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Research Article

Remarks on the Pressure Regularity Criterion of the Micropolar Fluid Equations in Multiplier Spaces

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This study is devoted to investigating the regularity criterion of weak solutions of the micropolar fluid equations in $\mathbb{R}^3$. The weak solution of micropolar fluid equations is proved to be smooth on $(0,T]$ when the pressure $\pi(x,t)$ satisfies the following growth condition in the multiplier spaces $\dot{X}^r$, $\int_0^T \|\pi(s,\cdot)\|_{\dot{X}^r}^{2/(2-r)}/(1+\ln(e+\|\pi(s,\cdot)\|_{L^2}))$, $ds < \infty$, and $0 \leq r \leq 1$. The previous results on Lorentz spaces and Morrey spaces are obviously improved.

1. Introduction

Consider the Cauchy problem of the three-dimensional (3D) micropolar fluid equations with unit viscosities

$$\begin{gathered}\nabla \cdot u = 0,\\ \partial_t u - \Delta u - \nabla \times w + \nabla \pi + u \cdot \nabla u = 0,\\ \partial_t w - \Delta w - \nabla \nabla \cdot w + 2w - \nabla \times u + u \cdot \nabla w = 0\end{gathered} \tag{1.1}$$

associated with the initial condition:

$$u(x,0) = u_0, \qquad w(x,0) = w_0, \tag{1.2}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, and $w(x,t) = (w_1(x,t), w_2(x,t), w_3(x,t))$ are the unknown velocity vector field and the microrotation vector field. $\pi(x,t)$ is the unknown scalar pressure field. u_0 and w_0 represent the prescribed initial data for the velocity and microrotation fields.

Micropolar fluid equations introduced by Eringen [1] are a special model of the non-Newtonian fluids (see [2–6]) which is coupled with the viscous incompressible Navier-Stokes model, microrotational effects, and microrotational inertia. When the microrotation effects are neglected or $w = 0$, the micropolar fluid equations (1.1) reduce to the incompressible Navier-Stokes flows (see, e.g., [7, 8]):

$$\begin{gathered} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla \pi = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0. \end{gathered} \tag{1.3}$$

That is to say, Navier-Stokes equations are viewed as a subclass of the micropolar fluid equations.

Mathematically, there is a large literature on the existence, uniqueness and large time behaviors of solutions of micropolar fluid equations (see [9–15] and references therein); however, the global regularity of the weak solution in the three-dimensional case is still a big open problem. Therefore it is interesting and important to consider the regularity criterion of the weak solutions under some assumptions of certain growth conditions on the velocity or on the pressure.

On one hand, as for the velocity regularity criteria, by means of the Littlewood-Paley decomposition methods, Dong and Chen [16] proved the regularity of weak solutions under the velocity condition:

$$\nabla u \in L^q\left(0, T; \dot{B}^0_{p,r}\left(\mathbb{R}^3\right)\right) \tag{1.4}$$

with

$$\frac{2}{q} + \frac{3}{p} = 2, \quad \frac{3}{2} < p \leq \infty, \quad r \leq \frac{2p}{3}. \tag{1.5}$$

Moreover, the result is further improved by Dong and Zhang [17] in the margin case:

$$u \in L^{2/(1+r)}\left(0, T; B^r_{\infty,\infty}\left(\mathbb{R}^3\right)\right), \quad -1 < r < 1. \tag{1.6}$$

On the other hand, as for the pressure regularity criteria, Yuan [18] investigated the regularity criterion of weak solutions of the micropolar fluid equations in Lebesgue spaces and Lorentz spaces:

$$\begin{gathered} \pi \in L^q\left(0, T; L^p\left(\mathbb{R}^3\right)\right), \quad \text{for } \frac{2}{q} + \frac{3}{p} = 2, \ \frac{3}{2} < p < \infty, \\ \pi \in L^q\left(0, T; L^{p,\infty}\left(\mathbb{R}^3\right)\right), \quad \text{for } \frac{2}{q} + \frac{3}{p} = 2, \ \frac{3}{2} < p < \infty, \end{gathered} \tag{1.7}$$

where $L^{p,\infty}(\mathbb{R}^3)$ is the Lorents space (see the definitions in the next section).

Recently, Dong et al. [19] improved the pressure regularity of the micropolar fluid equations in Morrey spaces:

$$\pi \in L^q\left(0, T; \dot{M}_{p,r}\left(\mathbb{R}^3\right)\right), \tag{1.8}$$

where

$$\frac{2}{q} + \frac{3}{p} = 2, \quad \frac{3}{2} < p \leq \infty, \quad r \leq p. \tag{1.9}$$

Furthermore, Jia et al. [20] refined the regularity from Morrey spaces to Besov spaces:

$$\pi \in L^q\left(0, T; B^r_{p,\infty}\left(\mathbb{R}^3\right)\right) \tag{1.10}$$

with

$$\frac{2}{q} + \frac{3}{p} = 2 + r, \quad \frac{3}{2+r} < p < \infty, \ -1 < r \leq 1. \tag{1.11}$$

One may also refer to some interesting results on the regularity criteria of Newtonian and non-Newtonian fluid equations (see [21–27] and references therein).

The aim of the present study is to investigate the pressure regularity criterion of the three-dimensional micropolar fluid equations in the multiplier spaces which are larger than the Lebesgue spaces, Lorentz spaces, and Morrey spaces.

2. Preliminaries and Main Result

Throughout this paper, we use c to denote the constants which may change from line to line. $L^p(\mathbb{R}^3)$, $W^{k,p}(\mathbb{R}^3)$ with $k \in \mathbf{Z}$, $1 \leq p \leq \infty$ denote the usual Lebesgue space and Sobolev space. $\dot{H}^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ denote the fractional Sobolev space with

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} \left|\hat{f}\right|^2 d\xi\right)^{1/2}. \tag{2.1}$$

Consider a measurable function f and define for $t \geq 0$ the Lebesgue measure

$$m(f, t) := m\left\{x \in \mathbb{R}^3 : |f(x)| > t\right\} \tag{2.2}$$

of the set $\{x \in \mathbb{R}^3 : |f(x)| > t\}$. The Lorentz space is defined by $f \in L^{p,q}(\mathbb{R}^3)$ if and only if

$$\begin{aligned} \|f\|_{L^{p,q}} &= \left(\int_0^\infty t^q (m(f,t))^{q/p} \frac{dt}{t}\right)^{1/q} < \infty, \quad \text{for } 1 \leq q < \infty, \\ \|f\|_{L^{p,\infty}} &= \sup_{t \geq 0}\left(t (m(f,t))^{1/p}\right) < \infty, \quad \text{for } q = \infty. \end{aligned} \tag{2.3}$$

We defined $\dot{M}_{p,q}(\mathbb{R}^3)$, $1 \le p, q \le \infty$ the homogeneous Morrey space associated with norm

$$\|f\|_{\dot{M}_{p,q}} = \sup_{r>0} \sup_{x\in\mathbb{R}^3} r^{3/p-3/q} \left(\int_{|x-y|<r} |f(y)|^q dy \right)^{1/q} < \infty. \tag{2.4}$$

We now recall the definition and some properties of the multiplier space $\dot{X}^r$.

Definition 2.1 (see Lemarié-Rieusset [28]). For $0 \le r < 3/2$, the space $\dot{X}^r$ is defined as the space of $f(x) \in L^2_{\text{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}^r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \left(\int_{\mathbb{R}^3} |fg|^2 dx \right)^{1/2}. \tag{2.5}$$

According the above definition of the multiplier space, it is not difficult to verify the homogeneity properties. For all $x_0 \in \mathbb{R}^3$

$$\begin{aligned} \|f(\cdot + x_0)\|_{\dot{X}^r} &= \|f\|_{\dot{X}^r}, \\ \|f(\lambda)\|_{\dot{X}^r} &= \frac{1}{\lambda^r} \|f\|_{\dot{X}^r}, \quad \lambda > 0. \end{aligned} \tag{2.6}$$

When $r = 0$, it is clear that (see Lemarié-Rieusset [28])

$$\dot{X}^0 \cong \text{BMO}, \tag{2.7}$$

where BMO denotes the homogenous space of bounded mean oscillations associated with the norm

$$\|f\|_{\text{BMO}} = \sup_{x\in\mathbb{R}^3, r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f(y) - \frac{1}{|B_r(y)|} \int_{B_r(y)} f(z) dz \right| dy. \tag{2.8}$$

In particular, the following imbedding (see Lemarié-Rieusset [28])

$$L^p\left(\mathbb{R}^3\right) \subset L^{p,\infty}\left(\mathbb{R}^3\right) \subset \dot{M}_{p,q}\left(\mathbb{R}^3\right) \subset \dot{X}^r\left(\mathbb{R}^3\right) \subset \dot{B}^{-r}_{\infty,\infty}\left(\mathbb{R}^3\right), \quad p = \frac{3}{r} > q > 2 \tag{2.9}$$

holds true.

In order to state our main results, we recall the definition of the weak solution of micropolar flows (see, e.g., Łukaszewicz [9]).

Definition 2.2. Let $T > 0$, $(u_0, w_0) \in L^2(\mathbb{R}^3)$, and $\nabla \cdot u_0 = 0$. (u, w) is termed as a weak solution to the 3D micropolar flows (1.1) and (1.2) on $(0, T)$, if (u, w) satisfies the following properties:

(i) $(u, w) \in L^{\infty}(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$;

(ii) equations (1.1) and (1.2) are valid in the sense of distributions.

Our main results are now read as follows.

Theorem 2.3. *Suppose $T > 0$, $(u_0, w_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$, and $\nabla \cdot u_0 = 0$ in the sense of distributions. Assume that (u, w) is a weak solution of the 3D micropolar fluid flows (1.1) and (1.2) on $(0, T)$. If the pressure $\pi(x, t)$ satisfies the logarithmically growth condition:*

$$\int_0^T \frac{\|\pi(s,\cdot)\|_{\dot{X}^r}^{2/(2-r)}}{1 + \ln(e + \|\pi(s,\cdot)\|_{L^2})} ds < \infty, \quad 0 \le r \le 1, \tag{2.10}$$

then the weak solution (u, w) is regular on $(0, T]$.

Thanks to

$$\int_0^T \frac{\|\pi(s,\cdot)\|_{\dot{X}^r}^{2/(2-r)}}{1 + \ln(e + \|\pi(s,\cdot)\|_{L^2})} ds \le \int_0^T \|\pi(s,\cdot)\|_{\dot{X}^r}^{2/(2-r)} ds, \tag{2.11}$$

it is easy to deduce the following pressure regularity criterion of the three-dimensional micropolar equations (1.1) and (1.2).

Corollary 2.4. *On the substitution of the pressure condition (2.10) by the following conditions:*

$$\int_0^T \|\pi(s,\cdot)\|_{\dot{X}^r}^{2/(2-r)} ds < \infty, \quad 0 \le r \le 1, \tag{2.12}$$

the conclusion of Theorem 2.3 holds true.

Remark 2.5. According to the embedding relation (2.9), our results obviously largely improve the previous results (1.7) and (1.8). Moreover, it seems incomparable with the Besov space (1.10).

Remark 2.6. Furthermore, since we have no additional growth condition on the microrotation vector field $w(x, t)$, Theorem 2.3 is also valid for the pressure regularity problem of the three-dimensional Navier-Stokes equations (see, e.g., Zhou [29, 30]).

3. Proof of Theorem 2.3

In order to prove our main results, we first recall the following local existence theorem of the three-dimensional micropolar fluid equations (1.1) and (1.2).

Lemma 3.1 (see Dong et al. [19]). *Assume* $3 < p < \infty$ *and* $(u_0, w_0) \in L^p(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *in the sense of distributions. Then there exist a constant* $T > 0$ *and a unique strong solution* (u, w) *of the 3D micropolar fluid equations* (1.1) *and* (1.2) *such that*

$$u \in BC\left([0,T); L^p\left(\mathbb{R}^3\right)\right), \qquad t^{1/2}\nabla u \in BC\left([0,T); L^p\left(\mathbb{R}^3\right)\right). \tag{3.1}$$

By means of the local existence result, (1.1) and (1.2) with $(u_0, w_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ admit a unique L^4-strong solution (u, w) on a maximal time interval. For the notation simplicity, we may suppose that the maximal time interval is $[0, T)$. Thus, to prove Theorem 2.3, it remains to show that

$$\lim_{t \to T}(\|u(t)\|_4 + \|w(t)\|_4) < \infty. \tag{3.2}$$

This will lead to a contradiction to the estimates to be derived below. We now begin to follow these arguments.

Taking the inner product of the second equation of (1.1) with $u|u|^2$ and the third equation of (1.1) with $w|w|^2$, respectively, and integrating by parts, it follows that

$$\begin{gathered}
\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \||u|\nabla u\|_{L^2}^2 + \frac{1}{2}\left\|\nabla|u|^2\right\|_{L^2}^2 = \int_{\mathbb{R}^3}(\nabla \times w)\cdot u|u|^2 dx - \int_{\mathbb{R}^3} u \cdot \nabla\pi|u|^2 dx, \\
\frac{1}{4}\frac{d}{dt}\|w\|_{L^4}^4 + \||w|\nabla w\|_{L^2}^2 + \frac{1}{2}\||w|\nabla \cdot w\|_{L^2}^2 + \left\|\nabla|w|^2\right\|_{L^2}^2 = \int_{\mathbb{R}^3}(\nabla \times u)\cdot w|w|^2 dx - 2\|w\|_{L^4}^4,
\end{gathered} \tag{3.3}$$

where we have used the following identities due to the divergence free property of the velocity field u:

$$\begin{gathered}
\int_{\mathbb{R}^3}(u \cdot \nabla u)\cdot u|u|^2 dx = 0, \\
\int_{\mathbb{R}^3}(u \cdot \nabla w)\cdot w|w|^2 dx = 0, \\
(\nabla u)\cdot\left(\nabla\left(u|u|^2\right)\right) = |\nabla u|^2|u|^2 + \frac{1}{2}\left|\nabla|u|^2\right|^2, \\
(\nabla w)\cdot\left(\nabla\left(w|w|^2\right)\right) = |\nabla w|^2|w|^2 + \frac{1}{2}\left|\nabla|w|^2\right|^2, \\
\int_{\mathbb{R}^3}(\nabla \cdot w)\nabla\cdot\left(w|w|^2\right)dx = \int_{\mathbb{R}^3}|\nabla \cdot w|^2|w|^2 dx + \int_{\mathbb{R}^3}(\nabla \cdot w)w\cdot\nabla|w|^2 dx \\
\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla \cdot w|^2|w|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|w|^2\right|^2 dx.
\end{gathered} \tag{3.4}$$

Furthermore, applying Young inequality, Hölder inequality, and integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\nabla \times w) \cdot u|u|^2 dx + \int_{\mathbb{R}^3} (\nabla \times u) \cdot w|w|^2 dx - 2\|w\|_{L^4}^4 \\
&\quad \le \|w\|_{L^4} \||u|\nabla u\|_{L^2} \|u\|_{L^4} + \|u\|_{L^4} \||w|\nabla w\|_{L^2} \|w\|_{L^4} - 2\|w\|_4^4 \\
&\quad \le \|u\|_{L^4}^4 + \||w|\nabla w\|_{L^2}^2 + \||u|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.5}$$

Combining the above inequalities, it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4\right) + 2\||u|\nabla u\|_{L^2}^2 + 2\left\|\nabla |u|^2\right\|_{L^2}^2 \\
&\qquad \le C\left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4\right) + C\left|\int_{\mathbb{R}^3} u \cdot \nabla \pi |u|^2 dx\right|.
\end{aligned} \tag{3.6}$$

In order to estimate the last term of the right-hand side of (3.6), taking the divergence operator $\nabla\cdot$ to the first equation of (1.1) produces the expression of the pressure:

$$\pi = (-\Delta)^{-1} \nabla \cdot (u \cdot \nabla u). \tag{3.7}$$

Employing Calderón-Zygmund inequality and the divergence free condition of the velocity derives the estimate of the pressure:

$$\begin{aligned}
\|\pi\|_{L^r} &\le c\|u\|_{L^{2r}}^2, \\
\|\nabla \pi\|_{L^r} &\le c\|u \cdot \nabla u\|_{L^r}, \quad 1 < r < \infty.
\end{aligned} \tag{3.8}$$

Therefore, we estimate the pressure term as

$$\left|-\int_{\mathbb{R}^3} u \cdot \nabla \pi |u|^2 dx\right| \le \int_{\mathbb{R}^3} |\pi||u|\left|\nabla |u|^2\right| dx \le \frac{1}{2}\left[\int_{\mathbb{R}^3} \left|\nabla |u|^2\right|^2 dx + \int_{\mathbb{R}^3} |\pi|^2 |u|^2 dx\right]. \tag{3.9}$$

Now we estimate the integral

$$I =: \int_{\mathbb{R}^3} |\pi|^2 |u|^2 dx \tag{3.10}$$

on the right-hand side of (3.9). By the Hölder inequality and the Young inequality we have

$$\begin{aligned} I &\leq \left\| \pi |u|^2 \right\|_{L^2} \|\pi\|_{L^2} \\ &\leq \|\pi\|_{\dot{X}^r} \left\| |u|^2 \right\|_{\dot{H}^r} \|\pi\|_{L^2} \\ &\leq \|\pi\|_{\dot{X}^r} \left\| |u|^2 \right\|_{L^2}^{1-r} \left\| \nabla |u|^2 \right\|_{L^2}^{r} \|u\|_{L^4}^2 \\ &\leq \|\pi\|_{\dot{X}^r} \|u\|_{L^4}^{2(2-r)} \left\| \nabla |u|^2 \right\|_{L^2}^{r} \\ &\leq \epsilon \left\| \nabla |u|^2 \right\|_{L^2}^{2} + C(\epsilon) \|\pi\|_{\dot{X}^r}^{2/(2-r)} \|u\|_{L^4}^4, \end{aligned} \tag{3.11}$$

where we have used the following interpolation inequality:

$$\|f\|_{\dot{H}^r} \leq \|f\|_{L^2}^{1-r} \|\nabla f\|_{L^2}^{r}. \tag{3.12}$$

Hence, combining the above inequalities, we derive

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4 \right) + \||u|\nabla u\|_{L^2}^2 + \||w|\nabla w\|_{L^2}^2 \\ &\quad \leq C\left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4 \right) + C(\epsilon) \|\pi\|_{\dot{X}^r}^{2/(2-r)} \|u\|_{L^4}^4. \end{aligned} \tag{3.13}$$

Furthermore, we have the second term of the right-hand side of (3.13) rewritten as

$$\begin{aligned} C(\epsilon)\|\pi\|_{\dot{X}^r}^{2/(2-r)} \|u\|_{L^4}^4 &\leq C\left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4 \right) \|\pi\|_{\dot{X}^r}^{2/(2-r)} \\ &\leq C \frac{\|\pi(t,\cdot)\|_{\dot{X}^r}^{2/(2-r)}}{1+\ln(e+\|\pi(t,\cdot)\|_{L^2})} \left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4 \right) [1+\ln(e+\|\pi(t,\cdot)\|_{L^2})] \\ &\leq C \frac{\|\pi(t,\cdot)\|_{\dot{X}^r}^{2/(2-r)}}{1+\ln(e+\|\pi(t,\cdot)\|_{L^2})} \left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4 \right) \left[1+\ln\left(e+\|u(t,\cdot)\|_{L^4}^2\right)\right] \\ &\leq C \frac{\|\pi(t,\cdot)\|_{\dot{X}^r}^{2/(2-r)}}{1+\ln(e+\|\pi(t,\cdot)\|_{L^2})} \left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4 \right) \left[1+\ln\left(e+\|u\|_{L^4}^4 + \|w\|_{L^4}^4\right)\right]. \end{aligned} \tag{3.14}$$

Inserting (3.14) into (3.13) and applying the Gronwall inequality, one shows that

$$\begin{aligned} 1+\ln\left(\|u\|_{L^4}^4 + \|w\|_{L^4}^4 \right) &\leq \left(1+\ln\left(\|u_0\|_{L^4}^4 + \|w_0\|_{L^4}^4 \right) \right) \\ &\quad \times \exp\left\{ CT + \int_0^T \frac{\|\pi(s,\cdot)\|_{\dot{X}^r}^{2/(2-r)}}{1+\ln(e+\|\pi(s,\cdot)\|_{L^2})} ds \right\} \end{aligned} \tag{3.15}$$

which implies

$$\int_{\mathbb{R}^3} \left(|u(t)|^4 + |w(t)|^4 \right) dx < \infty, \quad 0 < t \leq T. \tag{3.16}$$

Hence we complete the proof of Theorem 2.3.

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