

Research Article

Periodic Solutions of Some Impulsive Hamiltonian Systems with Convexity Potentials

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We study the existence of periodic solutions of some second-order Hamiltonian systems with impulses. We obtain some new existence theorems by variational methods.

1. Introduction

Consider the following systems:

$$\begin{aligned} \ddot{u}(t) &= f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ \Delta \dot{u}(t_k) &= g_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \\ u(T) - u(0) &= \dot{u}(T) - \dot{u}(0) = 0, \end{aligned} \tag{1.1}$$

where $k \in \mathbb{Z}$, $u \in \mathbb{R}^n$, $\Delta \dot{u}(t_k) = \dot{u}(t_k^+) - \dot{u}(t_k^-)$ with $\dot{u}(t_k^\pm) = \lim_{t \rightarrow t_k^\pm} \dot{u}(t)$, $g_k(u) = \text{grad}_u G_k(u)$, $G_k \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for each $k \in \mathbb{Z}$, there exists an $m \in \mathbb{Z}$ such that $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, and we suppose that $f(t, u) = \text{grad}_u F(t, u)$ satisfies the following assumption.

(A) $F(t, x)$ is measurable in t for $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| + |f(t, x)| \leq a(|x|)b(t), \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and $t \in [0, T]$.

Many solvability conditions for problem (1.1) without impulsive effect are obtained, such as, the coercivity condition, the convexity conditions (see [1–4] and their references), the sublinear nonlinearity conditions, and the superlinear potential conditions. Recently, by using variational methods, many authors studied the existence of solutions of some second-order differential equations with impulses. More precisely, Nieto in [5, 6] considers linear conditions, [7–10] the sublinear conditions, and [11–16] the sublinear conditions and the other conditions. But to the best of our knowledge, except [7] there is no result about convexity conditions with impulsive effects. By using different techniques, we obtain different results from [7].

We recall some basic facts which will be used in the proofs of our main results. Let

$$H_T^1 = \left\{ u : [0, T] \rightarrow \mathbb{R}^n \text{ absolutely continuous; } u(0) = u(T), \dot{u}(t) \in L^2(0, T; \mathbb{R}^n) \right\}, \quad (1.3)$$

with the inner product

$$\langle u, v \rangle = \int_0^T (u(t), v(t)) dt + \int_0^T (\dot{u}(t), \dot{v}(t)) dt, \quad \forall u, v \in H_T^1, \quad (1.4)$$

where (\cdot, \cdot) denotes the inner product in \mathbb{R}^n . The corresponding norm is defined by

$$\|u\| = \left(\int_0^T (u(t), u(t)) dt + \int_0^T (\dot{u}(t), \dot{u}(t)) dt \right)^{1/2}, \quad \forall u \in H_T^1. \quad (1.5)$$

The space H_T^1 has some important properties. For $u \in H_T^1$, let $\bar{u} = (1/2T) \int_0^T u(t) dt$, and $\tilde{u} = u(t) - \bar{u}$. Then one has Sobolev's inequality (see Proposition 1.3 in [1]):

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt. \quad (1.6)$$

Consider the corresponding functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t)) dt + \sum_{k=1}^m G_k(u(t_k)). \quad (1.7)$$

It follows from assumption (A) and the continuity of g_k one has that φ is continuously differentiable and weakly lower semicontinuous on H_T^1 . Moreover, we have

$$\langle \varphi'(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t)) dt + \int_0^T (f(t, u(t)), v(t)) dt + \sum_{k=1}^m (g_k(u(t_k)), v(t_k)), \quad (1.8)$$

for $u, v \in H_T^1$ and φ' is weakly continuous and the weak solutions of problem (1.1) correspond to the critical points of φ (see [8]).

Theorem 1.1 ([2, Theorem 1.1]). *Suppose that V and W are reflexive Banach spaces, $\varphi \in C^1(V \times W, R)$, $\varphi(v, \cdot)$ is weakly upper semi-continuous for all $v \in V$, and $\varphi(\cdot, w) : V \rightarrow R$ is convex for all $w \in W$ and φ is weakly continuous. Assume that*

$$\varphi(0, w) \rightarrow -\infty \quad (1.9)$$

as $\|w\| \rightarrow \infty$ and for every $M > 0$,

$$\varphi(v, w) \rightarrow +\infty, \quad (1.10)$$

as $\|v\| \rightarrow \infty$ uniformly for $\|w\| \leq M$. Then φ has at least one critical point.

2. Main Results

Theorem 2.1. *Assume that assumption (A) holds. If further*

(H₁) $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$, and

(H₂) there exist $\eta, \theta > 0$ such that $G_k(x) \geq \eta|x| + \theta$, for all $x \in \mathbb{R}^n$, then (1.1) possesses at least one solution in H_T^1 .

Remark 2.2. (H₁) implies there exists a point \bar{x} for which

$$\int_0^T \nabla F(t, \bar{x}) dt = 0. \quad (2.1)$$

Proof of Theorem 2.1. It follows Remark 2.2, (1.6), and (H₂) that

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T (F(t, u(t)) - F(t, \bar{x})) dt + \int_0^T F(t, \bar{x}) dt + \sum_{k=1}^m G_k(u(t_k)) \\ &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), u(t) - \bar{x}) dt + \sum_{k=1}^m G_k(u(t_k)) \\ &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), \tilde{u}) dt + \sum_{k=1}^m \eta |\tilde{u} + \bar{u}| + m\theta \\ &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \left(\int_0^T |f(t, \bar{x})| dt \right) \|\tilde{u}\|_\infty + m\eta |\bar{u}| - m\eta \|\tilde{u}\|_\infty + m\theta \\ &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - C_0 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{1/2} + m\eta |\bar{u}| + m\theta, \end{aligned} \quad (2.2)$$

for all $u \in H_T^1$ and some positive constant C_0 . As $\|u\| \rightarrow \infty$ if and only if $(\|u\|^2 + \|\dot{u}\|_2^2)^{1/2} \rightarrow \infty$, we have $\varphi(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$. By Theorem 1.1 and Corollary 1.1 in [1], φ has a minimum point in H_T^1 , which is a critical point of φ . Hence, problem (1.1) has at least one weak solution. \square

Theorem 2.3. Assume that assumption (A) and (H_1) hold. If further

(H_3) there exist $\eta, \theta > 0$ and $\alpha \in (0, 2)$ such that $G_k(x) \leq \eta|x|^\alpha + \theta$ for all $x \in \mathbb{R}^n$ and

(H_4) there exist some $\beta > \alpha$ and $\gamma > 0$ such that

$$|x|^{-\beta} \int_0^T F(t, x) dt \leq -\gamma, \quad (2.3)$$

for $|x| \geq M$ and $t \in [0, T]$, where M is a constant, then (1.1) possesses at least one solution in H_T^1 .

Remark 2.4. We can find that our condition (H_4) is very different from condition (vii) in [7] since we prove this by the saddle point theorem substituted for the least action principle.

Proof of Theorem 2.3. We prove φ satisfies the (PS) condition at first. Suppose $\{u_n\}$ is such a sequence that $\{\varphi(u_n)\}$ is bounded and $\lim_{n \rightarrow \infty} \varphi'(u_n) = 0$. We will prove it has a convergent subsequence. By (H_3) and (1.6), we have

$$\begin{aligned} \sum_{k=1}^m G_k(u(t_k)) &\leq \sum_{k=1}^m \eta |\tilde{u}(t_k) + \bar{u}|^\alpha + m\theta \\ &\leq 4m\eta (|\tilde{u}(t_k)|^\alpha + |\bar{u}|^\alpha) + m\theta \\ &\leq C_1 \|\dot{u}\|_2^\alpha + C_2 |\bar{u}|^\alpha + C_3, \end{aligned} \quad (2.4)$$

for some positive constants C_1, C_2, C_3 . By Remark 2.2, (1.6), and (2.4), we have

$$\begin{aligned} \varphi(u_n) &= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T (F(t, u_n(t)) - F(t, \bar{x})) dt + \int_0^T F(t, \bar{x}) dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), u_n(t) - \bar{x}) dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), \tilde{u}_n) dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &\geq \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt - \left(\int_0^T |f(t, \bar{x})| dt \right) \|\tilde{u}_n\|_\infty - C_1 \|\dot{u}_n\|_2^\alpha - C_2 |\bar{u}_n|^\alpha - C_4 \\ &\geq \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt - C_5 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{1/2} - C_1 \|\dot{u}_n\|_2^\alpha - C_2 |\bar{u}_n|^\alpha - C_4, \end{aligned} \quad (2.5)$$

for some positive constants C_4, C_5 , which implies that

$$C |\bar{u}_n|^{\alpha/2} \geq \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{1/2} - C_6, \quad (2.6)$$

for some positive constants C, C_6 . By (1.6), the above inequality implies that

$$\|\tilde{u}_n\|_\infty \leq C_7 \left(|\bar{u}_n|^{\alpha/2} + 1 \right), \quad (2.7)$$

for the positive constant C_7 . The one has

$$|u_n(t)| \geq |\bar{u}_n| - |\tilde{u}_n| \geq |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geq |\bar{u}_n| - C_7 \left(|\bar{u}_n|^{\alpha/2} + 1 \right), \quad \forall t \in [0, T]. \quad (2.8)$$

If $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\bar{u}_n| \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (2.9)$$

By (2.8) and (2.9), we have

$$|u_n(t)| \geq \frac{1}{2} |\bar{u}_n|, \quad (2.10)$$

for all large n and every $t \in [0, T]$. By (2.10) and (H_4) , one has $|u_n(t)| \geq M$ for all large n . It follows from (H_4) , (2.4), (2.6), (2.7), and above inequality that

$$\begin{aligned} \varphi(u_n) &\leq \left(C |\bar{u}_n|^{\alpha/2} + C_6 \right)^2 - \int_0^T \gamma |u_n(t)|^\beta dt + C_2 \|\tilde{u}\|_\infty^\alpha + C_2 |\bar{u}|^\alpha + C_3 \\ &\leq \left(C |\bar{u}_n|^{\alpha/2} + C_6 \right)^2 - 2^{-\beta} |\bar{u}_n|^\beta T \gamma + C_8 \left(|\bar{u}_n|^{\alpha/2} + 1 \right)^\alpha + C_2 |\bar{u}|^\alpha + C_3, \end{aligned} \quad (2.11)$$

for large n and the positive constant C_8 , which contradicts the boundedness of $\varphi(u_n)$ since $\beta > \alpha$. Hence $(|\bar{u}_n|)$ is bounded. Furthermore, (u_n) is bounded by (2.6). A similar calculation to Lemma 3.1 in [9] shows that φ satisfies the (PS) condition. We now prove that φ satisfies the other conditions of the saddle point theorem. Assume that $\widetilde{H}_T^1 = \{u \in H_T^1 : \bar{u} = 0\}$, then $H_T^1 = \widetilde{H}_T^1 \oplus \mathbb{R}^n$. From above calculation, one has

$$\varphi(u) \geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - C_5 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{1/2} - C_1 \|\dot{u}\|_2^\alpha - C_4, \quad (2.12)$$

for all $u \in \widetilde{H}_T^1$, which implies that

$$\varphi(u) \rightarrow +\infty, \quad (2.13)$$

as $\|u\| \rightarrow \infty$ in \widetilde{H}_T^1 . Moreover, by (H_3) and (H_4) we have

$$\begin{aligned}\varphi(x) &= \int_0^T F(t, x) dt + \sum_{k=1}^m G_k(x) \\ &\leq -T\gamma|x|^\beta + m\eta|x|^\alpha + m\theta,\end{aligned}\tag{2.14}$$

for $|x| > M$, which implies that

$$\varphi(x) \longrightarrow -\infty,\tag{2.15}$$

as $|x| \rightarrow \infty$ in \mathbb{R}^n since $\beta > \alpha$. Now Theorem 2.3 is proved by (2.13), (2.15), and the saddle point theorem. \square

Theorem 2.5. *Assume that assumption (A) holds. Suppose that $F(t, \cdot), G_k(x)$ are concave and satisfy*

(H_5) $G_k(x) \leq -\eta|x| + \theta$ for some positive constant $\eta, \theta > 0$, then (1.1) possesses at least one solution in H_T^1 .

Proof of Theorem 2.5. Consider the corresponding functional φ on $\mathbb{R}^n \times \widetilde{H}_T^1$ given by

$$\varphi(u) = -\frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, u(t)) dt - \sum_{k=1}^m G_k(u(t_k)),\tag{2.16}$$

which is continuously differentiable, bounded, and weakly upper semi-continuous on H_T^1 . Similar to the proof of Lemma 3.1 in [2], one has that $\varphi(x + w)$ is convex in $x \in \mathbb{R}^n$ for every $w \in \widetilde{H}_T^1$. By the condition, we have $-G_k(x + w) \geq -2G_k((1/2)x) + G_k(-w)$. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\varphi(x + w) &= -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - \int_0^T F(t, x + w) dt - \sum_{k=1}^m G_k(x + w) \\ &\geq -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - \left(\int_0^T |f(t, \bar{x})| dt \right) \|w\|_\infty - \sum_{k=1}^m G_k(x + w) + C_9 \\ &\geq -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - C_0 \left(\int_0^T |\dot{w}|^2 dt \right)^{1/2} - 2G_k\left(\frac{1}{2}x\right) + G_k(-w) + C_9,\end{aligned}\tag{2.17}$$

which means $\varphi(x+w) \rightarrow +\infty$ as $|x| \rightarrow \infty$, uniformly for $w \in \widetilde{H}_T^1$ with $\|w\| \leq M$ by (H_5) and (1.6). On the other hand,

$$\begin{aligned} \varphi(w) &= -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - \int_0^T F(t, w) dt - \sum_{k=1}^m G_k(w) \\ &\leq -\frac{1}{2} \int_0^T |\dot{w}|^2 dt + C_0 \left(\int_0^T |\dot{w}|^2 dt \right)^{1/2} + m\eta \|w\|_\infty + C_9, \end{aligned} \quad (2.18)$$

which implies that $\varphi(w) \rightarrow -\infty$ as $\|w\| \rightarrow \infty \in \widetilde{H}_T^1$ by (H_5) and (1.6). We complete our proof by Theorem 1.1. \square

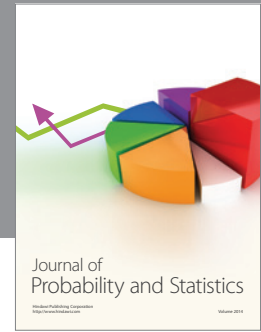
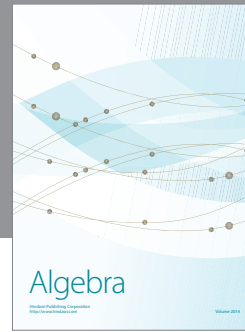
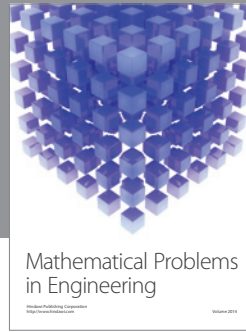
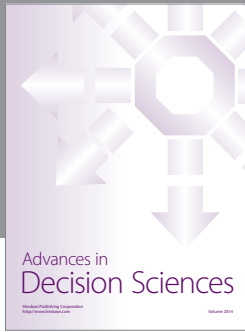
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