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Research Article

On Existence, Uniform Decay Rates, and Blow-Up for Solutions of a Nonlinear Wave Equation with Dissipative and Source

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This paper studies the blow-up and existence, and asymptotic behaviors of the solution of a nonlinear hyperbolic equation with dissipative and source terms. By using Galerkin procedure and the perturbed energy method, the local and global existence of solution is established. In addition, by the concave method, the blow-up of solutions can be obtained.

1. Introduction

In this paper, we investigate the following nonlinear wave equation:

$$\begin{gathered}
|u_t|^\rho u_{tt} + \Delta^2 u + |u_t|^m u_t + \gamma \Delta^2 u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) + |u|^{p-1}u, \quad (x,t) \in \Omega \times (0,T), \\
u = \frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, Δ is a Laplace operator, and $\partial u/\partial n|_{\partial\Omega}$ indicates derivative of u in outward normal direction of $\partial\Omega$. In addition to, if $n > 3$, $1 < p < (n+2)/(n-2)$ and if $n = 1, 2$, then $p > 1$. $\gamma \geq 0$ is a constant, $\sigma_i(s)$ $(i = 1, \ldots, N)$ are given in (A1) later.

In 1968, Greenberg et al. [1] first suggested and studied the following equation:

$$u_{tt} - u_{xxt} = \sigma(u_x)_x. \tag{1.2}$$

Under the condition $\sigma'(s) > 0$ and higher smooth conditions on $\sigma(s)$ and initial data, they claimed the global existence of classical solutions for the initial boundary value problem of (1.2).

The multidimensional form of the following:

$$u_{tt} - \Delta u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, t, u_{x_i}) = f(x, t) \tag{1.3}$$

was first studied by Clement [2, 3]. Exploiting the monotone operator method, he obtained the global existence of weak solutions for the initial boundary value problem of (1.2).

Our model comes from [4]. In [4], Yang has studied the global existence, asymptotic behavior, and blow-up of solutions for a nonlinear wave equations with dissipative term:

$$u_{tt} + \Delta^2 u + \lambda u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}), \tag{1.4}$$

with the same initial and boundary conditions as that of (1.1). In our model, we add damping and source terms which enhance the difficulty of proving the existence and decay of solution of (1.2).

More related studies of the damped hyperbolic equation with dissipative term or damping term can be found in papers [5–15].

The paper is organized as follows. In Section 2, we present some notations, and results needed later and main results. Section 3 contains the statement and the proofs of the decay of solutions. Section 4 gives the statement and the proofs of the blow-up of solutions.

2. Preliminaries

We first introduce the following abbreviations:

$$\begin{gathered} Q_T = \Omega \times (0, T), \qquad L_p = L_p(\Omega), \qquad W^{m,p} = W^{m,p}(\Omega), \\ H^m = W^{m,2}, \qquad H_0^m = W_0^{m,2}, \qquad \|\cdot\|_2 = \|\cdot\|_{L_2}, \qquad \|\cdot\|_{1,2} = \|\cdot\|_{H_0^1}. \end{gathered} \tag{2.1}$$

Let $(\cdot, \cdot)$ denote the L_2-inner product. We denote the dual of $W_0^{1,p}$ by $W^{-1,p'}$, with $p' = p/(p-1)$, and $\|\cdot\|_{-1,p'} = \|\cdot\|_{W^{-1,p'}}$. Now we make the following assumptions:

(A1) $\sigma_i \in C^1(R), \sigma_i(s)s > 0$ and

$$C_1|s|^r \leq |\sigma_i(s)| \leq C_2|s|^r, \tag{2.2}$$

for some $r \geq 1$, if $n = 1,2$, else if $n \geq 3$, then $r \leq n/(n-2)$, so that $\|\nabla u\|_{2r} \leq B_1\|\Delta u\|_2$, $i = 1,2,\ldots,N$, where B_1 is the optimal embedding constant.

(A2) The initial data

$$u_0 \in H_0^2, \qquad u_1 \in H_0^1. \tag{2.3}$$

(A3) $m < \rho + 1$; If $n > 3$, then $0 < \rho < m < 4/(n-2)$ and if $n = 1,2$, then $0 < \rho < m$.

(A4) If $n > 3$, $1 < p < (n+2)/(n-2)$ and if $n = 1,2$, then $p > 1$. Without loss of generality, we assume that $r < p$.

And

(B1) $\sigma_i \in C^1(R)$, $\sigma_i(s)s < 0$, and

$$C_1|s|^r \leq |\sigma_i(s)| \leq C_2|s|^r, \tag{2.4}$$

for some $r \geq 1$, if $n = 1,2$, else if $n \geq 3$, then $r \leq n/(n-2)$, so that $\|\nabla u\|_{2r} \leq B_1\|\Delta u\|_2$, $i = 1,2,\ldots,N$. where B_1 is the optimal embedding constant, and $C_2 < (r+1)/(p+1)$.

(B2) If $n > 3$, then $m < r-1 < p-1 < 4/(n-2)$ and if $n = 1,2$, then $0 < m < r-1 < p-1$.

(B3) $\rho + 1 < r$, and if $n > 3$, then $\rho < 4/(n-2)$ and if $n = 1,2$, then $0 < \rho$.

Throughout this paper, we use the embedding $H_0^2(\Omega) \hookrightarrow L^q(\Omega)$ which implies $\|u\|_q \leq B_2\|\Delta u\|_2$ when

$$2 \leq q \leq \frac{2n}{n-2} \quad \text{if } n \geq 3, \qquad q \geq 1 \quad \text{if } n = 1,2, \tag{2.5}$$

where B_2 is an optimal embedding constant. We introduce the following functionals:

$$A_i(s) = \int_0^s \sigma_i(\eta)\,d\eta, \tag{2.6}$$

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\Delta u\|_2^2 + \sum_{i=1}^{N}\int_\Omega A_i(u_{x_i})\,dx - \frac{\|u\|_{p+1}^{p+1}}{p+1}, \tag{2.7}$$

$$J(t) = \frac{1}{2}\|\Delta u\|_2^2 - \frac{C_2}{r+1}\|\nabla u\|_{r+1}^{r+1} - \frac{\|u\|_{p+1}^{p+1}}{p+1}, \tag{2.8}$$

$$I(t) = \|\Delta u\|_2^2 - C_2\|\nabla u\|_{r+1}^{r+1} - \|u\|_{p+1}^{p+1}. \tag{2.9}$$

Because $r < p$, we have

$$J(t) = \frac{I(t)}{r+1} + \frac{r-1}{2(r+1)}\|\Delta u\|_2^2 + \frac{p-r}{(p+1)(r+1)}\|u\|_{p+1}^{p+1}. \tag{2.10}$$

Theorem 2.1. *Assume that (A1)–(A4) hold, then problem* (1.1) *has a unique solution* u *satisfying*

$$u \in L_\infty\left([0,T]; H_0^2\right) \cap W^{1,\infty}([0,T]; L_{\rho+2});$$
$$u_t \in L_2\left([0,T]; H_0^2\right) \cap L^\infty([0,T]; L_{\rho+2}), \tag{2.11}$$

where $T < 1$.

Theorem 2.2. *Assume that (A1)–(A4) hold,* u *is the local solution of the problem* (1.1)*. And*

$$C_0\left(\left(\frac{2(r+1)}{r-1}E(0)\right)^{(r-1)/2} + \left(\frac{2(r+1)}{r-1}E(0)\right)^{(p-1)/2}\right) < 1,$$
$$I(0) > 0, \tag{2.12}$$

where $C_0 = \max\{B_1C_2, B_2\}$*, then* u *is a global bounded solution, moreover,*

$$E(t) \leq \frac{M}{1+t}\left(1 + t^{1/(m+2)} + t^{(m-\rho)/(m+2)} + t^{1/2}\right), \quad \forall t \in [0,\infty), \tag{2.13}$$

$$\lim_{t\to\infty} \|\Delta u\|_2^2 = \lim_{t\to\infty} \|u_t\|_{\rho+2}^{\rho+2} = 0, \tag{2.14}$$

where $M > 0$ *is a constant.*

Theorem 2.3. *Assume that (A1)–(A4) and*

$$\frac{pB_2}{p+1}\left(\frac{2(r+1)}{r-1}\right)^{(p+1)/2}(E(0))^{(p-1)/2} < 1 \tag{2.15}$$

hold, u *is the local solution of the problem* (2.7)*. Consider* (2.12) *are satisfied, then*

$$E(t) \leq Ke^{-\kappa t}, \quad \forall t \in [0,\infty), \tag{2.16}$$

where $K, \kappa > 0$ *are constants.*

Remark 2.4. When $\gamma > 0$, we will use perturbed energy method, which is different to the method of the proof of Theorem 2.2, to prove Theorem 2.3.

Theorem 2.5. *Assume that (B1)–(B3), (A4) hold, and there exist some* $(u_0, u_1) \in H^2(\Omega) \times L^{\rho+2}(\Omega)$*, then the solution of the problem* (1.1) *blows-up at the limited time* $T^* > 0$.

3. Decay of Solutions

In this section, we prove Theorems 2.1–2.3. First, we give the following Lemma.

Lemma 3.1 (see [2]). *Let Ω be any bounded domain in $\mathbf{R}^N$, $\{\omega_k\}_{k=1}^{\infty}$ be an orthogonal basis in $L_2(\Omega)$. Then for any $\varepsilon > 0$, there exists a positive number N_ε such that*

$$\|u\| \leq \left(\sum_{k=1}^{N_\varepsilon}(u,\omega_k)^2\right)^{1/2} + \varepsilon\|u\|_{1,p}, \tag{3.1}$$

for all $u \in W_0^{1,p}$ $(2 \leq p < \infty)$.

Proof of Theorem 2.1. We look for approximate solutions $u^n(t)$ of problem (1.1) of the form

$$u^n(t) := \sum_{j=1}^{n} T_{jn}(t)\omega_j, \tag{3.2}$$

where $\{\omega_j\}_{j=1}^{n}$ is an orthogonal basis in H_0^2, and also in L_2, and the coefficients $\{T_{jn}\}_{j=1}^{n}$ satisfy $T_{jn}(t) = (u^n(t),\omega_j)$ with

$$\begin{gathered}\left(\left|u_t^n(t)\right|^\rho u_{tt}^n(t),\omega_j\right) + \left(\Delta^2 u^n(t),\omega_j\right) + \left(\left|u_t^n\right|^m u_t^n(t),\omega_j\right) + \gamma\left(\Delta^2 u_t^n(t),\omega_j\right) \\ = \sum_i\left(\frac{\partial}{\partial x_i}\sigma_i\left(u_{x_i}^n(t)\right),\omega_j\right) + \left(|u^n(t)|^{p-1}u^n(t),\omega_j\right), \quad t > 0,\ j = 1,\ldots,n, \\ u^n(0) = u_0^n, \qquad u_t^n(0) = u_1^n.\end{gathered} \tag{3.3}$$

Since C_0^∞ is dense in H_0^2 and L_2, we choose $u_0^n, u_1^n \in C_0^\infty$ such that

$$u_0^n \longrightarrow u_0 \quad \text{in } H_0^2, \quad u_1^n \longrightarrow u_1 \quad \text{in } H_0^1 \quad \text{as } n \longrightarrow \infty. \tag{3.4}$$

The above system of o.d.e. has a local solution $u^n(t)$ defined in some interval $[0,T_n)$. The following will prove that the T_n can be substituted by some $T > 0$.

Multiply (3.3) by $T_{jn}'(t)$ and summing up about j, we get

$$\begin{gathered}\frac{d}{dt}\left[\frac{1}{\rho+2}\left\|u_t^n\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\Delta u^n\|_2^2\right] + \left\|u_t^n\right\|_{m+2}^{m+2} + \gamma\left\|\Delta u_t^n\right\|_2^2 \\ = \sum_{i=1}^{N}\left(\sigma_i^n(u_{x_i}),\left(u_{x_i}^n\right)_t\right) + \left(|u^n|^{p-1}u^n,u_t^n\right).\end{gathered} \tag{3.5}$$

A simple integration of (3.5) over $(0,t)$ leads to

$$\begin{aligned}
&\frac{1}{\rho+2}\|u_t^n\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\Delta u^n\|_2^2+\int_0^t\left(\|u_s^n\|_{m+2}^{m+2}+\gamma\|u_s^n\|_2^2\right)ds\\
&\quad=C_3+\sum_{i=1}^{N}\int_0^t\left(\sigma_i^n(u_{x_i}),\left(u_{x_i}^n\right)_s\right)ds+\int_0^t\left(|u^n|^{p-1}u^n,u_s^n\right)ds,
\end{aligned}\tag{3.6}$$

where $C_3=1/(\rho+2)\|u_1^n\|_{\rho+2}^{\rho+2}+(1/2)\|\Delta u_0^n\|_2^2>0$.

We now estimate the last two terms at the right-hand side of (3.6). Using Hölder inequality, Young's inequality, and the embedding theorem, we know there exist $q,w>0$, satisfying that

$$\frac{1}{q}+\frac{1}{w}+\frac{n-2}{2n}=1,\tag{3.7}$$

where

$$\begin{aligned}
(r-1)q&=\frac{2n}{n-2},\qquad \frac{1}{w}=\frac{n+2}{2n}-\frac{(r-1)(n-2)}{2n}\geq\frac{n-2}{2n},\\
(p-1)q&=\frac{2n}{n-2},\qquad \frac{1}{w}=\frac{n+2}{2n}-\frac{(p-1)(n-2)}{2n}\geq\frac{n-2}{2n}.
\end{aligned}\tag{3.8}$$

Consider the following:

$$\begin{aligned}
\left|\sum_{i=1}^{N}\int_0^t\left(\sigma_i(u_{x_i}^n),\left(u_{x_i}^n\right)_s\right)ds\right|&\leq C_2\int_0^t\int_\Omega|\nabla u^n|^{r-1}|\nabla u^n||\nabla u_s^n|dx\,ds\\
&\leq C_2\int_0^t\|\nabla u^n\|_{(r-1)q}^{r-1}\|\nabla u^n\|_w\|\nabla u_s^n\|_{2n/(n-2)}ds\\
&\leq C_2C\int_0^t\|\Delta u^n\|_2^{r-1}\|\Delta u^n\|_2\|\Delta u_s^n\|_2ds\\
&= C_2C\int_0^t\|\Delta u^n\|_2^{r}\|\Delta u_s^n\|_2ds\\
&\leq C_2C\left(\frac{1}{2\epsilon}\int_0^t\|\Delta u^n\|_2^{2r}ds+\frac{\epsilon}{2}\int_0^t\|\Delta u_s^n\|_2^2ds\right)\\
&\leq C_2C\left(\frac{1}{2\epsilon}\left(\int_0^t\|\Delta u^n\|_2^{2p}ds\right)^{r/p}\right.\\
&\qquad\left.+\frac{1}{2\epsilon}\left(\int_0^t 1\,ds\right)^{(p-r)/p}+\frac{\epsilon}{2}\int_0^t\|\Delta u_s^n\|_2^2ds\right)
\end{aligned}$$

$$\leq C_2 C \frac{1}{2\epsilon}\frac{p}{r}\left(\int_0^t \|\Delta u^n\|_2^{2p} ds\right) + C_2 C \frac{1}{2\epsilon}\frac{p-r}{p} + \frac{\epsilon}{2} C_2 C \int_0^t \|\Delta u_s^n\|_2^2 ds, \quad \forall \epsilon > 0, \tag{3.9}$$

assuming that $t < 1$.

Similarly,

$$\begin{aligned}\int_0^t \left(|u^n|^{p-1}u^n, u_s^n\right) ds &\leq \int_0^t \|u^n\|_{(p-1)q}^{p-1}\|u^n\|_w \|u_s^n\|_{2n/(n-2)} ds \\ &\leq C \int_0^t \|\Delta u^n\|_2^{p-1}\|\Delta u^n\|_2 \|\Delta u_s^n\|_2 ds \\ &\leq C\frac{1}{2\epsilon}\int_0^t \|\Delta u^n\|_2^{2p} ds + \frac{\epsilon}{2} C \int_0^t \|\Delta u_s^n\|_2^2 ds, \quad \forall \epsilon > 0.\end{aligned} \tag{3.10}$$

Using (3.6)–(3.10), we have

$$\begin{aligned}&\int_0^t \|u_s^n\|_{m+2}^{m+2} ds + \gamma \int_0^t \|\Delta u_s^n\|_2^2 ds + \frac{1}{2}\|\Delta u^n\|_2^2 + \frac{1}{\rho+2}\left\|u_t^n\right\|_{\rho+2}^{\rho+2} \\ &\quad \leq C_4 + C(\varepsilon)\int_0^t \|\Delta u^n\|_2^{2p} ds + \frac{\epsilon}{2} C \int_0^t \|\Delta u_s^n\|_2^2.\end{aligned} \tag{3.11}$$

Choosing $\epsilon = (1/C)\gamma$ in (3.11), we have

$$\begin{aligned}&\int_0^t \|u_s^n\|_{m+2}^{m+2} ds + \frac{1}{2}\|\Delta u^n\|_2^2 + \frac{1}{\rho+2}\left\|u_t^n\right\|_{\rho+2}^{\rho+2} + \frac{\gamma}{2}\int_0^t \|\Delta u_s^n\|_2^2 ds \\ &\quad \leq C_4 + C\int_0^t \|\Delta u^n\|_2^{2p} ds,\end{aligned} \tag{3.12}$$

where $C_4 = C_3 + ((C_2 C)/2\epsilon)((p-r)/p)$. Assuming $Y_n(t) = 2(C_4 + C\int_0^t \|\Delta u^n\|_2^{2p})$, we have

$$Y_n'(t) \leq 2C Y_n^p. \tag{3.13}$$

A simple integration of (3.13) over $(0, t)$ leads to

$$Y_n(t) \leq \left[Y_n^{1-p}(0) - 2C(p-1)t\right]^{-1/(p-1)}; \tag{3.14}$$

this implies that

$$\|\Delta u^n\|_2^2 + \frac{2}{\rho+2}\|u_t^n\|_{\rho+2}^{\rho+2} \le \left[Y_n^{1-p}(0) - 2C(p-1)t\right]^{-1/(p-1)}. \tag{3.15}$$

Though $Y_n(t)$ may blow up, there exists $0 < T < \min\{1, T_n\}$ satisfying

$$\|\Delta u^n\|_2^2 + \|u_t^n\|_{\rho+2}^{\rho+2} \le C, \quad \forall t \in [0,T], \tag{3.16}$$

where C is independent of n.

Moreover,

$$\int_0^t \|u_s^n\|_{m+2}^{m+2} ds + \int_0^t \|\Delta u_s^n\|_2^2 ds \le C. \tag{3.17}$$

By (3.17),

$$\begin{aligned}\int_0^t \left\||u_s^n(s)|^m u_s^n(s)\right\|_{(m+2)/(m+1)}^{(m+2)/(m+1)} ds &\le \int_0^t \|u_s^n(s)\|_{m+2}^{m+2} ds \\ &\le C, \quad t \in [0,T].\end{aligned} \tag{3.18}$$

By (3.16),

$$\left\||u^n(s)|^{p-1}u^n(s)\right\|_{(p+1)/p}^{(p+1)/p} \le \|u^n(s)\|_{p+1}^{p+1} \le \|\Delta u^n(s)\|_2^{p+1} \le C. \tag{3.19}$$

From (3.16)–(3.19), we have

$$\begin{aligned} u^n &\in L_\infty\left([0,T]; H_0^2\right); \\ u_t^n &\in L_2\left([0,T]; H_0^2\right) \cap L_\infty([0,T]; L_{\rho+2}). \end{aligned} \tag{3.20}$$

So the solution $u^n(t)$ of problem (3.3) exists on $[0,T]$ for each n. On the other hand, we can extract a subsequence from u^n, still denoted by u^n, such that

$$u^n \longrightarrow u \quad \text{weak}^* \text{ in } L_\infty\left([0,T]; H_0^2\right), \tag{3.21}$$

$$u_t^n \longrightarrow u_t \quad \text{weak}^* \text{ in } L_\infty([0,T]; L_{\rho+2}) \cap L_2\left([0,T]; H_0^2\right), \tag{3.22}$$

as $n \to \infty$. By (3.21), the Sobolev embedding theorem and the continuity of $\sigma_i(s)$, for $t \in [0,T]$,

$$\begin{aligned}
\nabla u^n(t) &\longrightarrow \nabla u(t) \quad \text{strongly in } L_2, \text{ a.e. on } \Omega,\\
\sigma_i\left(u^n_{x_i}(t)\right) &\longrightarrow \sigma_i(u_{x_i}(t)), \quad \text{a.e. on } \Omega,\ i = 1,\dots N,
\end{aligned} \tag{3.23}$$

as $n \to \infty$. By Lemma 3.1, (3.21)-(3.22), for any $\varepsilon > 0$, there exist positive constant $N_{1\varepsilon}$ and $N_{2\varepsilon}$ independent of u^n and u^n_t, respectively, such that, as $n \to \infty$,

$$\begin{aligned}
\|u^n(t) - u(t)\| &\le \left[\sum_{k=1}^{N_{1\varepsilon}} (u^n - u, \omega_k)^2\right]^{1/2} + \varepsilon \|u^n(t) - u(t)\|_{1,2} \le M\varepsilon,\\
\int_0^T \|u^n_t(s) - u_t(s)\|_2^2 ds &\le 2\left(\sum_{k=1}^{N_{2\varepsilon}} \int_0^T \left(u^n_t(s) - u_t(s), \omega_k\right)^2 ds \right.\\
&\qquad \left. + \varepsilon^2 \int_0^T \|u^n_t(s) - u_t(s)\|_{1,2}^2 ds\right) \le M\varepsilon^2.
\end{aligned} \tag{3.24}$$

By the arbitrariness of ε we get

$$\begin{aligned}
u^n &\longrightarrow u \quad \text{strongly in } L_\infty([0,T]; L_2), \text{ a.e. on } Q_T,\\
u^n_t &\longrightarrow u_t \quad \text{strongly in } L_2(Q_T), \text{ a.e. on } Q_T.
\end{aligned} \tag{3.25}$$

From the continuity of $|u_t|^m u_t$ and (3.25) we know that $|u^n_t|^m u^n_t \to |u_t|^m u_t$ a.e. on Q_T. With the same methods used above we easily get $|u^n|^{p-1}u^n \to |u|^{p-1}u$ a.e. on Q_T. Integrating (3.3) over $(0,t)$, $t < T$ gets

$$\begin{aligned}
&\frac{1}{\rho+1}\left(\left|u^n_t(t)\right|^{\rho+1}, \omega_j\right) + \int_0^t (\Delta u^n(s), \Delta\omega_j) ds + \int_0^t (|u^n_s|^m u^n_s(s), \omega_j) ds + \int_0^t \gamma(\Delta u^n_s(s), \Delta\omega_j) ds\\
&\quad = \sum_i \int_0^t \left(\frac{\partial}{\partial x_i}\sigma_i\left(u^n_{x_i}(s)\right), \omega_j\right) ds + \int_0^t \left(|u^n(s)|^{p-1} u^n(s), \omega_j\right) ds, \quad t > 0,\ j = 1,\dots,n.
\end{aligned} \tag{3.26}$$

Exploiting (3.16)-(3.17), we have

$$\begin{aligned}
&\int_0^t (\Delta u^n(s), \Delta\omega_j) ds \le C \int_0^t \|\Delta u^n\|_2^2 ds \le CT,\\
&\left(\left|u^n_t(t)\right|^m u^n_t(t), \omega_j\right) \le \left\|u^n_t\right\|_{\rho+2}^{m+1} \left\|\omega_j\right\|_{((\rho+2)/(m+1))'} \le \left\|u^n_t\right\|_{\rho+2}^{\rho+2} \left\|\omega_j\right\|_{((\rho+2)/(m+1))'}^{((\rho+2)/(m+1))'} \le C,
\end{aligned}$$

$$\int_0^t (|u_s^n(s)|^m u_s^n(s), \omega_j) ds \le \int_0^t \left(\|u_s^n\|_{m+2}^{m+2} + \|\omega_j\|_{m+2}^{m+2} \right) ds \le CT,$$
$$\int_0^t \left(|u^n(s)|^{p-1} u^n(s), \omega_j \right) ds \le CT. \tag{3.27}$$

Let $n \to \infty$ in (3.26) and we deduce from (3.23), (3.27) and the Lebesgue-dominated convergence theorem that

$$\frac{1}{\rho+1}\left(|u_t(t)|^{\rho+1}, \omega_j \right) + \int_0^t (\Delta u(s), \Delta\omega_j) ds + \int_0^t (|u_s(t)|^m u_s(s), \omega_j) ds + \int_0^t \gamma(\Delta u_s(s), \Delta\omega_j) ds$$
$$= \sum_i \int_0^t \left(\frac{\partial}{\partial x_i} \sigma_i(u_{x_i}(s)), \omega_j \right) ds + \int_0^t \left(|u(s)|^{p-1} u(s), \omega_j \right) ds, \quad t > 0,\ j = 1, \ldots, n. \tag{3.28}$$

This implies $u \in L_\infty([0,T]; H_0^2)$ is a local weak solution of problem (1.1). The proof of Theorem 2.1 is completed. □

Secondly, we prove Theorem 2.2. First we give two lemmas. It is easy to prove what follows.

Lemma 3.2. *The modified energy functional satisfies, along solutions of* (1.1)*,*

$$E'(t) = -\|u_t\|_{m+2}^{m+2} - \gamma\|\Delta u_t\|_2^2 \le 0. \tag{3.29}$$

Lemma 3.3. *Assume that (A1)–(A3) hold, satisfying*

$$C_0\left(\left(\frac{2(r+1)}{r-1} E(0) \right)^{(r-1)/2} + \left(\frac{2(r+1)}{r-1} E(0) \right)^{(p-1)/2} \right) < 1,$$
$$I(0) > 0. \tag{3.30}$$

Then $I(t) > 0$.

Proof. Since $I(0) > 0$, then there exists (by continuity) $T_* \le T$ such that $I(t) > 0$, for all $t \in [0, T_*)$, this gives

$$J(t) = \frac{I(t)}{r+1} + \frac{r-1}{2(r+1)}\|\Delta u\|_2^2 + \frac{p-r}{(p+1)(r+1)}\|u\|_{p+1}^{p+1}$$
$$\ge \frac{r-1}{2(r+1)}\|\Delta u\|_2^2 + \frac{p-r}{(p+1)(r+1)}\|u\|_{p+1}^{p+1}. \tag{3.31}$$

By using (2.8), (2.9), (3.31), and Lemma 3.2, we easily have

$$\|\Delta u\|_2^2 \le \frac{2(r+1)}{r-1} J(t) \le \frac{2(r+1)}{r-1} E(t) \le \frac{2(r+1)}{r-1} E(0), \quad \forall t \in [0, T_*). \tag{3.32}$$

We then exploit (2.12), and (3.32) to obtain

$$\begin{aligned} C_2\|\nabla u\|_{r+1}^{r+1} &\le B_1 C_2 \|\Delta u\|_2^{r+1} \le B_1 C_2 \|\Delta u\|_2^{r-1} \|\Delta u\|_2^2 \le B_1 C_2 \left(\frac{2(r+1)}{r-1} E(0)\right)^{(r-1)/2} \|\Delta u\|_2^2, \\ \|u\|_{p+1}^{p+1} &\le B_2 \|\Delta u\|_2^{p+1} \le B_2 \left(\frac{2(r+1)}{r-1} E(0)\right)^{(p-1)/2} \|\Delta u\|_2^2, \quad \forall t \in [0, T_*). \end{aligned} \tag{3.33}$$

Using (3.33), we have

$$\begin{aligned} C_2\|\nabla u\|_{r+1}^{r+1} + \|u\|_{p+1}^{p+1} &\le B_1 C_2 \left(\frac{2(r+1)}{r-1} E(0)\right)^{(r-1)/2} \|\Delta u\|_2^2 + B_2 \left(\frac{2(r+1)}{r-1} E(0)\right)^{(p-1)/2} \|\Delta u\|_2^2 \\ &\le C_0 \left(\left(\frac{2(r+1)}{r-1} E(0)\right)^{(r-1)/2} + \left(\frac{2(r+1)}{r-1} E(0)\right)^{(p-1)/2} \right) \|\Delta u\|_2^2 \\ &< \|\Delta u\|_2^2, \quad \forall t \in [0, T_*), \end{aligned} \tag{3.34}$$

where $C_0 = \max\{B_2, B_1 C_2\}$.

Therefore,

$$I(t) = \|\Delta u\|_2^2 - C_2 \|\nabla u\|_{r+1}^{r+1} - \|u\|_{p+1}^{p+1} > 0, \tag{3.35}$$

for all $t \in [0, T_*]$. By repeating this procedure, and using the fact that

$$\begin{aligned} &\lim_{t \to T_*} C_0 \left(\left(\frac{2(r+1)}{r-1} E(t)\right)^{(r-1)/2} + \left(\frac{2(r+1)}{r-1} E(t)\right)^{(p-1)/2} \right) \\ &\quad \le \lim_{t \to T_*} C_0 \left(\frac{2(r+1)}{r-1} E(0)\right)^{(r-1)/2} + \left(\frac{2(r+1)}{r-1} E(0)\right)^{(p-1)/2} < 1, \end{aligned} \tag{3.36}$$

the proof is completed. □

Lemma 3.4. *Assume that (A1)–(A4) hold,* (2.12) *satisfy. Then the solution is global existence. More, exist positive constant* $M > 0$ *has*

$$\|\Delta u\|_2^2 + \|u_t\|_{p+2}^{p+2} \le M; \qquad \int_0^t \|u_s\|_{m+2}^{m+2} ds \le M; \qquad \int_0^t \|\Delta u_s\|_2^2 ds \le M, \quad \forall t \in [0, \infty). \tag{3.37}$$

Proof. It suffices to show that

$$\|\Delta u\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} \tag{3.38}$$

is bounded independently of t. To achieve this, we use (2.9), (3.31), and Lemma 3.2 to get

$$\begin{aligned} E(0) &= E(t) + \int_0^t \|u_s\|_{m+2}^{m+2} ds + \gamma \int_0^t \|\Delta u_s\|_2^2 ds \\ &\geq J(t) + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \int_0^t \|u_s\|_{m+2}^{m+2} ds + \gamma \int_0^t \|\Delta u_s\|_2^2 ds \\ &\geq \frac{r-1}{2(r+1)}\|\Delta u\|_2^2 + \frac{p-r}{(p+1)(r+1)}\|u\|_{p+1}^{p+1} \\ &\quad + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \int_0^t \|u_s\|_{m+2}^{m+2} ds + \gamma \int_0^t \|\Delta u_s\|_2^2 ds \\ &\geq \frac{r-1}{2(r+1)}\|\Delta u\|_2^2 + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \int_0^t \|u_s\|_{m+2}^{m+2} ds + \gamma \int_0^t \|\Delta u_s\|_2^2 ds. \end{aligned} \tag{3.39}$$

Since $I(t), J(t)$ are positive. Therefore,

$$\|\Delta u\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} \leq CE(0). \tag{3.40}$$

Moreover,

$$\int_0^t \|u_s\|_{m+2}^{m+2} ds \leq E(0); \qquad \gamma \int_0^t \|\Delta u_s\|_2^2 ds \leq E(0), \tag{3.41}$$

where C is a positive constant, which depends only on r. □

Lemma 3.5. *Assume that (A1)–(A4) hold,* (2.12) *satisfy. Then exist* $C > 0$ *has*

$$\int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds \leq Ct^{(m-\rho)/(m+2)}; \tag{3.42}$$

$$\int_0^t \|\Delta u\|_2^2 ds \leq C \int_0^t I(s) ds. \tag{3.43}$$

Proof. Using Lemma 3.4, we have

$$\begin{aligned}\int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds &\le \left(\int_0^t \|u_s\|_{\rho+2}^{m+2} ds\right)^{(\rho+2)/(m+2)} \left(\int_0^t 1\, ds\right)^{(m-\rho)/(m+2)} \\ &\le C\left(\int_0^t \|u_s\|_{m+2}^{m+2} ds\right)^{(\rho+2)/(m+2)} t^{(m-\rho)/(m+2)} \\ &\le Ct^{(m-\rho)/(m+2)}.\end{aligned} \tag{3.44}$$

Using (3.34), we have

$$\begin{aligned}C_2\|\nabla u\|_{r+1}^{r+1} + \|u\|_{p+1}^{p+1} &\le BC_0\left(\left(\frac{2(r+1)}{r-1}E(0)\right)^{(r-1)/2}\left(\frac{2(r+1)}{r-1}E(0)\right)^{(p-1)/2}\right)\|\Delta u\|_2^2 \\ &:= \eta\|\Delta u\|_2^2.\end{aligned} \tag{3.45}$$

So

$$(1-\eta)\|\Delta u\|_2^2 \le I(t); \tag{3.46}$$

the proof is complete. □

Lemma 3.6. *Assume that (A1)–(A4) hold,* (2.12) *satisfy. Then there exists a* $C > 0$, *having*

$$\int_0^t I(s) ds \le C\left(t^{1/2} + t^{(m-\rho)/(m+2)} + t^{1/(m+2)} + 1\right). \tag{3.47}$$

Proof. By multiplying the differential equation in (1.1) by u and integrating over Ω, using integration by parts (2.9) and assumption (A1), we obtain

$$\begin{aligned}0 &= \frac{d}{dt}\left(\frac{|u_t|^\rho u_t}{\rho+1}, u\right) - \frac{\|u_t\|_{\rho+2}^{\rho+2}}{\rho+1} + \|\Delta u\|_2^2 + \left(|u_t|^m u_t, u\right) \\ &\quad + \gamma\left(\Delta^2 u_t, u\right) + \sum_{i=1}^N (\sigma_i(u_{x_i}), u_{x_i}) - \|u\|_{p+1}^{p+1} \\ &\ge \frac{d}{dt}\left(\frac{|u_t|^\rho u_t}{\rho+1}, u\right) - \frac{\|u_t\|_{\rho+2}^{\rho+2}}{\rho+1} + \|\Delta u\|_2^2 + \left(|u_t|^m u_t, u\right) \\ &\quad + \gamma(\Delta u_t, \Delta u) - C_2\|\nabla u\|_{r+1}^{r+1} - \|u\|_{p+1}^{p+1} \\ &= \frac{d}{dt}\left(\frac{|u_t|^\rho u_t}{\rho+1}, u\right) - \frac{\|u_t\|_{\rho+2}^{\rho+2}}{\rho+1} + I(t) \\ &\quad + (\|u_t\|^m u_t, u) + \gamma(\Delta u_t, \Delta u).\end{aligned} \tag{3.48}$$

So

$$\int_0^t I(s)ds \le \|u_t\|_{\rho+2}^{\rho+1}\|u\|_{\rho+2} + \|u_1\|_{\rho+2}^{\rho+2}\|u_0\|_{\rho+2} + \int_0^t \frac{\|u_s\|_{\rho+2}^{\rho+2}}{\rho+1}ds + \int_0^t (\|u_s\|^m u_s, u)ds + \gamma\int_0^t (\Delta u_s, \Delta u)ds. \tag{3.49}$$

Using (A3), (3.42)-(3.43), we have

$$\begin{aligned}
&\|u_t\|_{\rho+2}^{\rho+1}\|u\|_{\rho+2} \le C\left(\|u_t\|_{\rho+2}^{\rho+2}\right)^{(\rho+1)/(\rho+2)}\|\Delta u\|_2 \le C,\\
&\int_0^t\int_\Omega |u_s|^m u_s u\,dx\,ds \le \int_0^t \|u_s\|_{m+2}^{m+1}\|u\|_{m+2}ds \le C\int_0^t \|u_s\|_{m+2}^{m+1}\|\Delta u\|_2 ds\\
&\qquad \le C\int_0^t \|u_s\|_{m+2}^{m+1}ds \le C\left(\int_0^t \|u_s\|_{m+2}^{m+2}\right)^{(m+1)/(m+2)} t^{1/(m+2)} \le Ct^{1/(m+2)},\\
&\gamma\int_0^t (\Delta u_s, \Delta u)ds \le \gamma\int_0^t \|\Delta u_s\|_2\|\Delta u\|_2 ds \le C\left(\int_0^t \|\Delta u_s\|_2^2 ds\right)^{1/2} t^{1/2} \le Ct^{1/2}.
\end{aligned} \tag{3.50}$$

Therefore,

$$\int_0^t I(s)ds \le M\left(1 + t^{1/(m+2)} + t^{(m-\rho)/(m+2)} + t^{1/2}\right). \tag{3.51}$$

□

Proof of Theorem 2.2. First, $t \mapsto (1+t)E(t)$ is also absolutely continuous, and we have

$$\frac{d}{dt}((1+t)E(t)) \le E(t). \tag{3.52}$$

A simple integration of (3.52) over $(0,t)$ leads to

$$\begin{aligned}
(1+t)E(t) &\le E(0) + \int_0^t E(s)ds\\
&\le E(0) + \frac{1}{\rho+2}\int_0^t \|u_s\|_{\rho+2}^{\rho+2}ds + \frac{1}{2}\int_0^t \|\Delta u\|_2^2 ds\\
&\quad + \sum_0^N \int_0^t\int_\Omega A_i(u_{x_i})dx\,ds - \frac{1}{p+1}\int_0^t \|u\|_{p+1}^{p+1}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq E(0) + \frac{1}{\rho+2}\int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds + \frac{1}{2}\int_0^t \|\Delta u\|_2^2 ds \\
&\quad + \sum_0^N \int_0^t \int_\Omega A_i(u_{x_i}) dx\, ds + \frac{1}{p+1}\int_0^t \|u\|_{p+1}^{p+1} ds \\
&\leq E(0) + \frac{1}{\rho+2}\int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds + \frac{1}{2}\int_0^t \|\Delta u\|_2^2 ds \\
&\quad + \frac{C_2}{r+1}\int_0^t \|\nabla u\|_{r+1}^{r+1} ds + \frac{1}{p+1}\int_0^t \|u\|_{p+1}^{p+1} ds.
\end{aligned}
\tag{3.53}
$$

Using the upper inequality, (3.45), (3.46), and (3.52), we have

$$
\begin{aligned}
(1+t)E(t) &\leq E(0) + \frac{1}{\rho+2}\int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds + \frac{1}{2}\int_0^t \|\Delta u\|_2^2 ds + C\int_0^t \|\Delta u\|_2^2 ds \\
&\leq E(0) + \frac{1}{\rho+2}\int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds + C\int_0^t I(s) ds.
\end{aligned}
\tag{3.54}
$$

Apply (3.42), (3.47), we can get (2.13).

Using (3.31) and Lemma 3.3, we get $J(t) \geq 0, I(t) > 0$. (2.13) implies $\lim t \to \infty E(t) = 0$. So when $t \to \infty$, we have $\|u_t\|_{\rho+2}^{\rho+2} \to 0$ and $J(t) \to 0$. It is that (2.14) is satisfied. Theorem 2.2 is complete.

Following we will prove Theorem 2.3. For this purpose we set

$$
L(t) := E(t) + \varepsilon\Psi(t),
\tag{3.55}
$$

where ε is a positive constant and

$$
\Psi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\, dx.
\tag{3.56}
$$

□

Lemma 3.7. *Let ε be small enough. Then there exist two positive constants α_1 and α_2 such that*

$$
\alpha_1 L(t) \leq E(t) \leq \alpha_2 L(t).
\tag{3.57}
$$

Proof. By Lemma 3.4 and Young's inequality, a direct computation gives

$$
\begin{aligned}
L(t) &\leq E(t) + \frac{\varepsilon}{\rho+2}\int_\Omega |u_t|^{\rho+2} dx + \frac{\varepsilon}{\rho+2}\int_\Omega |u|^{\rho+2} dx \\
&\leq E(t) + \frac{\varepsilon}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{\varepsilon B_2}{\rho+2}\left(\frac{2(r+1)E(0)}{r-1}\right)^{\rho/2}\|\Delta u\|_2^2
\end{aligned}
$$

$$\leq E(t) + \varepsilon E(t) + \frac{\varepsilon B_2}{\rho+2}\left(\frac{2(r+1)}{r-1}\right)^{(\rho+2)/2} E(0)^{\rho/2}E(t)$$
$$\leq \frac{1}{\alpha_1}E(t). \tag{3.58}$$

Similarly, we have

$$\begin{aligned} L(t) &\geq E(t) - \frac{\varepsilon}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx - \frac{\varepsilon}{\rho+2}\int_\Omega |u|^{\rho+2}dx \\ &\geq E(t) - \frac{\varepsilon}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} - \frac{\varepsilon B_2}{\rho+2}\left(\frac{2(r+1)E(0)}{r-1}\right)^{\rho/2}\|\Delta u\|_2^2 \\ &\geq E(t) - \varepsilon E(t) - \frac{\varepsilon B_2}{\rho+2}\left(\frac{2(r+1)}{r-1}\right)^{(\rho+2)/2} E(0)^{\rho/2}E(t) \\ &\geq \frac{1}{\alpha_2}E(t), \end{aligned} \tag{3.59}$$

provided that ε is small enough. □

Lemma 3.8. *Assume that the conditions of Theorem 2.3 hold, then the function*

$$\Psi(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u \, dx \tag{3.60}$$

satisfies, along the solution of (1.1),

$$\Psi'(t) \leq -\frac{1}{4}E(t) + \mu\|u_t\|_{m+2}^{m+2} + \omega\|\Delta u\|_2^2. \tag{3.61}$$

Proof. Applying equations of (1.1), we see

$$\begin{aligned} \Psi'(t) &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \int_\Omega |u_t|^\rho u_{tt} u \, dx \\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \int_\Omega \left(-\Delta^2 u - |u_t|^m u_t + \sum_{i=1}^N \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) - \gamma\Delta^2 u_t + |u|^{p-1}u\right) u \, dx \\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - \|\Delta u\|_2^2 - (|u_t|^m u_t, u) - \sum_{i=1}^N (\sigma_i(u_{x_i}), u_{x_i}) - \gamma(\Delta u_t, \Delta u) + \|u\|_{p+1}^{p+1} \\ &= -\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} - \|\Delta u\|_2^2 - \sum_{i=1}^N \int_\Omega A_i(u_{x_i})dx + \sum_{i=1}^N \int_\Omega (A_i(u_{x_i}) - (\sigma_i(u_{x_i}), u_{x_i}))dx \end{aligned}$$

$$
\begin{aligned}
&-\left(|u_t|^m u_t, u\right) - \gamma(\Delta u_t, \Delta u) + \|u\|_{p+1}^{p+1} + \frac{2\rho+3}{(\rho+2)(\rho+1)}\|u_t\|_{\rho+2}^{\rho+2} \\
\le\; & -E(t) - \left(|u_t|^m u_t, u\right) - \gamma(\Delta u_t, \Delta u) + \frac{2\rho+3}{(\rho+2)(\rho+1)}\|u_t\|_{\rho+2}^{\rho+2} \\
&+ \frac{p}{p+1}\|u\|_{p+1}^{p+1} + \sum_{i=1}^{N}\int_\Omega \left(A_i(u_{x_i}) - \sigma_i(u_{x_i})u_{x_i}\right)dx.
\end{aligned}
\tag{3.62}
$$

Exploiting the assumption (A1) and Young's inequality, we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega \left(A_i(u_{x_i}) - \sigma_i(u_{x_i})u_{x_i}\right)dx \le 0. \\
&\frac{p}{p+1}\|u\|_{p+1}^{p+1} \le \frac{pB_2^{p+1}}{p+1}\|\Delta u\|_2^{p+1} \le \frac{pB_2^{p+1}}{p+1}\left(\frac{2(r+1)}{r-1}\right)^{(p+1)/2}(E(0))^{(p-1)/2}E(t) \\
&\left|\left(|u_t|^m u_t, u\right)\right| \le \frac{1}{(m+2)\delta^{(m+2)/(m+1)}}\|u_t\|_{m+2}^{m+2} + \frac{\delta^{m+2}}{m+2}\|u_t\|_{m+2}^{m+2} \\
&\qquad \le \frac{1}{(m+2)\delta^{(m+2)/(m+1)}}\|u_t\|_{m+2}^{m+2} + \frac{(\delta B_2)^{m+2}}{m+2}\left(\frac{2(r+1)E(0)}{r-1}\right)^{m/2}\|\Delta u\|_2^2 \\
&\qquad \le \frac{1}{(m+2)\delta^{(m+2)/(m+1)}}\|u_t\|_{m+2}^{m+2} \\
&\qquad\quad + \frac{(\delta B_2)^{m+2}}{m+2}\left(\frac{2(r+1)}{r-1}\right)^{(m+2)/2}(E(0))^{m/2}E(t), \quad \forall \delta > 0 \\
&\left|\gamma(\Delta u_t, \Delta u)\right| \le \frac{\gamma\eta^{-1}}{2}\|\Delta u_t\|_2^2 + \frac{\gamma\eta}{2}\|\Delta u\|_2^2, \quad \forall \eta > 0.
\end{aligned}
\tag{3.63}
$$

Exploiting (3.63) and (3.62), we get

$$
\begin{aligned}
\Psi'(t) \le & -\Bigg[1 - \frac{pB_2^{p+2}}{p+1}\left(\frac{2(r+1)}{r-1}\right)^{(p+1)/2}(E(0))^{(p-1)/2} \\
& - \frac{(\delta B_2)^{m+2}}{m+2}\left(\frac{2(r+1)}{r-1}\right)^{(m+2)/2}(E(0))^{m/2} - \gamma\eta\frac{2(r+1)}{r-1}\Bigg]E(t) \\
& + \frac{1}{(m+2)\delta^{(m+2)/(m+1)}}\|u_t\|_{m+2}^{m+2} + \left(\gamma\eta^{-1} + \frac{2\rho+3}{(\rho+1)(\rho+2)}\left((\rho+2)E(0)\right)^{\rho/(\rho+2)}B_2^2\right)\|\Delta u\|_2^2.
\end{aligned}
\tag{3.64}
$$

Choosing δ satisfies

$$\frac{(\delta B_2)^{m+2}}{m+2}\left(\frac{2(r+1)E(0)}{r-1}\right)^{m/2}=\frac{1}{4}\left[1-\frac{pB_2^{p+2}}{p+1}\left(\frac{2(r+1)}{r-1}\right)^{(p+1)/2}(E(0))^{(p-1)/2}\right] \tag{3.65}$$

and η satisfies

$$\gamma\eta\frac{2(r+1)}{r-1}=\frac{1}{4}\left[1-\frac{pB_2^{p+2}}{p+1}\left(\frac{2(r+1)}{r-1}\right)^{(p+1)/2}(E(0))^{(p-1)/2}\right], \tag{3.66}$$

at the above; the proof of (3.61) is completed. □

Proof of Theorem 2.3. Using (3.61) and Lemma 3.7, we have

$$\begin{aligned}
L'(t) &= E'(t)+\varepsilon\Psi'(t)\\
&\leq \left(-\|u_t\|_{m+2}^{m+2}-\gamma\|\Delta u_t\|\right)-\frac{\varepsilon}{4}E(t)+\varepsilon\mu\|u_t\|_{m+2}^{m+2}+\varepsilon\omega\|\Delta u_t\|\\
&= -\frac{\varepsilon}{4}E(t)-(\gamma-\varepsilon\omega)\|\Delta u_t\|-(1-\varepsilon\mu)\|u_t\|_{m+2}^{m+2}\\
&\leq -\frac{\varepsilon\alpha_1}{4}L(t)-(\gamma-\varepsilon\omega)\|\Delta u_t\|-(1-\varepsilon\mu)\|u_t\|_{m+2}^{m+2}.
\end{aligned} \tag{3.67}$$

Choosing ε satisfies $\varepsilon\leq\min\{\gamma/\omega,1/\mu\}$. So we have

$$L'(t)\leq-\frac{\varepsilon\alpha_1}{4}L(t),\quad \forall t\geq 0. \tag{3.68}$$

A simple integration of (3.68) over $(0,t)$ leads to

$$L(t)\leq L(0)e^{-(\varepsilon\alpha_1/4)t},\quad \forall t\geq 0. \tag{3.69}$$

Exploiting Lemma 3.7 again, we have

$$E(t)\leq\alpha_2L(0)e^{-(\varepsilon\alpha_1/4)t}:=Ke^{-\kappa t},\quad \forall t\geq 0, \tag{3.70}$$

where $K,\kappa>0$ are constants. The proof of Theorem 2.3 is complete. □

4. Blow-Up of Solutions

Proof of Theorem 2.5. Assuming that the solution of (1.1) is global, we have

$$E'(t)=-\|u_t\|_{m+2}^{m+2}-\gamma\|\Delta u_t\|_2^2\leq 0. \tag{4.1}$$

So

$$\|\Delta u\|_2^2 \le C_3; \tag{4.2}$$

we set

$$Q(t) := -\int_0^t E(s)ds + (ot + \omega)\int_\Omega u_0^2 dx, \tag{4.3}$$

where o, ω are constants and will be given later.

Consider

$$Q'(t) = -E(t) + o\int_\Omega u_0^2 dx \ge o\int_\Omega u_0^2 dx - E(0). \tag{4.4}$$

Choosing o satisfies the following condition in (4.4):

$$o\int_\Omega u_0^2 dx - E(0) = Q'(0) > 0, \tag{4.5}$$

so we have

$$Q'(t) \ge Q'(0) > 0, \quad \forall t \in [0, T]. \tag{4.6}$$

Moreover, we have

$$Q'(t) - Q'(0) = E(0) - E(t) = -\int_0^t E'(s)ds = \int_0^t \left(\|u_s\|_{m+2}^{m+2} + \gamma\|\Delta u_s\|_2^2\right)ds. \tag{4.7}$$

Define

$$K(t) := Q^{1-\gamma}(t) + \frac{\varepsilon}{\rho+1}\int_0^t\int_\Omega |u_s|^\rho u_s u\, dx\, ds, \tag{4.8}$$

where $\varepsilon > 0$ will be given later, and

$$0 < \gamma \le \min\left\{\frac{r-\rho-1}{(\rho+2)(r+1)}, \frac{p-m-1}{(p+1)(m+1)}, \frac{r-m-1}{(m+1)(r+1)}\right\}. \tag{4.9}$$

Multiplying (1.1) by u and a direct computation yield

$$\begin{aligned}
K'(t) &= (1-\gamma)Q^{-\gamma}Q'(t) + \frac{\varepsilon}{\rho+1}\int_{\Omega}|u_1|^{\rho}u_1u_0dx\,ds + \frac{\varepsilon}{\rho+1}\int_0^t\int_{\Omega}\left(|u_s|^{\rho}u_su\right)_s dx\,ds \\
&= (1-\gamma)Q^{-\gamma}Q'(t) + \frac{\varepsilon}{\rho+1}\int_{\Omega}|u_1|^{\rho}u_1u_0dx\,ds + \frac{\varepsilon}{\rho+1}\int_0^t\int_{\Omega}|u_s|^{\rho+2}dx\,ds \\
&\quad + \varepsilon\int_0^t\int_{\Omega}|u_s|^{\rho}u_{ss}u\,dx\,ds \\
&= (1-\gamma)Q^{-\gamma}Q'(t) + \frac{\varepsilon}{\rho+1}\int_{\Omega}|u_1|^{\rho}u_1u_0dx\,ds + \frac{\varepsilon}{\rho+1}\int_0^t\int_{\Omega}|u_s|^{\rho+2}dx\,ds \\
&\quad - \varepsilon\int_0^t\int_{\Omega}\Delta^2uu\,dx\,ds - \varepsilon\int_0^t\int_{\Omega}|u_s|^{m}u_su\,dx\,ds - \varepsilon\gamma\int_0^t\int_{\Omega}\Delta^2u_su\,dx\,ds \\
&\quad + \varepsilon\sum_{i=1}^{N}\int_0^t\int_{\Omega}\frac{\partial}{\partial x_i}\sigma_i(u_{x_i})u\,dx\,ds + \varepsilon\int_0^t\int_{\Omega}|u|^{p-1}uu\,dx\,ds \\
&= (1-\gamma)Q^{-\gamma}Q'(t) + \frac{\varepsilon}{\rho+1}\int_{\Omega}|u_1|^{\rho}u_1u_0dx\,ds + \frac{\varepsilon}{\rho+1}\int_0^t\int_{\Omega}|u_s|^{\rho+2}dx\,ds \\
&\quad - \varepsilon\int_0^t\int_{\Omega}|\Delta u|^2dx\,ds - \varepsilon\gamma\int_0^t\int_{\Omega}\Delta u_s\Delta u\,dx\,ds - \varepsilon\int_0^t\int_{\Omega}|u_s|^{m}u_su\,dx\,ds \\
&\quad - \varepsilon\sum_{i=1}^{N}\int_0^t\int_{\Omega}\sigma_i(u_{x_i})u_{x_i}dx\,ds + \varepsilon\int_0^t\|u\|_{p+1}^{p+1}ds.
\end{aligned} \tag{4.10}$$

Exploiting (4.7) and (B1), we have

$$\begin{aligned}
\gamma\int_0^t\int_{\Omega}\Delta u_s\Delta u\,dx\,ds &\le \frac{\gamma}{\eta^2}\int_0^t\|\Delta u_s\|_2^2ds + \eta^2\gamma\int_0^t\|\Delta u\|_2^2ds \\
&\le \frac{\gamma}{\eta^2}\left(Q'(t)-Q'(0)\right) + \eta^2\gamma C_3T,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\int_0^t\int_{\Omega}|u_s|^{m}u_su\,dx\,ds &\le \frac{\varsigma^{m+2}}{m+2}\int_0^t\int_{\Omega}|u|^{m+2}dx\,ds + \frac{m+1}{m+2}\varsigma^{-((m+1)/(m+2))}\int_0^t\int_{\Omega}|u_s|_{m+2}^{m+2}dx\,ds \\
&\le \frac{\varsigma^{m+2}}{m+2}\int_0^t\int_{\Omega}|u|^{m+2}dx\,ds + \frac{m+1}{m+2}\varsigma^{-((m+1)/(m+2))}\left(Q'(t)-Q'(0)\right)
\end{aligned} \tag{4.12}$$

$$-\int_0^t\int_{\Omega}\sigma_i(u_{x_i})u_{x_i}dx\,ds \ge C_4\int_0^t\|\nabla u\|_{r+1}^{r+1}ds. \tag{4.13}$$

Using (4.10)–(4.13), we have

$$\begin{aligned}
K'(t) \geq\ & (1-\gamma)Q^{-\gamma}Q'(t) + \frac{\varepsilon}{\rho+1}\int_{\Omega}|u_1|^{\rho}u_1u_0\,dx + \frac{\varepsilon}{\rho+1}\int_0^t\int_{\Omega}|u_s|^{\rho+2}dx\,ds \\
& + \varepsilon C_4\int_0^t\|\nabla u\|_{r+1}^{r+1}ds - \varepsilon\int_0^t\|\Delta u\|_2^2ds + \varepsilon\int_0^t\|u\|_{p+1}^{p+1}ds - \varepsilon\eta^2\gamma TC_3 \\
& - \varepsilon\left(\frac{m+1}{m+2}\varsigma^{-(m+1)/(m+2)} + \frac{\gamma}{\eta^2}\right)Q'(t) - \varepsilon\frac{\varsigma^{m+2}}{m+2}\int_0^t\int_{\Omega}|u|^{m+2}dx\,ds \\
& + \varepsilon\left(\frac{m+1}{m+2}\varsigma^{-(m+1)/(m+2)} + \frac{\gamma}{\eta^2}\right)Q'(0).
\end{aligned} \tag{4.14}$$

Choosing ς, η satisfies

$$\varsigma^{-((m+2)/(m+1))} = M_1Q^{-\gamma}(t), \qquad \frac{1}{\eta^2} = M_2Q^{-\gamma}(t). \tag{4.15}$$

Then we have

$$\begin{aligned}
K'(t) \geq\ & \left[(1-\gamma) - \varepsilon\frac{m+1}{m+2}M_1 - \varepsilon\gamma M_2\right]Q^{-\gamma}Q'(t) + \frac{\varepsilon}{\rho+1}\int_{\Omega}|u_1|^{\rho}u_1u_0\,dx\,ds \\
& + \frac{\varepsilon}{\rho+1}\int_0^t\int_{\Omega}|u_s|^{\rho+2}dx\,ds + \varepsilon\left(\frac{m+1}{m+2}M_1 + \varepsilon\gamma M_2\right)Q^{-\gamma}Q'(0) \\
& - \varepsilon\frac{M_1^{-m-1}}{m+2}Q^{\gamma(m+1)}\int_0^t\int_{\Omega}|u|^{m+2}dx\,ds - \varepsilon M_2^{-1}Q^{\gamma}\gamma TC_3 - \varepsilon\int_0^t\|\Delta u\|_2^2ds \\
& + \varepsilon\int_0^t\|u\|_{p+1}^{p+1}ds + \varepsilon C_4\int_0^t\|\nabla u\|_{r+1}^{r+1}ds.
\end{aligned} \tag{4.16}$$

A simple computing implies

$$\begin{aligned}
Q^{\gamma} &= \left[-\int_0^t E(s)ds + (ot+\omega)\int_{\Omega}u_0^2dx\right]^{\gamma} \\
&\leq \left[\int_0^t\left(\frac{\|u\|_{p+1}^{p+1}}{p+1} + C_2\|\nabla u\|_{r+1}^{r+1}\right)ds + (oT+\omega)\|u_0\|_2^2\right]^{\gamma} \\
&\leq 2^{\gamma-1}C_5\left[(oT+\omega)^{\gamma}\|u_0\|_2^{2\gamma} + \left(\int_0^t\left(\|u\|_{p+1}^{p+1} + \|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma}\right].
\end{aligned} \tag{4.17}$$

Using (4.17), (B2), embedding theorem, and Hölder inequality, we can get

$$
\begin{aligned}
Q^{\gamma(m+1)} &\leq \left[2^{\gamma-1}C_5\left[(oT+\omega)^{\gamma}\|u_0\|_2^{2\gamma}+\left(\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma}\right]\right]^{(m+1)} \\
&\leq 2^{(\gamma-1)(m+1)-1}C_6\left[\left(\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma(m+1)}\right. \\
&\qquad \left. + (oT+\omega)^{\gamma(m+1)}\|u_0\|_2^{2\gamma(m+1)}\right],
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
\alpha\int_0^t\int_\Omega |u|^{m+2}dx\,ds &\leq \alpha\int_0^t\left(\int_\Omega |u|^{p+1}dx\right)^{(m+2)/(p+1)}ds|\Omega|^{(p-m-1)/(p+1)} \\
&\leq \alpha|\Omega|^{(p-m-1)/(p+1)}T^{(p-m-1)/(p+1)}\left(\int_0^t \|u\|_{p+1}^{p+1}ds\right)^{(m+2)/(p+1)} \\
&=: C_8\left(\int_0^t \|u\|_{p+1}^{p+1}ds\right)^{(m+2)/(p+1)},
\end{aligned}
\tag{4.19}
$$

where $C_8 = \alpha|\Omega|^{(p-m-1)/(p+1)}T^{(p-m-1)/(p+1)}$.

Consider

$$
\begin{aligned}
(1-\alpha)\int_0^t\int_\Omega |u|^{m+2}dx\,ds &\leq (1-\alpha)B\int_0^t \|\nabla u\|_2^{m+2}ds \\
&\leq (1-\alpha)B\left(\int_0^t \|\nabla u\|_2^{r+1}ds\right)^{(m+2)/(r+1)} t^{(r-m-1)/(r+1)} \\
&\leq C_7\left(\int_0^t \|\nabla u\|_{r+1}^{r+1}ds\right)^{(m+2)/(r+1)}.
\end{aligned}
\tag{4.20}
$$

Using (4.18)–(4.20), (B1), we get

$$
\begin{aligned}
&Q^{\gamma(m+1)}\int_0^t\int_\Omega |u|^{m+2}dx\,ds \\
&\quad\leq 2^{(\gamma-1)(m+1)-1}C_6\left[\left(\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma(m+1)} + (oT+\omega)^{\gamma(m+1)}\|u_0\|_2^{2\gamma(m+1)}\right] \\
&\qquad\times\left[C_8\left(\int_0^t \|u\|_{p+1}^{p+1}ds\right)^{(m+2)/(p+1)} + C_7\left(\int_0^t \|\nabla u\|_{r+1}^{r+1}ds\right)^{(m+2)/(r+1)}\right]
\end{aligned}
$$

$$\leq 2^{(\gamma-1)(m+1)-1}C_6\Bigg[C_8\left(\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma(m+1)+((m+2)/(p+1))}$$
$$+(oT+\omega)^{\gamma(m+1)}\|u_0\|_2^{2\gamma(m+1)}$$
$$\times C_8\left(\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma(m+1)+((m+2)/(p+1))}\Bigg]$$
$$+2^{(\gamma-1)(m+1)-1}C_6\Bigg[C_7\left(\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma(m+1)+((m+2)/(r+1))}$$
$$+(oT+\omega)^{\gamma(m+1)}\|u_0\|_2^{2\gamma(m+1)}$$
$$\times C_7\left(\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right)^{\gamma(m+1)+((m+2)/(r+1))}\Bigg], \tag{4.21}$$

Using (4.9), (4.17), and Young inequality, we have

$$Q^{\gamma(m+1)}\int_0^t\int_\Omega |u|^{m+2}dx\,ds \leq C_9\left(1+\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right). \tag{4.22}$$

Similarly, we have

$$TC_3Q^\gamma \leq C_{10}\left(1+\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)ds\right). \tag{4.23}$$

Additionally, we choose ε satisfying

$$\varepsilon \leq \frac{1-\gamma}{((m+1)/(m+2))M_1+\gamma M_2}. \tag{4.24}$$

Using (4.16), (4.21)–(4.24), we get

$$K'(t) \geq \frac{\varepsilon}{\rho+1}\int_\Omega |u_1|^\rho u_1u_0dx\,ds+\frac{\varepsilon}{\rho+1}\int_0^t\int_\Omega |u_s|^{\rho+2}dx\,ds$$
$$-\varepsilon\int_0^t\|\Delta u\|_2^2ds-\varepsilon\left[\frac{M_1^{(-m-1)}}{m+2}C_9+C_{10}\frac{\gamma}{M_2}\right]$$
$$\times\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)-\varepsilon\left[\frac{M_1^{(-m-1)}}{m+2}C_9+C_{10}\frac{\gamma}{M_2}\right]$$
$$+\varepsilon\min\{1,C_4\}\int_0^t\left(\|\nabla u\|_{r+1}^{r+1}+\|u\|_{p+1}^{p+1}\right)+\xi Q(t)$$

$$
\begin{aligned}
&+\xi\sum_{i=1}^{N}\int_0^t\int_\Omega\int_0^{u_{x_i}}\sigma_i(\tau)d\tau\,dx\,ds\\
&-\xi\int_0^t\frac{\|u\|_{p+1}^{p+1}}{p+1}ds+\frac{\xi}{\rho+1}\int_0^t\|u_s\|_{\rho+2}^{\rho+2}ds\\
&+\frac{\xi}{2}\int_0^t\|\Delta u\|_2^2ds-\xi(ot+\omega)\|u_0\|_2^2,\\
&\int_0^t\int_\Omega\int_0^{u_{x_i}}\sigma_i(\tau)d\tau\,dx\,ds-\int_0^t\frac{\|u\|_{p+1}^{p+1}}{p+1}ds\\
&\geq-\frac{C_2}{r+1}\int_0^t\|\nabla u\|_{r+1}^{r+1}ds-\int_0^t\frac{\|u\|_{p+1}^{p+1}}{p+1}ds\\
\geq\ &-\frac{1}{p+1}\int_0^t\left(\|\nabla u\|_{r+1}^{r+1}+\|u\|_{p+1}^{p+1}\right)ds.
\end{aligned}
\tag{4.25}
$$

Using (4.25), we get

$$
\begin{aligned}
K'(t)\geq{}&\xi Q(t)+\left(\frac{\xi}{\rho+2}+\frac{\varepsilon}{\rho+1}\right)\int_0^t\|u_s\|_{\rho+2}^{\rho+2}ds+\varepsilon\left(\frac{\xi}{2\varepsilon}-1\right)\int_0^t\|\Delta u\|_2^2ds\\
&+\varepsilon\left(\min\{1,C_4\}-\frac{1}{p+1}\frac{\xi}{\varepsilon}-\frac{M_1^{-m-1}}{m+2}C_9-C_{10}\frac{\gamma}{M_2}\right)\int_0^t\left(\|u\|_{p+1}^{p+1}+\|\nabla u\|_{r+1}^{r+1}\right)\\
&+\varepsilon\left[\frac{1}{\rho+1}\int_\Omega|u_1|^\rho u_1u_0dx-\frac{\xi}{\varepsilon}(oT+\omega)\int_0^t u_0^2dx-\frac{M_1^{-m-1}}{m+2}C_9-C_{10}\frac{\gamma}{M_2}\right],
\end{aligned}
\tag{4.26}
$$

choosing $2\varepsilon<\xi<(p+1)\varepsilon$, u_0,u_1 satisfying

$$
\begin{aligned}
&\min\{1,C_4\}-\frac{1}{p+1}\frac{\xi}{\varepsilon}>0,\\
&\frac{1}{\rho+1}\int_\Omega|u_1|^\rho u_1u_0dx-\frac{\xi}{\varepsilon}(oT+\omega)\int_0^t u_0^2dx>0.
\end{aligned}
\tag{4.27}
$$

Then we choose M_1,M_2 big enough, satisfying

$$
\begin{aligned}
&\min\{1,C_4\}-\frac{1}{p+1}\frac{\xi}{\varepsilon}-\frac{M_1^{-m-1}}{m+2}C_9-C_{10}\frac{\gamma}{M_2}>0\\
&\frac{1}{\rho+1}\int_\Omega|u_1|^\rho u_1u_0dx-\frac{\xi}{\varepsilon}(oT+\omega)\int_0^t u_0^2dx-\frac{M_1^{-m-1}}{m+2}C_9-C_{10}\frac{\gamma}{M_2}>0.
\end{aligned}
\tag{4.28}
$$

Using the two above inequalities, we have

$$K'(t) \geq C\varepsilon\left[Q(t) + \int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds + \int_0^t \|\Delta u\|_2^2 ds + \int_0^t \left(\|\nabla u\|_{r+1}^{r+1} + \|u\|_{p+1}^{p+1}\right) ds + \nu\right], \tag{4.29}$$

where $\nu > 0$. So

$$K(t) > K(0) \geq \left(\omega \int_\Omega u_0^2 dx\right)^{1-\gamma} > 0. \tag{4.30}$$

Using (B1)–(B3), (A4), and Hölder and Young inequalities, we have

$$\begin{aligned} \left|\frac{1}{\rho+1}\int_0^t\int_\Omega |u_s|^\rho u_s u \, dx \, ds\right| &\leq \frac{1}{\rho+1}\int_0^t \|u_s\|_{\rho+2}^{\rho+1}\|\nabla u\|_{r+1} ds \\ &\leq \frac{C}{\rho+1}\int_0^t \left(\|u_s\|_{\rho+2}^{(\rho+1)\mu} + \|\nabla u\|_{r+1}^{\theta}\right) ds, \end{aligned} \tag{4.31}$$

where $(1/\mu) + (1/\theta) = 1$.

So we have

$$\begin{aligned} \left|\frac{1}{\rho+1}\int_0^t\int_\Omega |u_s|^\rho u_s u \, dx \, ds\right|^{1/(1-\gamma)} &\leq C\left(\int_0^t \|u_s\|_{\rho+2}^{(\rho+1)\mu} ds\right)^{1/(1-\gamma)} + \left(\int_0^t \|\nabla u\|_{r+1}^{\theta} ds\right)^{1/(1-\gamma)} \\ &\leq C\left(\int_0^t \|u_s\|_{\rho+2}^{(\rho+1)\mu\cdot((\rho+2)/((\rho+1)\mu))} ds\right)^{\mu(\rho+1)/(1-\gamma)(\rho+2)} \\ &\quad + C\left(\int_0^t \|\nabla u\|_{r+1}^{\theta\cdot((1+r)/\theta)} ds\right)^{\theta/(1+r)(1-\gamma)} \\ &= C\left(\int_0^t \|u_s\|_{\rho+2}^{\rho+2} ds\right)^{\mu(\rho+1)/(1-\gamma)(\rho+2)} \\ &\quad + C\left(\int_0^t \|\nabla u\|_{r+1}^{1+r} ds\right)^{\theta/(1+r)(1-\gamma)}. \end{aligned} \tag{4.32}$$

Using (4.9), (4.32) and choosing μ satisfy $\mu(\rho+1)/((\rho+2)(1-\gamma)) = 1$, then $\theta/(1+r)(1-\gamma) < 1$.

So

$$\left|\frac{1}{\rho+1}\int_0^t\int_\Omega |u_s|^\rho u_s u\,dx\,ds\right|^{1/(1-\gamma)} \leq C\left(\int_0^t \|u_s\|_{\rho+2}^{\rho+2}ds + \int_0^t \|\nabla u\|_{r+1}^{1+r}ds\right) + \beta$$
$$\leq C\left(\int_0^t \|u_s\|_{\rho+2}^{\rho+2}ds + \int_0^t \|\nabla u\|_{r+1}^{1+r}ds + \int_0^t \|\Delta u\|_2^2 ds + \int_0^t \|u\|_{p+1}^{p+1}\right) + \beta, \tag{4.33}$$

where $\beta > 0$ is a constant. Finally, we can easily get

$$K^{1/(1-\gamma)} \leq 2^{1/(1-\gamma)}\left(Q(t) + \varepsilon^{1/(1-\gamma)}\left|\frac{1}{\rho+1}\int_0^t\int_\Omega |u_s|^\rho u_s u\,dx\,ds\right|^{1/(1-\gamma)}\right). \tag{4.34}$$

Combining (4.29) and (4.33)-(4.34), we have

$$K'(t) \geq CK^{1/(1-\gamma)}(t), \quad t \leq T, \tag{4.35}$$

for some constant $C > 0$. Integrating the above inequality in $(0, t)$, we get

$$K^{1/(1-\gamma)} \geq \frac{1}{\left(K^{-\gamma/(1-\gamma)}(0) - ct\right)^{1/\gamma}}, \quad \forall t \leq T. \tag{4.36}$$

The above inequality implies $K(t)$ blows-up on some time T^*. Since u exists globally, so we have

$$\frac{\varepsilon}{\rho+1}\int_0^t\int_\Omega |u_s|^\rho u_s u\,dx\,ds < \infty. \tag{4.37}$$

And we know that $K(t) \to \infty$, as $t \to T^*$, so

$$Q^{1-\gamma}(t) \longrightarrow \infty; \tag{4.38}$$

this implies $Q(t) \to \infty$, that is to say $-\int_0^t E(s)ds \to \infty$. Because

$$-\int_0^t E(s)ds \leq -tE(t), \tag{4.39}$$

we know

$$E(t) \longrightarrow -\infty, \quad \text{as } t \longrightarrow T^*. \tag{4.40}$$

This contradicts with the assumption that u is a global solution. So the solution of (1.1) blows-up on time T^*. □

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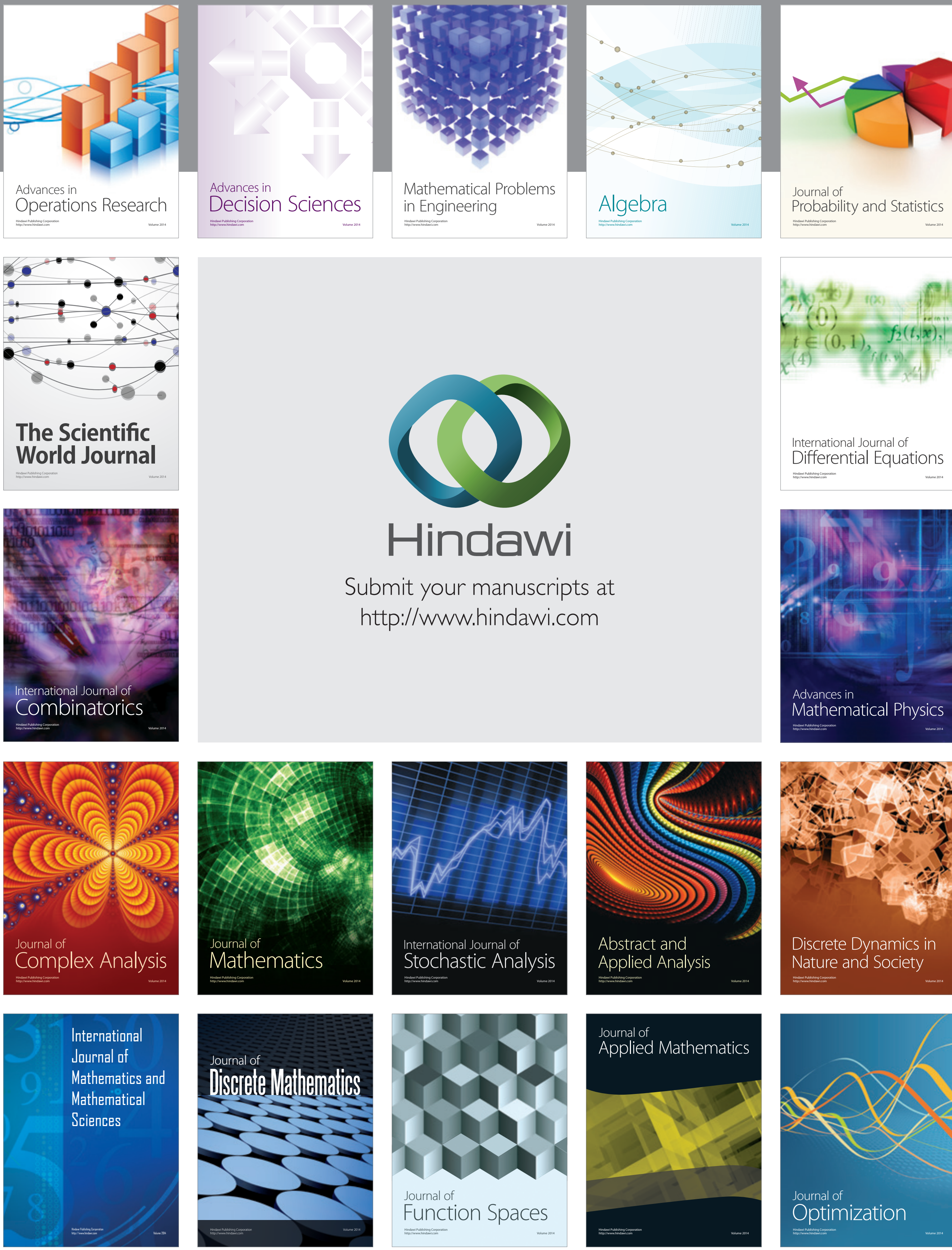
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