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Research Article

Basis Properties of Eigenfunctions of Second-Order Differential Operators with Involution

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We study the basis properties of systems of eigenfunctions and associated functions for one kind of generalized spectral problems for a second-order ordinary differential operator.

1. Introduction

Let us consider the partial differential equation with involution

$$w_t(t,x) = \alpha w_{xx}(t,x) + w_{xx}(t,-x), \quad -1 < x < 1, \; t > 0. \tag{1.1}$$

If the initial conditions

$$w(0,x) = f(x) \tag{1.2}$$

and the boundary conditions

$$\alpha_j w_x(t,-1) + \beta_j w_x(t,1) + \alpha_{j1} w(t,-1) + \beta_{j1} w(t,1) = 0, \quad j = 1,2 \tag{1.3}$$

are given, then the solving of this equation by Fourier's method leads to the problem of expansion of function $f(x)$ into series of eigenfunctions of spectral problem

$$\begin{gathered} -u''(-x) + \alpha u''(x) = \lambda u(x), \\ \alpha_j u'(-1) + \beta_j u'(1) + \alpha_{j1} u(-1) + \beta_j u(1) = 0, \quad j = 1,2. \end{gathered} \tag{1.4}$$

If the function $f(x) \in L^2(-1,1)$, then the question about basis property of eigenfunctions of spectral problem for second-order ordinary differential operator with involution raises.

Work of many researchers is devoted to the study of differential equations [1–5]. Various aspects of functionally differential equations with involution are studied in [6, 7]. The spectral problems for the double differentiation operator with involution are studied in [8–11] and the issues Riesz basis property of eigenfunctions in terms of coefficients of boundary conditions were considered.

This kind of spectral problems arises in the theory of solvability of differential equations in partial derivatives with an involution [7, page 265].

Results presented below are a continuation of studies of one of the authors in [9–11].

2. General Boundary Value Problem

In this paper, we study the spectral problem of the form

$$Lu \equiv -u''(-x) + \alpha u''(x) + \beta u'(x) + \gamma u'(-x) + \eta u(-x) = \lambda u(x), \tag{2.1}$$

$$\begin{aligned} &\alpha_1 u'(-1) + \beta_1 u'(1) + \alpha_{11} u(-1) + \beta_{11} u(1) = 0, \\ &\alpha_2 u'(-1) + \beta_2 u'(1) + \alpha_{21} u(-1) + \beta_{21} u(1) = 0, \end{aligned} \tag{2.2}$$

where α, β, γ, η, α_i, β_i, α_{ij}, β_{ij} are some complex numbers.

By direct calculation, one can verify that the square of the operator is in the form

$$\begin{aligned} L^2 u = {}& \left(1+\alpha^2\right) u^{IV}(x) - 2\alpha u^{IV}(-x) + 2\alpha\gamma u'''(-x) + 2\alpha\beta u'''(x) + 2\alpha\eta u''(-x) \\ &+ \left(-2\eta + \beta^2 - \gamma^2\right) u''(x) + \eta^2 u(x). \end{aligned} \tag{2.3}$$

Since it is assumed that Lu belongs to domain of operator L also, then function Lu satisfies boundary-value conditions (2.2)

$$\begin{aligned} &\alpha_1 (Lu)'(-1) + \beta_1 (Lu)'(1) + \alpha_{11} (Lu)(-1) + \beta_{11} (Lu)(1) = 0, \\ &\alpha_2 (Lu)'(-1) + \beta_2 (Lu)'(1) + \alpha_{21} (Lu)(-1) + \beta_{21} (Lu)(1) = 0. \end{aligned} \tag{2.4}$$

That is, the operator L^2 is generated by previous differential expression and boundary-value conditions (2.2) and (2.4).

The expression $L^2 u$ is an ordinary differential expression for $\alpha = 0$.

Therefore, applying the method in [8–10] we can obtain the following statement (the result).

Theorem 2.1. *If $\alpha = 0$, then the eigenfunctions of the generalized spectral problem* (2.1) *and* (2.2) *form a Riesz basis of the space $L_2(-1,1)$ in the following cases:*

(1) $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$;

(2) $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$, $|\alpha_1| + |\beta_1| > 0$, $\alpha_1^2 \neq \beta_2^2$, $\alpha_{21}^2 \neq \beta_{21}^2$,

(3) $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$; $\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \neq 0$.

The root vectors of operators A and A^2 coincide under some conditions (see, for instance, [10]). Therefore, we can consider the square of the operator L which is an ordinary differential operator. It is well known [12–14] that eigenfunctions of ordinary differential operator of even order with strongly regular boundary value conditions form a Riesz basis. As in [10], from here it is possible to deduce correctness of Theorem 2.1.

This technique is not applicable for a = 0 since L^2u is not an ordinary differential operator. Therefore, we consider this case separately.

3. General Solution of Special Type Equation

Let the operator L be given by the differential expression with an involution

$$Lu = -u''(-x) + \alpha u''(x), \tag{3.1}$$

and boundary conditions (2.2).

We consider the spectral problem $Lu = \lambda u(x)$ with periodic, antiperiodic boundary conditions, with the boundary conditions of Dirichlet and Sturm type. In these cases, it is possible to compute all the eigenvalues and eigenfunctions explicitly. The basis of our statements is the following.

Theorem 3.1. *If $a^2 \neq 1$, then the general solution of equation*

$$-u''(-x) + \alpha u''(x) = \lambda u(x), \tag{3.2}$$

where λ is the spectral parameter, has the form

$$u(x) = A\cos\sqrt{\frac{\lambda}{1-\alpha}}x + B\sin\sqrt{\frac{\lambda}{-1-\alpha}}x, \tag{3.3}$$

where A and B are arbitrary complex numbers.

If $\alpha^2 = 1$ and $\lambda \neq 0$, then (3.2) has only the trivial solution.

Proof. It is easy to see that functions (3.3) are solutions of (3.2). Let us prove the absence of other solutions.

Any function $u(x)$ can be represented as a sum of even and odd functions. Substituting this representation into (3.2) and into $-u''(x) + \alpha u''(-x) = \lambda u(-x)$, we conclude that the functions $u_1(x)$ and

$$\begin{gathered}
-(1-\alpha)u_1''(x) = \lambda u_1(x), \\
-(-1-\alpha)u_2''(x) = \lambda u_2(x).
\end{gathered} \tag{3.4}$$

□

4. The Dirichlet Problem

Consider the spectral problem (3.2) $a^2 \neq 1$ with boundary conditions

$$u(-1) = 0, \qquad u(1) = 0. \tag{4.1}$$

Note that the spectral problem (3.2) and (4.1) is self-adjoint for real α. We calculate the eigenvalues and eigenfunctions of the Dirichlet problem (3.2) and (4.1). Using Theorem 3.1, it is easy to see that the spectral problem (3.2) and (4.1) has two sequences of simple eigenvalues.

If $\alpha \notin \{(8k^2 + 4k + 1)/(4k + 1) : k \in Z\}$, then corresponding eigenfunctions are given by the formulas

$$u_{k1}(x) = \cos\left(\frac{\pi}{2} + k\pi\right)x, \quad k = 0, 1, 2, \ldots, \qquad u_{k2}(x) = \sin k\pi x, \quad k = 1, 2, \ldots. \tag{4.2}$$

If $\alpha \notin (8k^2 + 4k + 1)/(4k + 1)$ for some $k_0 \in Z$, then the eigenfunctions of the spectral problem (3.2) and (4.1) are given by

$$\begin{gathered}
u_{k1}(x) = \cos\left(\frac{\pi}{2} + k\pi\right)x, \quad k = 0, 1, 2, \ldots, \qquad u_{k2}(x) = \sin k\pi x, \quad k = 1, 2, \ldots, \ k \neq k_0, \\
u_{k_0 1}(x) = \cos\left(\frac{\pi}{2} + k_0\pi\right)x + \sin\sqrt{\frac{1-\alpha}{-1-\alpha}}\left(\frac{\pi}{2} + k_0\pi\right)x, \\
u_{k_0 2}(x) = \sin k_0\pi x + \cos\sqrt{\frac{-1-\alpha}{1-\alpha}}k_0\pi x.
\end{gathered} \tag{4.3}$$

Theorem 4.1. *If $a^2 \neq 1$, then the system of eigenfunctions of the spectral problem* (3.2) *and* (4.1), *which is given above, forms an orthonormal basis of the space $L_2(-1, 1)$.*

Proof. For real values of α, the spectral problem (3.2) and (4.1) is self-adjoint. Therefore, the system (4.1), as a system of eigenfunctions self-adjoint operator, is an orthonormal. Analogously, the case $\alpha = (8k_0^2 + 4k_0 + 1)/(4k_0 + 1)$, $k_0 \in Z$, is considered. Also note that every orthonormal basis is automatically a Riesz basis.

The system (4.2) does not depend on α, hence Theorem 4.1 is proved. □

5. Periodic and Antiperiodic Problem

Now consider the spectral problem (3.2) with the periodic boundary conditions

$$u(-1) = u(1), \qquad u'(-1) = u'(1). \tag{5.1}$$

It follows immediately from Theorem 3.1 that the eigenfunctions of the spectral problem (3.2) and (5.1) are given by

$$(\lambda_{k1})^2 = -(1+\alpha)k^2\pi^2, \qquad (\lambda_{k2})^2 = (1-\alpha)k^2\pi^2. \tag{5.2}$$

They are simple and correspond to the eigenfunctions

$$u_{k1}(x) = \sin k\pi x, \quad k = 0,1,2,\ldots, \qquad u_{k2}(x) = \cos k\pi x, \quad k = 0,1,2,\ldots. \tag{5.3}$$

Similarly, the eigenvalues and eigenfunctions of the spectral problem with antiperiodic boundary conditions

$$u(-1) = -u(1), \qquad u'(-1) = -u'(1) \tag{5.4}$$

are calculated.

In this case, there are two series of eigenvalues also

$$\begin{aligned}(\lambda_{k1})^2 &= (1-\alpha)\left(\frac{\pi}{2}+k\pi\right), \quad k = 0,1,2,\ldots,\\ (\lambda_{k2})^2 &= (-1-\alpha)\left(\frac{\pi}{2}+k\pi\right), \quad k = 0,1,2,\ldots.\end{aligned} \tag{5.5}$$

They correspond to the eigenfunctions

$$u_{k1} = \cos\left(\frac{\pi}{2}+k\pi\right)x, \quad k = 1,2,\ldots, \qquad u_{k2} = \sin\left(\frac{\pi}{2}+k\pi\right)x, \quad k = 0,1,2,\ldots. \tag{5.6}$$

Theorem 5.1. *If $\alpha^2 \neq 1$, then the systems of eigenfunctions of the spectral problem* (3.2) *with periodic or antiperiodic boundary conditions form orthonormal bases of the space $L_2(-1,1)$.*

The proof is analogous to the proof of Theorem 4.1. Also note that for periodic conditions the eigenfunctions form the classical orthonormal basis of $L_2(-1,1)$.

Analogously, it is possible to check that the eigenfunctions of spectral problems (3.2), $\alpha^2 \neq 1$, with boundary conditions of Sturm type

$$u'(-1) = 0, \quad u'(1) = 0 \tag{5.7}$$

and with nonself-adjoint boundary conditions

$$u(-1) = 0, \quad u'(-1) = u'(1) \tag{5.8}$$

form orthonormal bases of $L_2(-1, 1)$.

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