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Research Article

Some Results on Strictly Pseudocontractive Nonself-Mappings and Equilibrium Problems in Hilbert Spaces

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An equilibrium problem and a strictly pseudocontractive nonself-mapping are investigated. Strong convergence theorems of common elements are established based on hybrid projection algorithms in the framework of real Hilbert spaces.

1. Introduction

Bifunction equilibrium problems which were considered by Blum and Oettli [1] have intensively been studied. It has been shown that the bifunction equilibrium problem covers fixed point problems, variational inequalities, inclusion problems, saddle problems, complementarity problem, minimization problem, and the Nash equilibrium problem; see [1–4] and the references therein. Iterative methods have emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization; see [5–16] and the references therein. In this paper, we investigate an equilibrium problem and a strictly pseudocontractive nonself-mapping based on hybrid projection algorithms. Strong convergence theorems of common elements lie in the solution set of the equilibrium problem and the fixed point set of the strictly pseudocontractive nonself-mapping.

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and F a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. In this paper, we consider the following equilibrium problem:

$$\text{Find } x\in C \text{ such that } F(x,y)\geq 0,\quad \forall y\in C. \tag{1.1}$$

The set of such an $x\in C$ is denoted by $EP(F)$, that is,

$$EP(F)=\{x\in C : F(x,y)\geq 0,\ \forall y\in C\}. \tag{1.2}$$

Given a mapping $T : C \to H$, let $F(x,y)=\langle Tx, y-x\rangle$ for all $x,y\in C$. Then $z\in EP(F)$ if and only if $\langle Tz, y-z\rangle\geq 0$ for all $y\in C$, that is, z is a solution of the variational inequality.

To study the equilibrium problem (1.1), we may assume that F satisfies the following conditions:

(A1) $F(x,x)=0$ for all $x\in C$;

(A2) F is monotone, that is, $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$;

(A3) for each $x,y,z\in C$,

$$\limsup_{t\downarrow 0} F(tz+(1-t)x, y)\leq F(x,y); \tag{1.3}$$

(A4) for each $x\in C$, $y\mapsto F(x,y)$ is convex and lower semicontinuous.

If H is an Euclidean space, then we see that $F(x,y)=x-y$ is a simple example satisfying the above assumptions. See [2] for more details. Let $S : D(S) \to R(S)$, where $D(S)$ and $R(S)$ denote the domain and the range of the mapping S. If $D(S)=R(S)$, then the mapping S is said to be a self-mapping. If $D(S)\neq R(S)$, then the mapping S is said to be a nonself-mapping. Let $S : C \to H$ be a nonself-mapping. In this paper, we use $F(S)$ to denote the fixed point set of S. Recall the following definitions. S is said to be nonexpansive if

$$\|Sx-Sy\|\leq\|x-y\|,\quad \forall x,y\in C. \tag{1.4}$$

S is said to be strictly pseudocontractive if there exists a constant $\kappa\in[0,1)$ such that

$$\|Sx-Sy\|^2\leq\|x-y\|^2+\kappa\|(I-S)x-(I-S)y\|^2,\quad \forall x,y\in C. \tag{1.5}$$

For such a case, S is also said to be κ-strict pseudocontraction. It is clear that (1.5) is equivalent to

$$\langle Sx-Sy, x-y\rangle\leq\|x-y\|^2-\frac{1-\kappa}{2}\|(x-Sx)-(y-Sy)\|^2,\quad \forall x,y\in C. \tag{1.6}$$

S is said to be pseudocontractive if

$$\|Sx-Sy\|^2\leq\|x-y\|^2+\|(I-S)x-(I-S)y\|^2,\quad \forall x,y\in C. \tag{1.7}$$

It is clear that (1.7) is equivalent to

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{1.8}$$

The class of κ-strict pseudocontractions which was introduced by Browder and Petryshyn [17] in 1967 has been considered by many authors. It is easy to see that the class of strict pseudocontractions falls into the one between the class of nonexpansive mappings and the class of pseudocontractions. For studying the class of strict pseudocontractions, Zhou [18] proposed the following convex combination method: define a mapping $S_t : C \to H$ by

$$S_t x = tx + (1 - t)Sx, \quad \forall x \in C. \tag{1.9}$$

He showed that S_t is nonexpansive if $t \in [\kappa, 1)$; see [18] for more details.

Recently, many authors considered the problem of finding a common element in the fixed point set of a nonexpansive mapping and in the solution set of the equilibrium problem (1.1) based on iterative methods; see, for instance, [19–27].

In 2007, Tada and Takahashi [23] considered an iterative method for the equilibrium problem (1.1) and a nonexpansive nonself-mapping. To be more precise, they obtained the following results.

Theorem TT. *Let C be a closed convex subset of a real Hilbert space H, let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4), and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$, and let*

$$\begin{aligned} u_n \in C \ \text{ such that } \ & f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ w_n &= (1 - \alpha_n)x_n + \alpha_n S u_n, \\ C_n &= \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \end{aligned} \tag{1.10}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, 1]$, for some $a \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. *Then the sequence $\{x_n\}$ converges strongly to $P_{F(S)\cap EP(f)}(x)$.*

We remark that the iterative process (1.10) is called the hybrid projection iterative process. Recently, the hybrid projection iterative process which was first considered by Haugazeau [28] in 1968 has been studied for fixed point problem of nonlinear mappings and equilibrium problems by many authors. Since the sequence generated in the hybrid projection iterative process depends on the sets C_n and Q_n, the hybrid projection iterative process is also known as "CQ" iterative process; see [29] and the reference therein.

Recently, Takahashi et al. [30] considered the shrinking projection process for the fixed point problem of nonexpansive self-mapping. More precisely, they obtain the iterative sequence monontonely without the help of the set Q_n; see [30] for more details.

In this paper, we reconsider the same shrinking projection process for the equilibrium problem (1.1) and a strictly pseudocontractive nonself-mapping. We show that the sequence generated in the proposed iterative process converges strongly to some common element in the fixed point set of a strictly pseudocontractive nonself-mapping and in the solution set of the equilibrium problem (1.1). The main results presented in this paper mainly improved the corresponding results in Tada and Takahashi [23].

2. Preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let P_C be the metric projection from H onto C. That is, for $x \in H$, $P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$. We know that the mapping P_C is firmly nonexpansive, that is,

$$\left\|P_C x - P_C y\right\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

The following lemma can be found in [1, 2].

Lemma 2.1. *Let C be a nonempty closed convex subset of H, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Further, define

$$T_r x = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\} \tag{2.3}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single valued;*

(b) *T_r is firmly nonexpansive, that is,*

$$\left\|T_r x - T_r y\right\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H; \tag{2.4}$$

(c) *$F(T_r) = EP(F)$;*

(d) *$EP(F)$ is closed and convex.*

Lemma 2.2 (see [18]). *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $S : C \to H$ a strict pseudocontraction. Then the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \overline{x}$ and $x_n - Sx_n \to 0$, then $\overline{x} \in F(S)$.*

Lemma 2.3 (see [18]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to H$ a κ-strict pseudocontraction. Define a mapping $S_\alpha x = \alpha x + (1 - \alpha)Sx$ for all $x \in C$. If $\alpha \in [\kappa, 1)$, then the mapping S_α is a nonexpansive mapping such that $F(S_\alpha) = F(S)$.*

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ which satisfy (A1)–(A4), respectively. Let $S : C \to H$ be a κ-strict pseudocontraction. Assume that $\digamma := EP(F_1) \cap EP(F_2) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in H, \\ C_1 = H, \\ z_n = \gamma_n u_n + (1-\gamma_n) v_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\beta_n z_n + (1-\beta_n) S z_n), \\ C_{n+1} = \{ w \in C_n : \|y_n - w\| \leq \|x_n - w\| \}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 0, \end{gathered} \tag{Δ}$$

where u_n is chosen such that

$$F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C, \tag{3.1}$$

and v_n is chosen such that

$$F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n \rangle \geq 0, \quad \forall v \in C. \tag{3.2}$$

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$, $\kappa \leq \beta_n \leq b < 1$, $0 < c \leq \gamma_n \leq d < 1$;

(b) $0 < e \leq r_n$, $0 < f \leq s_n$

for some $a, b, c, d, e, f \in \mathbb{R}$. Then the sequence $\{x_n\}$ generated in the (Δ) converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\digamma} x_1$.

Proof. First, we show that C_n is closed and convex for each $n \geq 1$. It is easy to see that C_n is closed for each $n \geq 1$. We only show that C_n is convex for each $n \geq 1$. Note that $C_1 = H$ is convex. Suppose that C_m is convex for some positive integer m. Next, we show that C_{m+1} is convex for the same m. Note that $\|y_n - w\| \leq \|x_n - w\|$ is equivalent to

$$2\langle x_n - y_n, w \rangle \leq \|x_n\|^2 - \|y_n\|^2. \tag{3.3}$$

Take w_1 and w_2 in C_{m+1}, and put $\overline{w} = t w_1 + (1-t) w_2$. It follows that $w_1 \in C_m, w_2 \in C_m$,

$$\begin{gathered} 2\langle x_n - y_n, w_1 \rangle \leq \|x_n\|^2 - \|y_n\|^2, \\ 2\langle x_n - y_n, w_2 \rangle \leq \|x_n\|^2 - \|y_n\|^2. \end{gathered} \tag{3.4}$$

Combining (3.4), we can obtain that $2\langle x_n - y_n, \overline{w}\rangle \le \|x_n\|^2 - \|y_n\|^2$, that is, $\|y_n - \overline{w}\| \le \|x_n - \overline{w}\|$. In view of the convexity of C_m, we see that $\overline{w} \in C_m$. This shows that $\overline{w} \in C_{m+1}$. This concludes that C_n is closed and convex for each $n \ge 1$.

Define a mapping $S_n : C \to H$ by $S_n x = \beta_n x + (1-\beta_n)Sx$ for all $x \in C$. It follows from Lemma 2.3 that S_n is nonexpansive and $F(S_n) = F(S)$ for all $n \ge 1$. Next, we show that $\mathcal{F} \subset C_n$ for all $n \ge 1$. It is easy to see that $\mathcal{F} \subset C_1 = H$. Suppose that $\mathcal{F} \subset C_k$ for some integer $k \ge 1$. We intend to claim that $\mathcal{F} \subset C_{k+1}$ for the same k. For any $p \in \mathcal{F} \subset C_k$, we have

$$\begin{aligned}
\|y_k - p\| &\le \alpha_k\|x_k - p\| + (1-\alpha_k)\|S_k z_k - p\| \\
&\le \alpha_k\|x_k - p\| + (1-\alpha_k)\|z_k - p\| \\
&\le \alpha_k\|x_k - p\| + (1-\alpha_k)\left(\gamma_k\|u_k - p\| + (1-\gamma_k)\|v_k - p\|\right) \\
&= \alpha_k\|x_k - p\| + (1-\alpha_k)\left(\gamma_k\|T_{r_k}x_k - p\| + (1-\gamma_k)\|T_{s_k}x_k - p\|\right) \\
&\le \alpha_k\|x_k - p\| + (1-\alpha_k)\left(\gamma_k\|x_k - p\| + (1-\gamma_k)\|x_k - p\|\right) \\
&= \|x_k - p\|.
\end{aligned} \tag{3.5}$$

This shows that $p \in C_{k+1}$. This proves that $\mathcal{F} \subset C_n$ for all $n \ge 1$.

Since $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have that

$$\begin{aligned}
0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\
&\le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.
\end{aligned} \tag{3.6}$$

It follows that

$$\|x_1 - x_n\| \le \|x_1 - x_{n+1}\|. \tag{3.7}$$

On the other hand, for any $p \in \mathcal{F} \subset C_n$, we see that $\|x_1 - x_n\| \le \|x_1 - p\|$. In particular, we have

$$\|x_1 - x_n\| \le \|x_1 - P_{\mathcal{F}}x_1\|. \tag{3.8}$$

This shows that the sequence $\{x_n\}$ is bounded. In view of (3.7), we see that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. It follows from (3.6) that

$$\begin{aligned} &\|x_n - x_{n+1}\|^2 \\ &\quad = \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &\quad = \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &\quad = \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &\quad = \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &\quad \le \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2. \end{aligned} \tag{3.9}$$

This yields that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.10}$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we see that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|. \tag{3.11}$$

It follows that

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_n - x_{n+1}\|. \tag{3.12}$$

From (3.10), we obtain that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.13}$$

On the other hand, we have

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)S_n z_n\| = (1-\alpha_n)\|x_n - S_n z_n\|. \tag{3.14}$$

From the restriction (a), we obtain from (3.13) that

$$\lim_{n\to\infty}\|x_n - S_n z_n\| = 0. \tag{3.15}$$

For any $p \in \mathcal{F}$, we have that

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2\\ &\le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle\\ &= \langle u_n - p, x_n - p\rangle\\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2\right).\end{aligned} \tag{3.16}$$

This implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{3.17}$$

In a similar way, we get that

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|v_n - x_n\|^2. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_nz_n - p\|^2\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left(\gamma_n\|u_n - p\|^2 + (1-\gamma_n)\|v_n - p\|^2\right)\\ &\le \|x_n - p\|^2 - (1-\alpha_n)\gamma_n\|u_n - x_n\|^2 - (1-\alpha_n)(1-\gamma_n)\|v_n - x_n\|^2.\end{aligned} \tag{3.19}$$

This implies that

$$\begin{aligned}(1-\alpha_n)\gamma_n\|u_n - x_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2\\ &\le (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|.\end{aligned} \tag{3.20}$$

In view of the restriction (a), we obtain from (3.13) that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.21}$$

It also follows from (3.19) that

$$\begin{aligned}(1-\alpha_n)(1-\gamma_n)\|v_n - x_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2\\ &\le (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|.\end{aligned} \tag{3.22}$$

In view of the restriction (a), we obtain from (3.13) that

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.23}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$. From (3.21) and (3.23), we see that $u_{n_i} \rightharpoonup q$ and $v_{n_i} \rightharpoonup q$, respectively. From (3.21) and the restriction (b), we see that

$$\lim_{n\to\infty}\frac{\|u_n - x_n\|}{r_n} = 0. \tag{3.24}$$

Now, we are in a position to show that $q \in EP(F_1)$. Note that

$$F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C. \tag{3.25}$$

From (A2), we see that

$$\frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq F_1(u, u_n). \tag{3.26}$$

Replacing n by n_i, we arrive at

$$\left\langle u - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F_1(u, u_{n_i}). \tag{3.27}$$

In view of (3.24) and (A4), we get that

$$F_1(u, q) \leq 0, \quad \forall u \in C. \tag{3.28}$$

For any t with $0 < t \leq 1$ and $u \in C$, let $u_t = tu + (1-t)q$. Since $u \in C$ and $q \in C$, we have $u_t \in C$ and hence $F_1(u_t, q) \leq 0$. It follows that

$$0 = F_1(u_t, u_t) \leq tF_1(u_t, u) + (1-t)F_1(u_t, q) \leq tF_1(u_t, u), \tag{3.29}$$

which yields that

$$F_1(u_t, u) \geq 0, \quad \forall u \in C. \tag{3.30}$$

Letting $t \downarrow 0$, we obtain from (A3) that

$$F_1(q, u) \geq 0, \quad \forall u \in C. \tag{3.31}$$

This means that $q \in EP(F_1)$. In the same way, we can obtain that $q \in EP(F_2)$. Next, we show that $q \in F(S)$. Note that

$$\|z_n - x_n\| \leq \gamma_n \|u_n - x_n\| + (1-\gamma_n)\|v_n - x_n\|. \tag{3.32}$$

It follows from (3.21) and (3.23) that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.33}$$

On the other hand, we have

$$\begin{aligned} \|x_n - S_n x_n\| &\leq \|S_n x_n - S_n z_n\| + \|S_n z_n - x_n\| \\ &\leq \|x_n - z_n\| + \|S_n z_n - x_n\|. \end{aligned} \tag{3.34}$$

It follows from (3.15) and (3.33) that

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{3.35}$$

Note that

$$\begin{aligned} \|Sx_n - x_n\| &\leq \|Sx_n - S_n x_n\| + \|S_n x_n - x_n\| \\ &\leq \beta_n \|Sx_n - x_n\| + \|S_n x_n - x_n\|, \end{aligned} \tag{3.36}$$

which yields that

$$(1-\beta_n)\|Sx_n - x_n\| \leq \|S_n x_n - x_n\|. \tag{3.37}$$

This implies from the restriction (a) and (3.35) that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.38}$$

It follows from Lemma 2.2 that $q \in F(S)$. This shows that $q \in \mathcal{F}$. Since $\overline{x} = P_{\mathcal{F}} x_1$, we obtain that

$$\begin{aligned} \|x_1 - \overline{x}\| &\leq \|x_1 - q\| \leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - \overline{x}\|, \end{aligned} \tag{3.39}$$

which yields that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - q\| = \|x_1 - \overline{x}\|. \tag{3.40}$$

It follows that $\{x_{n_i}\}$ converges strongly to $\overline{x}$. Therefore, we can conclude that the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathcal{F}} x_1$. This completes the proof. □

From Theorem 3.1, we have the following results.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $S : C \to H$ be a κ-strict pseudocontraction. Assume that $\mathcal{F} := EP(F) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$
\begin{gathered}
x_1 \in H, \\
C_1 = H, \\
y_n = \alpha_n x_n + (1-\alpha_n)\big(\beta_n u_n + (1-\beta_n) S u_n\big), \\
C_{n+1} = \{ w \in C_n : \|y_n - w\| \leq \|x_n - w\| \}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 0,
\end{gathered}
\tag{3.41}
$$

where u_n is chosen such that

$$
F(u_n, u) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C. \tag{3.42}
$$

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$, $\kappa \leq \beta_n \leq b < 1$;

(b) $0 < e \leq r_n$

for some $a, b, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}} x_1$.

Proof. Putting $F_1 \equiv F_2 \equiv F$ and $r_n \equiv s_n$ in Theorem 3.1, we see that $z_n \equiv u_n$. From the proof of Theorem 3.1, we can conclude the desired conclusion immediately. □

Remark 3.3. Corollary 3.2 improves Theorem TT in the following aspects.

(1) From the viewpoint of mappings, the class of nonexpansive mappings is extended to the class of strict pseudocontractions.

(2) From the viewpoint of computation, the set Q_n is removed.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to H$ be a κ-strict pseudocontraction with fixed points. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_1 \in H, \\
C_1 = H, \\
y_n = \alpha_n x_n + (1-\alpha_n)\left(\beta_n P_C x_n + (1-\beta_n) S P_C x_n\right), \\
C_{n+1} = \left\{w \in C_n : \left\|y_n - w\right\| \le \|x_n - w\|\right\}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 0.
\end{gathered} \tag{3.43}$$

Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the restrictions $0 \le \alpha_n \le a < 1$ and $\kappa \le \beta_n \le b < 1$ for some $a, b \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{F(S)} x_1$.

Proof. Putting $F_1 \equiv F_2 \equiv 0$ and $r_n \equiv s_n \equiv 1$, we can obtain from Theorem 3.1 the desired conclusion easily.

If S is nonexpansive and $\beta_n \equiv 0$, then Corollary 3.4 is reduced to the following. □

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to H$ be a nonexpansive mapping with fixed points. Let $\{\alpha_n\}$ be a sequence in $[0,1]$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_1 \in H, \\
C_1 = H, \\
y_n = \alpha_n x_n + (1-\alpha_n) S P_C x_n, \\
C_{n+1} = \left\{w \in C_n : \left\|y_n - w\right\| \le \|x_n - w\|\right\}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 0.
\end{gathered} \tag{3.44}$$

Assume that the control sequence $\{\alpha_n\}$ satisfies the restriction $0 \le \alpha_n \le a < 1$ for some $a \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{F(S)} x_1$.

Recently, many authors studied the following convex feasibility problem (CFP):

$$\text{finding an } \ x \in \bigcap_{m=1}^{N} C_m, \tag{3.45}$$

where $N \ge 1$ is an integer and each C_m is a nonempty closed and convex subset of a real Hilbert space H. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration, computer tomography, and radiation therapy treatment planning.

Next, we consider the case that each C_m is the solution set of an equilibrium problem.

Theorem 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4) for each $1 \leq m \leq N$. Assume that $\mathcal{F} := \cap_{m=1}^{N} EP(F_m) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\gamma_{(n,1)}\}, \ldots,$ and $\{\gamma_{(n,N)}\}$ be sequences in $[0,1]$. Let $r_{(n,1)}, \ldots,$ and $r_{(n,N)}$ be sequences in $(0,\infty)$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in H, \\ C_1 = H, \\ F_m\left(u_{(n,m)}, u\right) + \frac{1}{r_{(n,m)}}\left\langle u - u_{(n,m)}, u_{(n,m)} - x_n\right\rangle \geq 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{m=1}^{N}\gamma_{(n,m)}u_{(n,m)}, \\ C_{n+1} = \left\{w \in C_n : \left\|y_n - w\right\| \leq \left\|x_n - w\right\|\right\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 0. \end{gathered} \tag{3.46}$$

Assume that the control sequences satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$, $\sum_{m=1}^{N}\gamma_{(n,m)} = 1$, $0 < c_m \leq \gamma_{(n,m)} \leq d_m < 1$;

(b) $0 < e_m \leq r_{(n,m)}$,

where a, $c_1, \ldots, c_m$, $d_1, \ldots, d_m$, $e_1, \ldots, e_m \in \mathbb{R}$. Then the sequence $\{x_n\}$ generated in the above iterative process converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}x_1$.

Proof. Let S be the identity mapping and $\beta_n \equiv 0$, then we can obtain from Theorem 3.1 the desired conclusion easily. □

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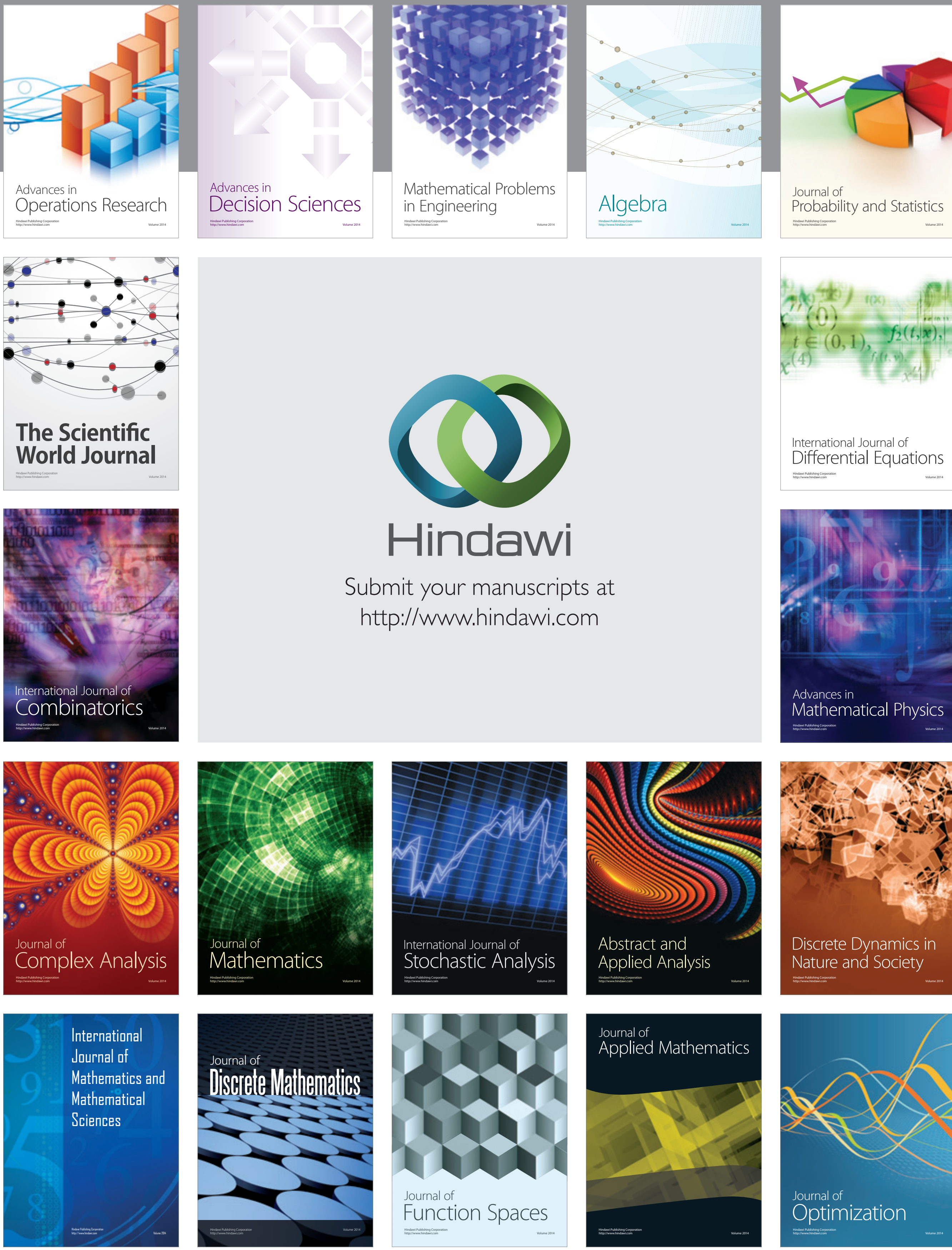
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