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Research Article

Asymptotic Convergence of the Solutions of a Dynamic Equation on Discrete Time Scales

J. Diblík,[1,2] M. Růžičková,[3] Z. Šmarda,[1] and Z. Šutá[3]

[1] *Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, 616 00 Brno, Czech Republic*
[2] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 602 00 Brno, Czech Republic*
[3] *Department of Mathematics, University of Žilina, 01026 Žilina, Slovakia*

Correspondence should be addressed to J. Diblík, diblik@feec.vutbr.cz

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The paper investigates a dynamic equation $\Delta y(t_n) = \beta(t_n)[y(t_{n-j}) - y(t_{n-k})]$ for $n \to \infty$, where k and j are integers such that $k > j \geq 0$, on an arbitrary discrete time scale $\mathbb{T} := \{t_n\}$ with $t_n \in \mathbb{R}$, $n \in \mathbb{Z}_{n_0-k}^{\infty} = \{n_0 - k, n_0 - k + 1, \ldots\}$, $n_0 \in \mathbb{N}$, $t_n < t_{n+1}$, $\Delta y(t_n) = y(t_{n+1}) - y(t_n)$, and $\lim_{n\to\infty} t_n = \infty$. We assume $\beta : \mathbb{T} \to (0, \infty)$. It is proved that, for the asymptotic convergence of all solutions, the existence of an increasing and asymptotically convergent solution is sufficient. Therefore, the main attention is paid to the criteria for the existence of an increasing solution asymptotically convergent for $n \to \infty$. The results are presented as inequalities for the function β. Examples demonstrate that the criteria obtained are sharp in a sense.

1. Introduction

We use the following notation: for an integer s, we define that $\mathbb{Z}_s^{\infty} := \{s, s+1, \ldots\}$, and if an integer $q \geq s$, we define $\mathbb{Z}_s^{q} := \{s, s+1, \ldots, q\}$.

Hilger initiated in [1, 2] the calculus of time scales in order to create a theory that unifies discrete and continuous analyses. He defined a time scale $\mathbb{T}$ as an arbitrary nonempty closed subset of real numbers. The theoretical background for time scales can be found in [3].

In this paper, we use discrete time scales. To be exact, we define a discrete time scale $\mathbb{T} = T(t)$ as an arbitrary unbounded increasing sequence of real numbers, that is, $T(t) := \{t_n\}$, where $t_n \in \mathbb{R}$, $n \in \mathbb{Z}_{n_0-k}^{\infty}$, $n_0 \in \mathbb{N}$, $k > 0$ is an integer, $t_n < t_{n+1}$, and $\lim_{n\to\infty} t_n = \infty$. For a fixed $v \in \mathbb{Z}_{n_0-k}^{\infty}$, we define a time scale $\mathbb{T}_v = T_v(t) := \{t_n\}$, where $n \in \mathbb{Z}_v^{\infty}$. Obviously, $T_{n_0-k}(t) = T(t)$. In addition, for integers s, q, $q \geq s \geq n_0 - k$, we define the set $\mathbb{T}_s^q = T_s^q(t) := \{t_s, t_{s+1}, \ldots, t_q\}$.

In the paper we study a dynamic equation

$$\Delta y(t_n) = \beta(t_n)\left[y(t_{n-j}) - y(t_{n-k})\right] \tag{1.1}$$

as $n \to \infty$. The difference is defined as usual: $\Delta y(t_n) := y(t_{n+1}) - y(t_n)$, integers k and j in (1.1) satisfy the inequality $k > j \geq 0$, and $\beta : \mathbb{T} \to \mathbb{R}^+ := (0, \infty)$. Without loss of generality, we assume that $t_{n_0-k} > 0$ (this is a technical detail, necessary for some expressions to be well defined). Throughout the paper, we adopt the notation $\sum_{i=k+1}^{k} \mathcal{B}(t_i) = 0$ where k is an integer and $\mathcal{B}$ denotes the function under consideration.

The results concern the asymptotic convergence of all solutions of (1.1). First we prove that, in the general case, the asymptotic convergence of all solutions is determined only by the existence of an increasing and bounded solution. Therefore, our effort is focused on developing criteria guaranteeing the existence of such solutions. The proofs of the results are based on comparing the solutions of (1.1) with those of an auxiliary inequality with the same left-hand and right-hand sides as in (1.1). We also illustrate general results using examples with particular time scales.

The problem concerning the asymptotic convergence of solutions in the continuous case, that is, in the case of delayed differential equations or other classes of equations, is a classical one and has attracted much attention recently (we refer, e.g., to the papers [4–11]).

The problem of the asymptotic convergence of solutions of discrete and difference equations with delay has not yet received much attention. Some recent results can be found, for example, in [12–19].

Comparing the known investigations with the results presented, we can see that our results give sharp sufficient conditions of the asymptotic convergence of solutions. This is illustrated by examples. Nevertheless, we are not concerned with computing the limits of the solutions as $n \to \infty$.

The paper is organized as follows. In Section 2, auxiliary definitions and results are collected. An auxiliary inequality is studied, and the relationship of its solutions with the solutions of (1.1) is derived. Section 3 contains results concerning the convergence of all solutions of (1.1). The criteria of existence of an increasing and convergent solution of (1.1) are established in Section 4. Examples illustrating the sharpness of the results derived are discussed as well.

2. Auxiliary Definitions and Results

Let $\mathcal{C} := \mathcal{C}(\mathbb{T}_{n_0-k}^{n_0}, \mathbb{R})$ be the space of discrete functions mapping the discrete interval $\mathbb{T}_{n_0-k}^{n_0}$ into $\mathbb{R}$. Let $v \in \mathbb{Z}_{n_0}^{\infty}$ be given. The function $y : \mathbb{T}_{v-k} \to \mathbb{R}$ is said to be a *solution of* (1.1) *on* $\mathbb{T}_{v-k}$ if it satisfies (1.1) for every $n \in \mathbb{Z}_v^{\infty}$. A solution y of (1.1) on $\mathbb{T}_{v-k}$ is *asymptotically convergent* if the limit $\lim_{n\to\infty} y(t_n)$ exists and is finite. For a given $v \in \mathbb{Z}_{n_0}^{\infty}$ and $\varphi \in \mathcal{C}$, we say that $y = y_{(t_v,\varphi)}$ is a *solution of* (1.1) defined by the initial conditions (t_v, φ) if $y_{(t_v,\varphi)}$ is a solution of (1.1) on $\mathbb{T}_{v-k}$ and $y_{(t_v,\varphi)}(t_{v+m}) = \varphi(t_m)$ for $m \in \mathbb{Z}_{-k}^{0}$.

2.1. *Auxiliary Inequality*

The inequality

$$\Delta\omega(t_n) \geq \beta(t_n)\left[\omega(t_{n-j}) - \omega(t_{n-k})\right] \tag{2.1}$$

is a helpful tool in the analysis of solutions of (1.1). Let $v \in \mathbb{Z}_{n_0}^{\infty}$. The function $\omega : \mathbb{T}_{v-k} \to \mathbb{R}$ is said to be a *solution* of (2.1) on $\mathbb{T}_{v-k}$ if ω satisfies (2.1) for $n \in \mathbb{Z}_{v}^{\infty}$. A solution ω of (2.1) on $\mathbb{T}_{v-k}$ is *asymptotically convergent* if the limit $\lim_{n\to\infty}\omega(t_n)$ exists and is finite.

We give some properties of solutions of inequalities of type (2.1) to be used later on. We will also compare the solutions of (1.1) with those of (2.1).

Lemma 2.1. *Let $\varphi \in \mathcal{C}$ be increasing (nondecreasing, decreasing, nonincreasing) on $\mathbb{T}_{n_0-k}^{n_0}$. Then the solution $y_{(n_0,\varphi)}(t_n)$ of* (1.1)*, where $n \in \mathbb{Z}_{n_0}^{\infty}$ is increasing (nondecreasing, decreasing, nonincreasing) on $\mathbb{T}_{n_0}$, too.*

Lemma 2.2. *Let $\varphi \in \mathcal{C}$ be increasing (nondecreasing) and $\omega : \mathbb{T} \to \mathbb{R}$ be a solution of inequality* (2.1) *with $\omega(t_m) = \varphi(t_m)$, $m \in \mathbb{Z}_{n_0-k}^{n_0}$. Then, $\omega(t_n)$, where $n \in \mathbb{Z}_{n_0}^{\infty}$ is increasing (nondecreasing).*

The proofs of both lemmas above follow directly from the form of (1.1), (2.1), and from the properties $\beta(t_n) > 0$, $n \in \mathbb{Z}_{n_0-k}^{\infty}$, $k > j \geq 0$.

Theorem 2.3. *Let $\omega : \mathbb{T} \to \mathbb{R}$ be a solution of* (2.1) *on $\mathbb{T}$. Then there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of* (1.1) *on $\mathbb{T}$ such that*

$$y(t_n) \leq \omega(t_n) \tag{2.2}$$

holds for every $n \in \mathbb{Z}_{n_0-k}^{\infty}$. In particular, a solution $y_{(n_0,\phi)}$ of (1.1) *with $\phi \in \mathcal{C}$, defined by*

$$\phi(t_n) := \omega(t_n), \quad n \in \mathbb{Z}_{n_0-k}^{n_0}, \tag{2.3}$$

is such a solution.

Proof. Let $\omega(t_n)$ be a solution of (2.1) defined on $\mathbb{T}$. We will show that the solution $y(t_n) := y_{(n_0,\phi)}(t_n)$ of (1.1) with ϕ defined by (2.3) satisfies (2.2), that is,

$$y_{(n_0,\phi)}(t_n) \leq \omega(t_n) \tag{2.4}$$

for every $n \in \mathbb{Z}_{n_0-k}^{\infty}$. Let $W : \mathbb{T} \to \mathbb{R}$ be defined by

$$W(t_n) := \omega(t_n) - y(t_n). \tag{2.5}$$

Then $W(t_n) = 0$ if $n \in \mathbb{Z}_{n_0-k}^{n_0}$ and, in addition, W is a solution of (2.1) on $\mathbb{T}$. Lemma 2.2 implies that W is nondecreasing. Consequently,

$$W(t_n) = \omega(t_n) - y(t_n) \geq W(t_{n_0}) = \omega(t_{n_0}) - y(t_{n_0}) = 0, \tag{2.6}$$

and $y(t_n) \leq \omega(t_n)$ for all $n \geq n_0$. □

2.2. A Solution of Inequality (2.1)

Now we will construct a solution of (2.1). The result obtained will help us obtain sufficient conditions for the existence of an increasing and asymptotically convergent solution of (1.1) (see Theorem 4.1 below).

Lemma 2.4. *Let there exists a function $\varepsilon : \mathbb{T} \to \mathbb{R}^+$ such that*

$$\varepsilon(t_{n+1}) \geq \sum_{i=n-k+1}^{n-j} \beta(t_{i-1})\varepsilon(t_i) \tag{2.7}$$

for every $n \in \mathbb{Z}_{n_0}^{\infty}$. Then there exists a solution $\omega = \omega_\varepsilon$ of (2.1) *defined on $\mathbb{T}$ and having the form*

$$\omega_\varepsilon(t_n) := \sum_{i=n_0-k+1}^{n} \beta(t_{i-1})\varepsilon(t_i). \tag{2.8}$$

Proof. Assuming that ω_ε defined by (2.8) is a solution of (2.1) for $n \in \mathbb{Z}_{n_0}^{\infty}$, we will deduce the inequality for ε. We get

$$\begin{gathered}
\Delta\omega_\varepsilon(t_n) = \omega_\varepsilon(t_{n+1}) - \omega_\varepsilon(t_n) = \sum_{i=n_0-k+1}^{n+1} \beta(t_{i-1})\varepsilon(t_i) - \sum_{i=n_0-k+1}^{n} \beta(t_{i-1})\varepsilon(t_i) = \beta(t_n)\varepsilon(t_{n+1}), \\
\omega_\varepsilon(t_{n-j}) - \omega_\varepsilon(t_{n-k}) = \sum_{i=n_0-k+1}^{n-j} \beta(t_{i-1})\varepsilon(t_i) - \sum_{i=n_0-k+1}^{n-k} \beta(t_{i-1})\varepsilon(t_i) = \sum_{i=n-k+1}^{n-j} \beta(t_{i-1})\varepsilon(t_i).
\end{gathered} \tag{2.9}$$

We substitute ω_ε for ω in (2.1). Then, using (2.9), (2.1) turns into

$$\beta(t_n)\varepsilon(t_{n+1}) \geq \beta(t_n) \sum_{n-k+1}^{n-j} \beta(t_{i-1})\varepsilon(t_i). \tag{2.10}$$

Reducing the last inequality by $\beta(t_n)$, we obtain the desired inequality. □

2.3. Decomposition of a Function into the Difference of Two Increasing Functions

It is well-known that every absolutely continuous function is representable as the difference of two increasing absolutely continuous functions [20, page 318]. We will need a simple analogue of this result on discrete time scales under consideration.

Lemma 2.5. *Every function $\varphi \in \mathcal{C}$ can be decomposed into the difference of two increasing functions $\varphi_j \in \mathcal{C}$, $j = 1, 2$, that is,*

$$\varphi(t_n) = \varphi_1(t_n) - \varphi_2(t_n), \quad n \in \mathbb{Z}_{n_0-k}^{n_0}. \tag{2.11}$$

Proof. Let constants $M_n > 0$, $n \in \mathbb{Z}_{n_0-k}^{n_0}$ be such that

$$M_{n+1} > M_n + \max\{0, \varphi(t_n) - \varphi(t_{n+1})\} \tag{2.12}$$

is valid for each $n \in \mathbb{Z}_{n_0-k}^{n_0-1}$. We set

$$\begin{aligned} \varphi_1(t_n) &:= \varphi(t_n) + M_n, \quad n \in \mathbb{Z}_{n_0-k}^{n_0}, \\ \varphi_2(t_n) &:= M_n, \quad n \in \mathbb{Z}_{n_0-k}^{n_0}. \end{aligned} \tag{2.13}$$

It is obvious that (2.11) holds. Now we verify that both functions φ_j, $j = 1, 2$ are increasing. The first one should satisfy $\varphi_1(t_{n+1}) > \varphi_1(t_n)$ for $n \in \mathbb{Z}_{n_0-k}^{n_0-1}$, which means that

$$\varphi(t_{n+1}) + M_{n+1} > \varphi(t_n) + M_n \tag{2.14}$$

or

$$M_{n+1} > M_n + \varphi(t_n) - \varphi(t_{n+1}). \tag{2.15}$$

We conclude that the last inequality holds because, due to (2.12), we have

$$M_{n+1} > M_n + \max\{0, \varphi(t_n) - \varphi(t_{n+1})\} \geq M_n + \varphi(t_n) - \varphi(t_{n+1}). \tag{2.16}$$

The inequality $\varphi_2(t_{n+1}) > \varphi_2(t_n)$ obviously holds for every $n \in \mathbb{Z}_{n_0-k}^{n_0-1}$ due to (2.12) as well. □

2.4. Auxiliary Asymptotic Decomposition

The following lemma can be proved easily by induction. The symbol $\mathcal{O}$ (capital "*O*") stands for the Landau order symbol.

Lemma 2.6. *For fixed $r, \sigma \in \mathbb{R} \setminus \{0\}$, the asymptotic representation*

$$(n - r)^\sigma = n^\sigma \left[1 - \frac{\sigma r}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \tag{2.17}$$

holds for $n \to \infty$.

3. Convergence of All Solutions

The main result of this part is the statement that the existence of an increasing and asymptotically convergent solution of (1.1) implies the asymptotical convergence of all solutions.

Theorem 3.1. *If* (1.1) *has an increasing and asymptotically convergent solution on $\mathbb{Z}_{n_0-k}^{\infty}$, then all the solutions of* (1.1) *defined on $\mathbb{Z}_{n_0-k}^{\infty}$ are asymptotically convergent.*

Proof. First we prove that every solution defined by a monotone initial function is convergent. We will assume that a monotone initial function $\varphi \in \mathcal{C}$ is given. For definiteness, let φ be increasing or nondecreasing (the case when it is decreasing or nonincreasing can be considered in much the same way). By Lemma 2.1, the solution $y_{(n_0,\varphi)}$ is monotone, that is, it is either increasing or nondecreasing. We prove that $y_{(n_0,\varphi)}$ is convergent.

Denote the assumed increasing and asymptotically convergent solution of (1.1) as $y = Y(t_n)$, $n \in \mathbb{Z}_{n_0-k}^{\infty}$. Without loss of generality, we assume that $y_{(n_0,\varphi)} \not\equiv Y$ on $\mathbb{Z}_{n_0-k}^{\infty}$ since, in the opposite case, we can choose another initial function. Similarly, without loss of generality, we can assume

$$\Delta Y(t_n) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1}. \tag{3.1}$$

Hence, there is a constant $\gamma > 0$ such that

$$\Delta Y(t_n) - \gamma \Delta y(t_n) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1} \tag{3.2}$$

or

$$\Delta\big(Y(t_n) - \gamma y(t_n)\big) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1}. \tag{3.3}$$

This implies that the function $Y(t_n) - \gamma y(t_n)$ is increasing on $\mathbb{Z}_{n_0-k}^{n_0-1}$, and Lemma 2.1 implies that $Y(t_n) - \gamma y(t_n)$ is increasing on $\mathbb{Z}_{n_0-k}^{\infty}$. Thus,

$$Y(t_n) - \gamma y(t_n) > Y(t_{n_0}) - \gamma y(t_{n_0}), \quad n \in \mathbb{Z}_{n_0}^{\infty} \tag{3.4}$$

or

$$y(t_n) < y(t_{n_0}) + \frac{1}{\gamma}\big(Y(t_n) - Y(t_{n_0})\big), \quad n \in \mathbb{Z}_{n_0}^{\infty} \tag{3.5}$$

and, consequently, $y(t_n)$ is a bounded function on $\mathbb{Z}_{n_0-k}^{\infty}$ because of the boundedness of $Y(t_n)$. Obviously, in such a case, $y(t_n)$ is asymptotically convergent and has a finite limit.

Summarizing the previous section, we state that every monotone solution is convergent. It remains to consider a class of all nonmonotone initial functions. For the behavior of a solution $y_{(n_0,\varphi)}$ generated by a nonmonotone initial function $\varphi \in \mathcal{C}$, there are two possibilities: $y_{(n_0,\varphi)}$ is either eventually monotone and, consequently, convergent, or $y_{(n_0,\varphi)}$ is eventually nonmonotone.

Now we use the statement of Lemma 2.5 that every discrete function $\varphi \in \mathcal{C}$ can be decomposed into the difference of two increasing discrete functions $\varphi_j \in \mathcal{C}$, $j = 1, 2$. In accordance with the previous part of the proof, every function $\varphi_j \in \mathcal{C}$, $j = 1, 2$ defines an increasing and asymptotically convergent solution $y_{(n_0,\varphi_j)}$. Now it is clear that the solution $y_{(n_0,\varphi)}$ is asymptotically convergent. □

From Theorem 3.1, it follows that a crucial property assuring the asymptotical convergence of all solutions of (1.1) is the existence of a strictly monotone and asymptotically convergent solution. In the next part, we will focus our attention on the relevant criteria. Now, in order to finish this section, we need an obvious statement concerning the asymptotic convergence. From Lemma 2.1 and Theorem 2.3, we immediately derive the following result.

Theorem 3.2. *Let ω be an increasing and bounded solution of* (2.1) *on* $\mathbb{T}$. *Then there exists an increasing and asymptotically convergent solution y of* (1.1) *on* $\mathbb{T}$.

Combining the statements of Theorems 2.3, 3.1, and 3.2, we get a series of equivalent statements.

Theorem 3.3. *The following three statements are equivalent.*

(a) *Equation* (1.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}^{\infty}_{n_0-k}$.

(b) *All solutions of* (1.1) *defined on* $\mathbb{Z}^{\infty}_{n_0-k}$ *are asymptotically convergent.*

(c) *Inequality* (2.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}^{\infty}_{n_0-k}$.

4. Increasing Convergent Solutions of (1.1)

This part deals with the problem of detecting the existence of asymptotically convergent increasing solutions. We provide sufficient conditions for the existence of such solutions of (1.1).

The important theorem below is a consequence of Lemma 2.1, Theorem 2.3, and Lemma 2.4.

Theorem 4.1. *Let there exists a function $\varepsilon : \mathbb{T} \to \mathbb{R}^+$ satisfying*

$$\sum_{i=n_0-k+1}^{\infty} \beta(t_{i-1})\varepsilon(t_i) < \infty,$$
$$\varepsilon(t_{n+1}) \geq \sum_{i=n-k+1}^{n-j} \beta(t_{i-1})\varepsilon(t_i) \tag{4.1}$$

for every $n \in \mathbb{Z}^{\infty}_{n_0}$. Then the initial function

$$\varphi(t_n) := \sum_{i=n_0-k+1}^{n} \beta(t_{i-1})\varepsilon(t_i), \quad n \in \mathbb{Z}^{n_0}_{n_0-k} \tag{4.2}$$

defines an increasing and asymptotically convergent solution $y_{(t_{n_0},\varphi)}(t_n)$ of (1.1) *on* $\mathbb{T}$ *satisfying*

$$y_{(t_{n_0},\varphi)}(t_n) \leq \sum_{i=n_0-k+1}^{n} \beta(t_{i-1})\varepsilon(t_i) \tag{4.3}$$

for every $n \in \mathbb{Z}^{\infty}_{n_0}$.

Although Theorem 4.1 itself can serve as a source of various concrete criteria, later we will apply its following modification which can be used easily. Namely, assuming that β in (1.1) can be estimated by a suitable function, we can deduce that (1.1) has an increasing asymptotically convergent solution. We consider such a case.

Theorem 4.2. *Let there exist functions* $\beta^* : \mathbb{T} \to \mathbb{R}^+$ *and* $\varepsilon : \mathbb{T} \to \mathbb{R}^+$ *such that the inequalities*

$$\beta(t_n) \le \beta^*(t_n), \tag{4.4}$$

$$\varepsilon(t_{n+1}) \ge \sum_{i=n-k+1}^{n-j} \beta^*(t_{i-1})\varepsilon(t_i) \tag{4.5}$$

hold for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$, *and moreover*

$$\sum_{i=n_0-k+1}^{\infty} \beta^*(t_{i-1})\varepsilon(t_i) < \infty. \tag{4.6}$$

Then there exists an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}$ *of* (1.1) *satisfying*

$$y(t_n) \le \sum_{i=n_0-k+1}^{n} \beta(t_{i-1})\varepsilon(t_i) \tag{4.7}$$

for every $n \in \mathbb{Z}_{n_0}^{\infty}$. *Such a solution is defined, for example, by the initial function*

$$\varphi(t_n) := \sum_{i=n_0-k+1}^{n} \beta(t_{i-1})\varepsilon(t_i), \quad n \in \mathbb{Z}_{n_0-k}^{n_0}. \tag{4.8}$$

Proof. From (4.5) and (4.6), we get

$$\begin{gathered}\varepsilon(t_{n+1}) \ge \sum_{i=n-k+1}^{n-j} \beta^*(t_{i-1})\varepsilon(t_i) \ge \sum_{i=n-k+1}^{n-j} \beta(t_{i-1})\varepsilon(t_i),\\ \infty > \sum_{i=n_0-k+1}^{\infty} \beta^*(t_{i-1})\varepsilon(t_i) \ge \sum_{i=n_0-k+1}^{\infty} \beta(t_{i-1})\varepsilon(t_i).\end{gathered} \tag{4.9}$$

Then all assumptions of Theorem 4.1 are true. From its conclusion now follows the statement of Theorem 4.2. □

4.1. Some Special Criteria

It will be demonstrated by examples that, in many applications, the function β^* mentioned in Theorem 4.2 can have the form

$$\beta^*(t_n) = c - \gamma(t_n), \tag{4.10}$$

where c is a positive constant and $\gamma : \mathbb{T} \to \mathbb{R}^+$ is a suitable function such that $\gamma(t_n) < c$ (at least for all sufficiently large n) and

$$\lim_{n\to\infty} \gamma(t_n) = 0. \tag{4.11}$$

Below we carry on in this way and give sufficient conditions for the existence of increasing and asymptotically convergent solutions of (1.1) for general discrete time scale under consideration. For several special time scales, we derive such criteria in subsequent sections.

Theorem 4.3. *Let there exist constants $c > 0$, $p > 0$ and $\alpha > 0$ such that*

$$\beta(t_n) \le c - \frac{p}{t_n}\ , \tag{4.12}$$

$$\frac{1}{t_{n+1}^{\alpha}} \ge \sum_{i=n-k+1}^{n-j} \left[c - \frac{p}{t_{i-1}}\right] \frac{1}{t_i^{\alpha}} \tag{4.13}$$

hold for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$, and moreover

$$\sum_{i=n_0-k+1}^{\infty} \frac{1}{t_i^{\alpha}} < \infty. \tag{4.14}$$

Then there exists an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}^+$ of (1.1) satisfying

$$y(t_n) \le \sum_{i=n_0-k+1}^{n} \left[c - \frac{p}{t_{i-1}}\right] \frac{1}{t_i^{\alpha}} \tag{4.15}$$

for every $n \in \mathbb{Z}_{n_0}^{\infty}$. Such a solution is defined, for example, by the initial function

$$\varphi(t_n) := \sum_{i=n_0-k+1}^{n} \left[c - \frac{p}{t_{i-1}}\right] \frac{1}{t_i^{\alpha}}, \quad n \in \mathbb{Z}_{n_0-k}^{n_0}. \tag{4.16}$$

Proof. We will apply Theorem 4.2 with

$$\beta^*(t_n) := c - \frac{p}{t_n}, \qquad \varepsilon(t_n) := \frac{1}{t_n^{\alpha}}. \tag{4.17}$$

Inequality (4.5) turns into

$$\varepsilon(t_{n+1}) = \frac{1}{t_{n+1}^{\alpha}} \ge \sum_{i=n-k+1}^{n-j} \beta^*(t_{i-1})\varepsilon(t_i) = \sum_{i=n-k+1}^{n-j} \left[c - \frac{p}{t_{i-1}}\right] \frac{1}{t_i^{\alpha}} \tag{4.18}$$

and is true due to (4.13). Inequality (4.6) holds due to assumption (4.14) as well because $\lim_{n\to\infty} t_n = \infty$ and

$$\sum_{i=n_0-k+1}^{\infty} \beta^*(t_{i-1})\varepsilon(t_i) = \sum_{i=n_0-k+1}^{\infty} \left[c - \frac{p}{t_{i-1}}\right] \frac{1}{t_i^{\alpha}} < \infty. \tag{4.19}$$

Now, all assumptions of Theorem 4.2 are true, and its statement gives the statement of Theorem 4.3. □

Theorem 4.4. *Let there exist constants $c > 0$, $p > 0$ and $\alpha > 0$ such that the inequalities*

$$\beta(t_n) \leq c - \frac{p}{\ln t_n}, \tag{4.20}$$

$$\frac{1}{(\ln t_{n+1})^\alpha} \geq \sum_{i=n-k+1}^{n-j} \left[c - \frac{p}{\ln t_{i-1}}\right] \frac{1}{(\ln t_i)^\alpha} \tag{4.21}$$

hold for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$, and moreover

$$\sum_{i=n_0-k+1}^{\infty} \frac{1}{(\ln t_i)^\alpha} < \infty. \tag{4.22}$$

Then there exists an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}^+$ of (1.1) satisfying

$$y(t_n) \leq \sum_{i=n_0-k+1}^{n} \left[c - \frac{p}{\ln t_{i-1}}\right] \frac{1}{(\ln t_i)^\alpha} \tag{4.23}$$

for every $n \in \mathbb{Z}_{n_0}^{\infty}$. Such a solution is defined, for example, by the initial function

$$\varphi(t_n) := \sum_{i=n_0-k+1}^{n} \left[c - \frac{p}{\ln t_{i-1}}\right] \frac{1}{(\ln t_i)^\alpha}, \quad n \in \mathbb{Z}_{n_0-k}^{n_0}. \tag{4.24}$$

Proof. We will apply Theorem 4.2 with

$$\beta^*(t_n) := c - \frac{p}{\ln t_n}, \qquad \varepsilon(t_n) := \frac{1}{(\ln t_n)^\alpha}. \tag{4.25}$$

Inequality (4.5) turns into

$$\varepsilon(t_{n+1}) = \frac{1}{(\ln t_{n+1})^\alpha} \geq \sum_{i=n-k+1}^{n-j} \beta^*(t_{i-1})\varepsilon(t_i) = \sum_{i=n-k+1}^{n-j} \left[c - \frac{p}{\ln t_{i-1}}\right] \frac{1}{(\ln t_i)^\alpha} \tag{4.26}$$

and is true due to (4.21). Inequality (4.6) holds due to assumption (4.22) as well because $\lim_{n\to\infty} t_n = \infty$ and

$$\sum_{i=n_0-k+1}^{\infty} \beta^*(t_{i-1})\varepsilon(t_i) = \sum_{i=n_0-k+1}^{\infty} \left[c - \frac{p}{\ln t_{i-1}}\right] \frac{1}{(\ln t_i)^\alpha} < \infty. \tag{4.27}$$

Now, all assumptions of Theorem 4.2 are true, and its statement gives the statement of Theorem 4.4. $\square$

4.2. Time Scale $T(t) := \{n(1+\delta(n))\}$

Now, using Theorem 4.3, we derive sufficient conditions for the existence of an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}^+$ of (1.1) in the case when the time scale is defined as $\mathbb{T} = T(t) = \{t_n\}$, $t_n := n(1+\delta(n))$, where $\delta : \mathbb{T} \to \mathbb{R}$, $|\delta(n)| \le \delta^*$, $\delta^* \in (0,1)$, $n \in \mathbb{Z}_{n_0-k}^{\infty}$, and

$$\delta(n) = \mathcal{O}\left(\frac{1}{n^2}\right). \tag{4.28}$$

Theorem 4.5. *Let* (4.12) *be true for*

$$c := \frac{1}{k-j}, \qquad p := \frac{p^*(k+j+1)}{2(k-j)}, \tag{4.29}$$

where $p^* > 1$, *that is,*

$$\beta(t_n) \le \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)t_n} \tag{4.30}$$

holds for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$. *Let, moreover,* $\alpha \in (1, p^*)$. *Then there exists an increasing and asymptotically convergent solution* $y : \mathbb{T} \to \mathbb{R}^+$ *of* (1.1) *satisfying* (4.15) *for* $n \in \mathbb{Z}_{n_0}^{\infty}$. *Such a solution is defined, for example, by the initial function* (4.16).

Proof. We use Theorem 4.3 and assume (without loss of generality) that n_0 is sufficiently large for the asymptotic computations performed below to be correct. Let us verify that (4.13) holds. For the right-hand side $\mathcal{R}(t_n)$ of (4.13), we have

$$\begin{aligned}
\mathcal{R}(t_n) &= \sum_{i=n-k+1}^{n-j}\left[\frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)t_{i-1}}\right]\frac{1}{t_i^{\alpha}} \\
&= \frac{1}{k-j}\sum_{i=n-k+1}^{n-j}\frac{1}{t_i^{\alpha}} - \frac{p^*(k+j+1)}{2(k-j)}\sum_{i=n-k+1}^{n-j}\frac{1}{t_{i-1}t_i^{\alpha}} \\
&= \frac{1}{k-j}\sum_{i=n-k+1}^{n-j}\frac{1}{i^{\alpha}(1+\delta(i))^{\alpha}} \\
&\quad - \frac{p^*(k+j+1)}{2(k-j)}\sum_{i=n-k+1}^{n-j}\frac{1}{(i-1)(1+\delta(i-1))i^{\alpha}(1+\delta(i))^{\alpha}}.
\end{aligned} \tag{4.31}$$

Since $i \in \{n-k+1, n-k+2, \dots, n-j\}$ and $n \to \infty$, we can asymptotically decompose $\mathcal{R}(t_n)$ as $n \to \infty$ using decomposition formula (2.17) in Lemma 2.6. Applying this formula to the term $i^{-\alpha}$ in the first sum with $\sigma = -\alpha$ and with $r = n - i$, we get

$$\frac{1}{i^{\alpha}} = \frac{1}{(n-(n-i))^{\alpha}} = \frac{1}{n^{\alpha}}\left[1 + \frac{\alpha(n-i)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right]. \tag{4.32}$$

In addition to this, we have

$$\frac{1}{(1+\delta(i))^{\alpha}} = 1+\mathcal{O}\left(\frac{1}{i^2}\right) = 1+\mathcal{O}\left(\frac{1}{n^2}\right). \tag{4.33}$$

To estimate the second sum, we need only the first terms of the asymptotic decomposition and the order of accuracy, which can be computed easily without using Lemma 2.6. We also take into account that

$$\frac{1}{i-1} = \frac{1}{n-(n-i+1)} = \frac{1}{n}\cdot\frac{1}{1+(n-i+1)/n} = \frac{1}{n}\cdot\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \tag{4.34}$$

$$\frac{1}{1+\delta(i-1)} = 1+\mathcal{O}\left(\frac{1}{(i-1)^2}\right) = 1+\mathcal{O}\left(\frac{1}{n^2}\right). \tag{4.35}$$

Then we get

$$\begin{aligned}
\mathcal{R}(t_n) &= \frac{1}{(k-j)n^{\alpha}}\left[1+\mathcal{O}\left(\frac{1}{n^2}\right)\right]\sum_{i=n-k+1}^{n-j}\left[1+\frac{\alpha(n-i)}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&\quad - \frac{p^*(k+j+1)}{2(k-j)n^{\alpha+1}}\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right]\sum_{i=n-k+1}^{n-j}\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{(k-j)n^{\alpha}}\left[1+\frac{\alpha(k-1)}{n}+1+\frac{\alpha(k-2)}{n}+\cdots+1+\frac{\alpha j}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&\quad - \frac{p^*(k+j+1)}{2(k-j)n^{\alpha+1}}\left[1+1+\cdots+1+\mathcal{O}\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{(k-j)n^{\alpha+1}}\left[(k-j)n+\alpha(k-1)+\alpha(k-2)+\cdots+\alpha j+\mathcal{O}\left(\frac{1}{n}\right)\right] \\
&\quad - \frac{p^*(k+j+1)}{2(k-j)n^{\alpha+1}}\left[(k-j)+\mathcal{O}\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{n^{\alpha}}+\frac{\alpha}{(k-j)n^{\alpha+1}}\frac{(k+j-1)(k-j)}{2}-\frac{p^*(k+j+1)}{2(k-j)n^{\alpha+1}}(k-j)+\mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right),
\end{aligned} \tag{4.36}$$

and, finally,

$$\mathcal{R}(t_n) = \frac{1}{n^{\alpha}}+\frac{\alpha}{2n^{\alpha+1}}(k+j-1)-\frac{p^*(k+j+1)}{2n^{\alpha+1}}+\mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right). \tag{4.37}$$

A similar decomposition of the left-hand side $\mathcal{L}(t_n)$ in (4.13) leads to (we use the decomposition formula (2.17) in Lemma 2.6 with $\sigma = -\alpha$ and $r = -1$)

$$\mathcal{L}(t_n) = \frac{1}{t_{n+1}^{\alpha}} = \frac{1}{(n+1)^{\alpha}(1+\delta(n+1))^{\alpha}}$$
$$= \frac{1}{n^{\alpha}}\left[1 - \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)\right]\left[1 + O\left(\frac{1}{n^2}\right)\right] = \frac{1}{n^{\alpha}} - \frac{\alpha}{n^{\alpha+1}} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right). \tag{4.38}$$

Comparing $\mathcal{L}(t_n)$ and $\mathcal{R}(t_n)$, we see that, for $\mathcal{L}(t_n) \geq \mathcal{R}(t_n)$, it is necessary to match the coefficients of the terms $n^{-\alpha-1}$ because the coefficients of the terms $n^{-\alpha}$ are equal. It means that we need

$$-\alpha > \frac{1}{2}\,\alpha(k+j-1) - \frac{1}{2}\,p^*(k+j+1). \tag{4.39}$$

Simplifying this inequality, we get

$$\frac{1}{2}\,p^*(k+j+1) > \alpha + \frac{1}{2}\,\alpha(k+j-1)\ , \tag{4.40}$$

and, finally, $p^* > \alpha$. This inequality is assumed, and therefore (4.13) that holds n_0 is sufficiently large.

It remains to prove that (4.14) holds for $\alpha > 1$. But it is a well-known fact that the series

$$\sum_{i=n_0-k+1}^{\infty} \frac{1}{t_i^{\alpha}} = \sum_{i=n_0-k+1}^{\infty} \frac{1}{i^{\alpha}(1+\delta(i))^{\alpha}} \tag{4.41}$$

is convergent for $\alpha > 1$.

Thus, all assumptions of Theorem 4.3 are fulfilled and, from the conclusions, we deduce that all conclusions of Theorem 4.5 hold. □

4.3. Time Scale $T(t) := \{n\}$

The time scale $\mathbb{T} = T(t) = \{t_n\}, t_n := n$, where $n \in \mathbb{Z}_{n_0-k}^{\infty}$ is a particular case of the previous time scale defined in Section 4.2 if $\delta(n) = 0$ for every $n \in \mathbb{Z}_{n_0-k}^{\infty}$. Then (1.1) turns into

$$\Delta y(n) = \beta(n)\left[y(n-j) - y(n-k)\right] \tag{4.42}$$

and (4.30), which is crucial for the existence of an increasing and asymptotically convergent solution, takes the form

$$\beta(n) \leq \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n}, \quad n \in \mathbb{Z}_{n_0-k}^{\infty} \tag{4.43}$$

with a $p^* > 1$. Equation (4.42) has recently been considered in [12] and (4.43) coincides with (3.4) in [12, Theorem 3.3]. Thus, Theorem 4.5 can be viewed as a generalization of Theorem 3.3 in [12]. Moreover, using the following example, we will demonstrate that (4.43) is, in a sense, the best one.

Example 4.6. Consider (4.42), where

$$\beta(n) := \frac{1}{(n+1)\sum_{i=n-k+1}^{n-j} 1/i}. \tag{4.44}$$

It is easy to verify that (4.42) has a solution $y(n) = \sum_{i=1}^{n} 1/i$, which is the nth partial sum of harmonic series and, therefore, is divergent as $n \to \infty$. Now we asymptotically compare the function β with the right-hand side of (4.43). First we develop an asymptotic decomposition of β when $n \to \infty$. We get

$$\begin{aligned}
\beta(n) &= \frac{1}{(n+1)\sum_{i=n-k+1}^{n-j} 1/i} = \frac{1}{1+1/n} \cdot \frac{1}{\sum_{i=1}^{k-j}(1/(1+((i-k)/n)))} \\
&= \frac{1}{1+1/n} \cdot \frac{1}{\sum_{i=1}^{k-j}[1-(i-k)/n+\mathcal{O}(1/n^2)]} \\
&= \left[1-\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \cdot \frac{1}{k-j} \cdot \frac{1}{1-\sum_{i=1}^{k-j}[(i-k)/(k-j)n+\mathcal{O}(1/n^2)]} \\
&= \frac{1}{k-j} \cdot \left[1-\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \cdot \left[1+\sum_{i=1}^{k-j}\frac{i-k}{(k-j)n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&= \frac{1}{k-j} \cdot \left[1-\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \cdot \left[1-\frac{k}{n}+\frac{k-j+1}{2n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&= \frac{1}{k-j} \cdot \left[1-\frac{k}{n}+\frac{k-j+1}{2n}-\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&= \frac{1}{k-j} - \frac{k+j+1}{2(k-j)n} + \mathcal{O}\left(\frac{1}{n^2}\right).
\end{aligned} \tag{4.45}$$

Now, (4.43) requires that

$$\beta(n) = \frac{1}{k-j} - \frac{k+j+1}{2(k-j)n} + \mathcal{O}\left(\frac{1}{n^2}\right) \le \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n}. \tag{4.46}$$

The last will hold if

$$-\frac{k+j+1}{2(k-j)n} < -\frac{p^*(k+j+1)}{2(k-j)n}, \tag{4.47}$$

that is, if $p^* < 1$. This inequality is the opposite to $p^* > 1$ guaranteeing the existence of an increasing and asymptotically convergent solution. The example also shows that the criterion (4.43) is sharp in a sense. We end this part with a remark that Example 4.6 corrects the Example 4.4 in [12], where the case $j = 0$ and $k = 1$ was considered.

4.4. *Time Scale* $T(t) := \{q^n\}$, $q > 1$

We will focus our attention on the sufficient conditions for the existence of an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}^+$ of (1.1) if the time scale is defined as $\mathbb{T} = T(t) = \{t_n\}$, $t_n := q^n$, where $n \in \mathbb{Z}_{n_0-k}^{\infty}$ and $q > 1$. We will apply Theorem 4.4.

Theorem 4.7. *Let* (4.20) *hold for*

$$c := \frac{1}{k-j}, \qquad p := \frac{p^*(k+j+1)\ln q}{2(k-j)}, \tag{4.48}$$

where $p^* > 1$, that is, the inequality

$$\beta(t_n) \le \frac{1}{k-j} - \frac{p^*(k+j+1)\ln q}{2(k-j)\ln t_n} = \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n} \tag{4.49}$$

holds for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$. Let, moreover, $\alpha \in (1, p^*)$. Then there exists an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}^+$ of (1.1) *satisfying* (4.23) *for $n \in \mathbb{Z}_{n_0}^{\infty}$. Such a solution is defined, for example, by the initial function* (4.24).

Proof. We use Theorem 4.4 and assume (without loss of generality) that n_0 is sufficiently large for the asymptotic computations performed below to be correct. Let us verify (4.21). For the right-hand side $\mathcal{R}(t_n)$ of (4.21), we have

$$\begin{aligned}
\mathcal{R}(t_n) &= \sum_{i=n-k+1}^{n-j} \left[\frac{1}{k-j} - \frac{p^*(k+j+1)\ln q}{2(k-j)\ln t_{i-1}}\right] \frac{1}{(\ln t_i)^{\alpha}} \\
&= \frac{1}{k-j} \sum_{i=n-k+1}^{n-j} \frac{1}{(\ln t_i)^{\alpha}} - \frac{p^*(k+j+1)\ln q}{2(k-j)} \sum_{i=n-k+1}^{n-j} \frac{1}{(\ln t_{i-1})(\ln t_i)^{\alpha}} \\
&= \frac{1}{k-j} \sum_{i=n-k+1}^{n-j} \frac{1}{i^{\alpha}(\ln q)^{\alpha}} - \frac{p^*(k+j+1)\ln q}{2(k-j)} \sum_{i=n-k+1}^{n-j} \frac{1}{(i-1)i^{\alpha}(\ln q)^{\alpha+1}} \\
&= \frac{1}{(k-j)(\ln q)^{\alpha}} \sum_{i=n-k+1}^{n-j} \frac{1}{i^{\alpha}} - \frac{p^*(k+j+1)}{2(k-j)(\ln q)^{\alpha}} \sum_{i=n-k+1}^{n-j} \frac{1}{(i-1)i^{\alpha}} \\
&= [\text{we apply decompositions (4.32) and (4.34)}] \\
&= \frac{1}{(k-j)(\ln q)^{\alpha} n^{\alpha}} \sum_{i=n-k+1}^{n-j} \left[1 + \frac{\alpha(n-i)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&\quad - \frac{p^*(k+j+1)}{2(k-j)(\ln q)^{\alpha} n^{\alpha+1}} \sum_{i=n-k+1}^{n-j} \left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right].
\end{aligned} \tag{4.50}$$

Finally, applying some of the computations from the proof of Theorem 4.5, we get

$$\mathcal{R}(t_n) = \frac{1}{(\ln q)^{\alpha} n^{\alpha}} + \frac{\alpha}{2(\ln q)^{\alpha} n^{\alpha+1}}(k+j-1) - \frac{p^*(k+j+1)}{2(\ln q)^{\alpha} n^{\alpha+1}} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right). \tag{4.51}$$

and, for the left-hand side $\mathcal{L}(t_n)$ of (4.20),

$$\mathcal{L}(t_n) = \frac{1}{(\ln q)^{\alpha}(n+1)^{\alpha}} = \frac{1}{(\ln q)^{\alpha} n^{\alpha}} - \frac{\alpha}{(\ln q)^{\alpha} n^{\alpha+1}} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right). \tag{4.52}$$

Comparing $\mathcal{L}(t_n)$ and $\mathcal{R}(t_n)$, we see that, for $\mathcal{L}(t_n) \geq \mathcal{R}(t_n)$,

$$-\frac{\alpha}{(\ln q)^{\alpha}} > \frac{\alpha(k+j-1)}{2(\ln q)^{\alpha}} - \frac{p^*(k+j+1)}{2(\ln q)^{\alpha}} \tag{4.53}$$

is sufficient. Simplifying it, we get

$$p^*(k+j+1) > \alpha(k+j+1), \tag{4.54}$$

and, finally, $p^* > \alpha$. This inequality is assumed, and therefore (4.21) is valid if n_0 is sufficiently large.

It remains to prove that (4.22) holds for $\alpha > 1$. But it is a well-known fact that the series

$$\sum_{i=n_0-k+1}^{\infty} \frac{1}{(\ln t_i)^{\alpha}} = \sum_{i=n_0-k+1}^{\infty} \frac{1}{i^{\alpha}(\ln q)^{\alpha}} \tag{4.55}$$

is convergent for $\alpha > 1$.

Thus, all assumptions of Theorem 4.4 are true, and, from its conclusions, we deduce that all conclusions of Theorem 4.7 are true. □

Example 4.8. Consider (1.1), where

$$\beta(t_n) := \frac{1}{(n+1)\sum_{i=n-k+1}^{n-j}(\ln q)/(\ln t_i)} = \frac{1}{(n+1)\sum_{i=n-k+1}^{n-j} 1/i}. \tag{4.56}$$

Then it is easy to verify that (1.1) has a solution

$$y(t_n) = \sum_{i=1}^{n} \frac{\ln q}{\ln t_i} = \sum_{i=1}^{n} \frac{1}{i}, \tag{4.57}$$

which is the nth partial sum of harmonic series and, as such, is divergent as $n \to \infty$. Now we asymptotically compare the function β with the right-hand side of (4.49). Proceeding as in Example 4.6, we get

$$\beta(t_n) = \frac{1}{k-j} - \frac{k+j+1}{2(k-j)n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{4.58}$$

Inequality (4.49) is valid if

$$\beta(t_n) = \frac{1}{k-j} - \frac{k+j+1}{2(k-j)n} + \mathcal{O}\left(\frac{1}{n^2}\right) \le \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n}, \tag{4.59}$$

that is if $p^* < 1$. This inequality is the opposite to $p^* > 1$ guaranteeing the existence of an increasing and asymptotically convergent solution. Thus, the example also shows that criterion (4.49) is sharp in a sense.

4.5. A General Criterion for the Existence of an Increasing and Asymptotically Convergent Solution

Analysing two criteria for the existence of an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}^+$ of (1.1) expressed by (4.12) and (4.20), that is, by inequalities

$$\begin{aligned} \beta(t_n) &\le c - \frac{p}{t_n}, \\ \beta(t_n) &\le c - \frac{p}{\ln t_n} \end{aligned} \tag{4.60}$$

with suitable constants c and p, we can state the following. The first criterion (4.12) can successfully be used, for example, for the time scale $T(t) = \{t_n\}$, where $t_n = n$. In this case, as stated in Theorem 4.5, (4.30), that is,

$$\beta(t_n) \le \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)t_n} = \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n} \tag{4.61}$$

is assumed with a $p^* > 1$.

The second criterion (4.20) can successfully be used, for example, for the time scale $T(t) = \{t_n\}$ where $t_n = q^n$ and $q > 1$. Then, as stated in Theorem 4.7, (4.49), that is,

$$\beta(t_n) \le \frac{1}{k-j} - \frac{p^*(k+j+1)\ln q}{2(k-j)\ln t_n} = \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n} \tag{4.62}$$

is assumed with a $p^* > 1$. Comparing (4.61) and (4.62), we see that, although their left-hand sides are different due to different meaning of t_n in every case, their right-hand sides are identical.

The following result gives a criterion for every discrete time scale $T(t) = \{t_n\}$ with properties described in introduction.

Theorem 4.9. *Let*

$$\beta(t_n) \le \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n} \tag{4.63}$$

holds for all $n \in \mathbb{Z}^{\infty}_{n_0-k}$ and for a fixed $p^* > 1$. Let, moreover, $\alpha \in (1, p^*)$. Then there exists an increasing and asymptotically convergent solution $y : \mathbb{T} \to \mathbb{R}^+$ of (1.1) satisfying

$$y(t_n) \le \sum_{i=n_0-k+1}^{n} \left[\frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)(i-1)}\right]\frac{1}{i^{\alpha}} \tag{4.64}$$

for every $n \in \mathbb{Z}_{n_0}^{\infty}$*. Such a solution is defined, for example, by the initial function*

$$\varphi(t_n) := \sum_{i=n_0-k+1}^{n} \left[\frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)(i-1)}\right]\frac{1}{i^\alpha}, \quad n \in \mathbb{Z}_{n_0-k}^{n_0}. \tag{4.65}$$

Proof. We will apply Theorem 4.2 with

$$\beta^*(t_n) := \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n}, \qquad \varepsilon(t_n) := \frac{1}{n^\alpha}. \tag{4.66}$$

Inequality (4.5) turns into

$$\varepsilon(t_{n+1}) = \frac{1}{(n+1)^\alpha} \geq \sum_{i=n-k+1}^{n-j} \beta^*(t_{i-1})\varepsilon(t_i) = \sum_{i=n-k+1}^{n-j}\left[\frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)(i-1)}\right]\frac{1}{i^\alpha}. \tag{4.67}$$

Asymptotic decompositions of the left-hand and right-hand sides were used in the proof of Theorem 4.5 (if $\delta(n) = 0$, i.e., $t_n = n$ for every $n \in \mathbb{Z}_{n_0-k}^{\infty}$) and a similar decomposition was used in the proof of Theorem 4.7. Therefore, we will not repeat it. We will only state that the above inequality holds for $p^* > \alpha$. (4.6) holds as well because the series

$$\sum_{i=n_0-k+1}^{\infty}\left[\frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)(i-1)}\right]\frac{1}{i^\alpha} \tag{4.68}$$

is obviously convergent. □

Remark 4.10. Although Theorem 4.9 is a general result, it has a disadvantage in applications because of its implicit character. Unlike (4.61) and (4.62), where the left-hand and middle parts are explicitly expressed in terms of t_n, the right-hand side of the crucial inequality (4.63) cannot, in a general situation of arbitrary time scale $\{t_n\}$, be explicitly expressed using only the t_n terms. This is only possible if, for a given time scale, a function f is explicitly known such that $f(t_n) = n$. Then, (4.63) can be written in the form

$$\beta(t_n) \leq \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)f(t_n)} = \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n}. \tag{4.69}$$

Remark 4.11. On the other hand, in a sense, Theorem 4.9 gives the best possible result. Indeed, (1.1) with

$$\beta(t_n) := \frac{1}{(n+1)\sum_{i=n-k+1}^{n-j} 1/i} \tag{4.70}$$

has an increasing asymptotically divergent solution $y(t_n) = \sum_{i=1}^{n} 1/i$. An asymptotic decomposition of the right-hand side of (4.70) was performed in Example 4.6 and an increasing and asymptotically convergent solution exists if (4.63), that is,

$$\beta(t_n) = \frac{1}{(n+1)\sum_{i=n-k+1}^{n-j} 1/i} \leq \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n} \tag{4.71}$$

holds, or if

$$\beta(t_n) = \frac{1}{k-j} - \frac{k+j+1}{2(k-j)n} + \mathcal{O}\left(\frac{1}{n^2}\right) \le \frac{1}{k-j} - \frac{p^*(k+j+1)}{2(k-j)n}. \tag{4.72}$$

The last holds for $p^* < 1$. This inequality is the opposite to $p^* > 1$ guaranteeing the existence of an increasing and asymptotically convergent solution. Thus, the example shows that our general criterion is sharp in a sense.

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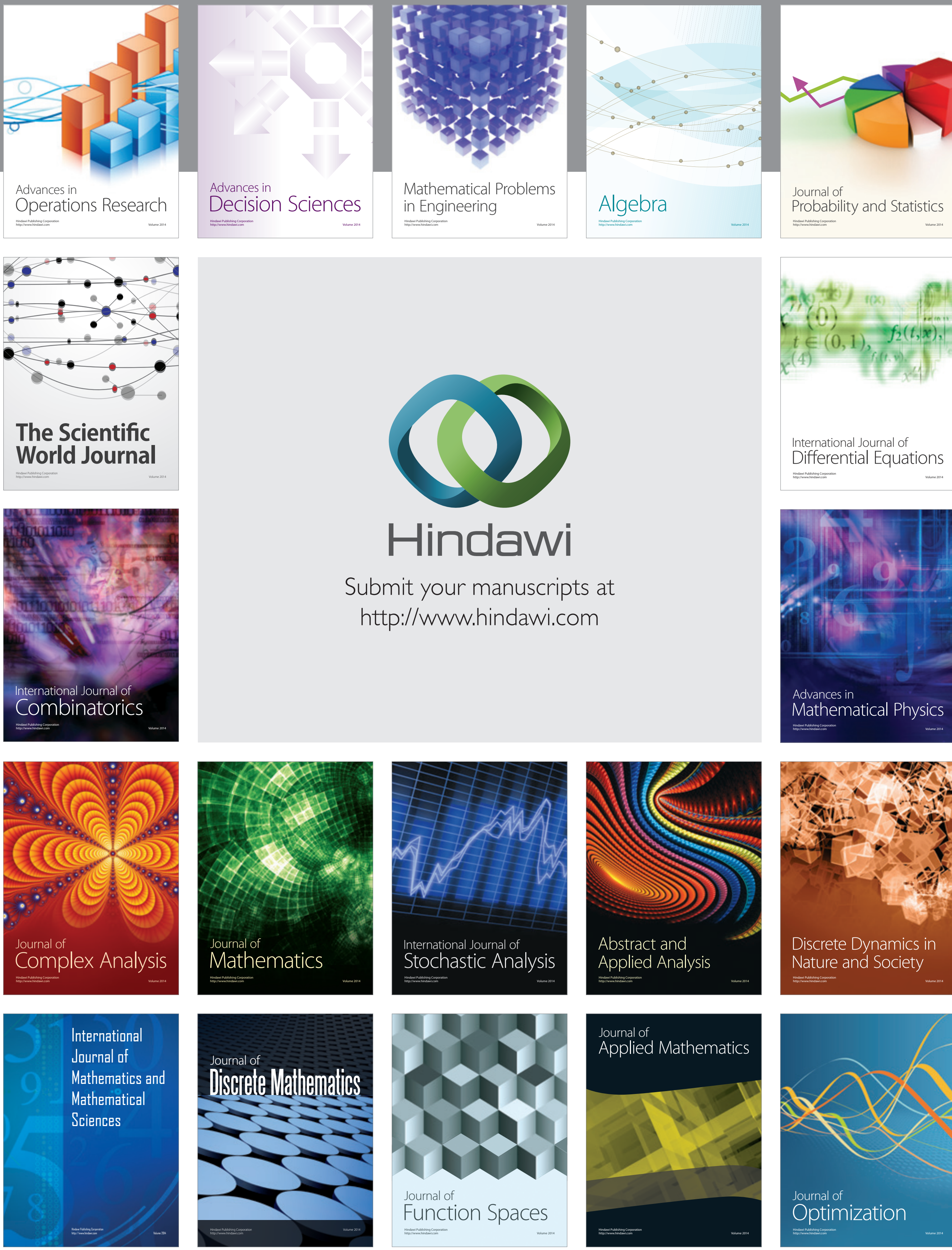
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