\cdot)) \| \theta(x^\circ(\cdot), y(\cdot)) \|^2 \left(\rho'^\ell \Lambda_\ell^{10} + \rho''^\ell \Lambda_\ell^{20} \right) \right).
$$

$$
\tag{3.6}
$$

According to hypothesis (c), we have

$$
\left(\int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_{x^\circ}(t)) \rangle + \langle \nu_\alpha(t), h(\pi_{x^\circ}(t)) \rangle \right] dt^\alpha \right.
$$

$$
\leqq \int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_y(t)) \rangle + \langle \nu_\alpha(t), h(\pi_y(t)) \rangle \right] dt^\alpha \Big)
$$

$$
\Longrightarrow \left(b(x^\circ(\cdot), y(\cdot)) \int_{\gamma_{t_0,t_1}} \left[\left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y}(\pi_y(t)) \right\rangle \right. \right. \right.
$$

$$
\left. + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y}(\pi_y(t)) \right\rangle \right\rangle + \langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)) \rangle,
$$

$$
\left\langle \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right\rangle
$$

$$
+ \left\langle D_{\theta\sigma}^2 \eta(\pi_{x^\circ}(t), \pi_y(t)) \right\rangle, \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y_{\theta\sigma}}(\pi_y(t)) \right\rangle
$$

$$
\left. \left. + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y_{\theta\sigma}}(\pi_y(t)) \right\rangle \right\rangle \right] dt^\alpha
$$

$$
\leqq -\rho''' b(x^\circ(\cdot), y(\cdot)) \|\theta(x^\circ(\cdot), y(\cdot))\|^2 \Big).
$$

(3.7)

Making the sum of the implications (3.6) and (3.7), it follows that

$$
\left(\Lambda_\ell^{10} F^\ell(x^\circ(\cdot)) - \Lambda_\ell^{20} K^\ell(x^\circ(\cdot)) \right.
$$

$$
+ \int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_{x^\circ}(t)) \rangle + \langle \nu_\alpha(t), h(\pi_{x^\circ}(t)) \rangle \right] dt^\alpha
$$

$$
- \int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_y(t)) \rangle + \langle \nu_\alpha(t), h(\pi_y(t)) \rangle \right] dt^\alpha \leqq 0 \Big)
$$

$$
\Longrightarrow \left(b(x^\circ(\cdot), y(\cdot)) \int_{\gamma_{t_0,t_1}} \left[\left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \Lambda_\ell^{10} \frac{\partial f_\alpha^\ell}{\partial y}(\pi_y(t)) - \Lambda_\ell^{20} \frac{\partial k_\alpha^\ell}{\partial y}(\pi_y(t)) \right. \right. \right.
$$

$$
\left. + \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y}(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y}(\pi_y(t)) \right\rangle \right\rangle
$$

$$
+ \left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \Lambda_\ell^{10} \frac{\partial f_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) - \Lambda_\ell^{20} \frac{\partial k_\alpha^\ell}{\partial y_\gamma}(\pi_y(t)) \right.
$$

$$+\left\langle \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right\rangle$$

$$+\left\langle D^2_{\theta\sigma}\eta(\pi_{x^\circ}(t), \pi_y(t)), \qquad \Lambda^{10}_\ell \frac{\partial f^\ell_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t)) - \Lambda^{20}_\ell \frac{\partial k^\ell_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t)) \right.$$

$$+\left\langle \mu_\alpha(t), \frac{\partial g}{\partial y_{\theta\sigma}}(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y_{\theta\sigma}}(\pi_y(t)) \right\rangle \right\rangle \Bigg] dt^\alpha$$

$$< -b(x^\circ(\cdot), y(\cdot)) \|\theta(x^\circ(\cdot), y(\cdot))\|^2 \left(\rho'^\ell \Lambda^{10}_\ell + \rho''^\ell \Lambda^{20}_\ell + \rho'''\right).$$

$$(3.8)$$

Since $b(x^\circ(\cdot), y(\cdot)) > 0$, we obtain

$$\int_{\gamma_{t_0,t_1}} \left[\left\langle \eta \pi_{x^\circ}(t), \pi_y(t), \frac{\partial V_\alpha}{\partial y \pi_y}(t), \lambda, \mu(t), \nu(t) \right\rangle \right.$$

$$+ \left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(\pi_y(t), \lambda, \mu(t), \nu(t)) \right\rangle$$

$$+ \left\langle D^2_{\theta\sigma}\eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t), \lambda, \mu(t), \nu(t)) \right\rangle \Bigg] dt^\alpha$$

$$< -\|\theta(x^\circ(\cdot), y(\cdot))\|^2 \left(\rho'^\ell \Lambda^{10}_\ell + \rho''^\ell \Lambda^{20}_\ell + \rho'''\right),$$

$$(3.9)$$

where

$$V_\alpha(\pi_y(\cdot), \lambda, \mu(\cdot), \nu(\cdot)) = \Lambda^{10}_\ell f^\ell_\alpha(\pi_y(t)) - \Lambda^{20}_\ell k^\ell_\alpha(\pi_y(t))$$

$$+ \langle \mu_\alpha(y(t)), g(\pi_y(t)) \rangle + \langle \nu_\alpha(t), h(\pi_y(t)) \rangle, \quad t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1, p}.$$

$$(3.10)$$

The following relations hold:

$$\left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(\pi_y(t), \lambda, \mu(t), \nu(t)) \right\rangle$$

$$= D_\gamma \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma}(\pi_y(t), \lambda, \mu(t), \nu(t)) \right\rangle$$

$$- \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), D_\gamma\left(\frac{\partial V_\alpha}{\partial y_\gamma}\right)(\pi_y(t), \lambda, \mu(t), \nu(t)) \right\rangle,$$

$$(3.11)$$

$$\left\langle D^2_{\theta\sigma}\eta(\pi_{x^\circ}(t),\pi_y(t)),\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle$$

$$= D_\theta D_\sigma\left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle$$

$$- D_\theta\left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),D_\sigma\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle$$

$$- D_\sigma\left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),D_\theta\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle$$

$$+ \left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),D^2_{\sigma\theta}\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle.$$

$$(3.12)$$

By replacing the relations (3.11) and (3.12), and using Euler-Lagrange PDE, the relation (3.9) becomes

$$\int_{\gamma_{t_0,t_1}}\left[D_\gamma\left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),\frac{\partial V_\alpha}{\partial y_\gamma}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle\right.$$

$$+ D_\theta D_\sigma\left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle$$

$$- D_\theta\left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle \qquad (3.13)$$

$$\left. - D_\sigma\left\langle \eta(\pi_{x^\circ}(t),\pi_y(t)),D_\theta\frac{\partial V_\alpha}{\partial y_{\theta\sigma}}(\pi_y(t),\lambda,\mu(t),\nu(t))\right\rangle\right]dt^\alpha$$

$$< -\|\theta(x^\circ(\cdot),y(\cdot))\|^2\left(\rho'^\ell\Lambda^{10}_\ell + \rho''^\ell\Lambda^{20}_\ell + \rho'''\right).$$

According to [15], §9, a total divergence is equal to a total derivative, therefore, the left-hand side of (3.13) becomes null, and replacing into the inequality (3.13), it follows that

$$0 < -\|\theta(x^\circ(\cdot),y(\cdot))\|^2\left(\rho'^\ell\Lambda^{10}_\ell + \rho''^\ell\Lambda^{20}_\ell + \rho'''\right). \qquad (3.14)$$

From hypothesis (e), the previous relation becomes $0 < 0$, which is false. Next, from relation (3.8), it follows that

$$0 \leqq \Lambda^{10}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{20}_\ell K^\ell(x^\circ(\cdot))$$

$$+ \int_{\gamma_{t_0,t_1}}\left[\langle\mu_\alpha(t),g(\pi_{x^\circ}(t))\rangle + \langle\nu_\alpha(t),h(\pi_{x^\circ}(t))\rangle\right]dt^\alpha \qquad (3.15)$$

$$- \int_{\gamma_{t_0,t_1}}\left[\langle\mu_\alpha(t),g(\pi_y(t))\rangle + \langle\nu_\alpha(t),h(\pi_y(t))\rangle\right]dt^\alpha.$$

Taking into account the inequality:

$$\int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_y(t))\rangle + \langle \nu_\alpha(t), h(\pi_y(t))\rangle\right] dt^\alpha \geqq 0, \tag{3.16}$$

the above-mentioned relation becomes

$$\Lambda_\ell^{10} F^\ell(x^\circ(\cdot)) - \Lambda_\ell^{20} K^\ell(x^\circ(\cdot)) > 0, \tag{3.17}$$

that is,

$$K^\ell(x^\circ(\cdot))K^\ell(y(\cdot)) \left[\frac{F^\ell(x^\circ(\cdot))}{K^\ell(x^\circ(\cdot))} - \frac{F^\ell(y(\cdot))}{K^\ell(y(\cdot))}\right] > 0. \tag{3.18}$$

Because $K^\ell(x^\circ(\cdot))K^\ell(y(\cdot)) > 0$, $\ell = \overline{1,r}$, we conclude that

$$\frac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))} - \frac{F(y(\cdot))}{K(y(\cdot))} \nleq (0,\dots,0), \tag{3.19}$$

or

$$\frac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))} \nleq \frac{F(y(\cdot))}{K(y(\cdot))}. \tag{3.20}$$

Therefore, the relation $\varpi(x^\circ(\cdot)) \leq \delta(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ is not satisfied, and this completes the proof. $\square$

We will finish our considerations by giving the statements of two results on direct duality and converse duality, respectively. Their proofs follow directly from the weak duality.

Theorem 3.3 (Direct Duality). *Let $x^\circ(\cdot)$ be an efficient solution of (MFP). Suppose that the hypotheses of Theorem are satisfied. Then, there are Λ^{10} and Λ^{20} in $\mathbb{R}^r$, and the smooth functions $\mu^\circ :$ $\Omega_{t_0,t_1} \to \mathbb{R}^{msp}$ and $\nu^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$, such that $(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ is an efficient solution of the dual (MFD), and $\varpi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$.*

We will present now a theorem concerning the converse duality, by changing some of the hypotheses.

Theorem 3.4 (Converse Duality). *Let $(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ be an efficient solution of the dual problem (MFD). Suppose the following conditions hold:*

(a) *$\overline{x}(\cdot)$ is an efficient solution of the primal problem (MFP);*

(b) *for any $\ell = \overline{1,r}$, one has*

$$F^\ell(x^\circ(\cdot)) > 0, \qquad K^\ell(x^\circ(\cdot)) > 0, \qquad \Lambda_\ell^{10} F^\ell(x^\circ(\cdot)) - \Lambda_\ell^{20} K^\ell(x^\circ(\cdot)) = 0; \tag{3.21}$$

(c) *for any $\ell = \overline{1,r}$, the functional $F^\ell(x(\cdot))$ is (ρ'^ℓ, b)-quasiinvex at the point $x^\circ(\cdot)$ and the functional $-K^\ell(x(\cdot))$ is (ρ''^ℓ, b)-quasiinvex at the point $x^\circ(\cdot)$, with respect to η and θ;*

(d) *the functional,*

$$\int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_x(t)) \rangle + \langle \nu_\alpha(t), h(\pi_x(t)) \rangle \right] dt^\alpha, \tag{3.22}$$

is (ρ''', b)-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ;

(e) *one of the functionals of (c), (d) is strictly (ρ'^ℓ, b)-, (ρ''^ℓ, b)-, or (ρ''', b)-quasiinvex with respect to η and θ, respectively;*

(f) *$\rho'^\ell \Lambda_\ell^{10} + \rho''^\ell \Lambda_\ell^{20} + \rho''' \geq 0$.*

Then, $\overline{x}(\cdot) = x^\circ(\cdot)$, and $\varpi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$.

4. Conclusions

In our previous work [12], we initiated an optimization theory for the second-order jet bundle. We considered the problem of minimization of vectors of curvilinear functionals (well known as mechanical work), thought as multitime multiobjective variational problem, subject to PDE and/or PDI constraints (limited resources). Within this framework, we introduced necessary conditions. As natural continuation of our results in [12], and strongly motivated by its possible applications in mechanics, the present work introduced a dual program theory for this class of problems (for related but complementary research, see [16, 17]).

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