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Research Article

A Predator-Prey Model with Functional Response and Stage Structure for Prey

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A predator-prey system with Holling type II functional response and stage structure for prey is presented. The local and global stability are studied by analyzing the associated characteristic transcendental equation and using comparison theorem. The existence of a Hopf bifurcation at the positive equilibrium is also studied. Some numerical simulations are also given to illustrate our results.

1. Introduction

Predator-prey system is a very classical and important mathematical model in population dynamics, there are many studies in progress nowadays (see [3–5]). In recent years, many scholars are giving their attention to predator-prey model with stage structure, and many results have been published. In [6, 7], the influences of prey or predator with stage structure to the state of the ecosystem are studied, respectively. A time-delay model of single-species growth incorporating stage structure as a reasonable generation of the logistic model was derived in [8]. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immatures and a reduced survival of immatures to their maturity. The model takes the form

$$\begin{aligned}\dot{x}_i(t) &= bx_m(t) - d_1x_i(t) - be^{-d_1\tau}x_m(t-\tau),\\ \dot{x}_m(t) &= be^{-d_1\tau}x_m(t-\tau) - d_2x_m(t) - \beta x_m^2(t),\end{aligned} \tag{1.1}$$

where $x_i(t)$ and $x_m(t)$ denote the immature and mature population densities, respectively; b represents the birth rate of the immature population; d_1 and d_2 are the death rate of immature and mature population, respectively; τ denotes the length of time from the birth to maturity. It is shown in [8] that the introduction of stage structure does not affect the permanence of the species, but, under some hypotheses, it may maximize the total carrying capacity of the population.

Based on the previous model, several predator-prey models with stage structure are developed and studied (see [9–15]). Recently, Song et al. [16] investigated the following model with stage structure for prey:

$$\begin{aligned}
\dot{x}_i(t) &= bx_m(t) - d_1x_i(t) - be^{-d_1\tau}x_m(t-\tau),\\
\dot{x}_m(t) &= be^{-d_1\tau}x_m(t-\tau) - \beta x_m^2(t) - \frac{a_1x_m(t)y(t)}{x_m(t)+cy(t)},\\
\dot{y}(t) &= \frac{a_2x_m(t)y(t)}{x_m(t)+cy(t)} - dy(t).
\end{aligned}\tag{1.2}$$

Their results showed that the positive equilibrium of system (1.2) was globally stable under some conditions.

On the other hand, the functional response plays an important role in all prey interactions. A functional response refers to the change in the density of prey attached per unit time per predator as the prey density changes. One significant component of the predator-prey relationship is the predator's functional responses. When the predator spends some time searching for prey and some time, exclusive of searching, processing each captured prey item (i.e., handling time), Holling type II is a useful functional response (see [17–21]). For example, Sun et al. [22] considered the following model with Holling type II functional response and stage structure for predator:

$$\begin{aligned}
\dot{x}(t) &= bx(t) - \beta x^2(t) - \frac{a_1x(t)y(t)}{1+cx(t)},\\
\dot{Y}(t) &= \frac{a_2x(t)y(t)}{1+cx(t)} - d_1Y(t) - a_2\frac{x(t-\tau)y(t-\tau)}{1+cx(t-\tau)}e^{-d_1\tau},\\
\dot{y}(t) &= a_2\frac{x(t-\tau)y(t-\tau)}{1+cx(t-\tau)}e^{-d_1\tau} - d_2y(t).
\end{aligned}\tag{1.3}$$

By analyzing the corresponding characteristic equations, the local stability of equilibria and existence of a Hopf bifurcation at the positive equilibrium are obtained in [22].

In this paper, we consider and study the following model with stage structure for prey:

$$\begin{aligned}
\dot{x}_i(t) &= bx_m(t) - d_1x_i(t) - be^{-d_1\tau}x_m(t-\tau),\\
\dot{x}_m(t) &= be^{-d_1\tau}x_m(t-\tau) - d_2x_m(t) - \beta x_m^2(t) - \frac{a_1x_m(t)y(t)}{1+cx_m(t)},\\
\dot{y}(t) &= \frac{a_2x_m(t)y(t)}{1+cx_m(t)} - dy(t).
\end{aligned}\tag{1.4}$$

For continuity of the initial conditions, we further require

$$x_i(t) > 0, \quad x_m(t) > 0, \quad y(t) > 0, \quad t \in [-\tau, 0],$$
$$x_i(0) = b\int_{-\tau}^{0} x_m(s)e^{d_1 s}ds. \tag{1.5}$$

The remainder of the paper is organized as follows: in the next section, results on positivity and boundedness of solutions are presented. In Section 3, through using geometric stability switch criteria [1] to analyze the corresponding characteristic equations, we discuss the local stability of equilibria. Further, we study the existence of a Hopf bifurcation at the positive equilibria. The global stability of boundary equilibrium is obtained by using comparison theorem in Section 4. In Section 5, we try to interpret our results by numerical simulation. At last, we give some discussions.

2. Positive and Bounded

In this section, we discuss the positivity and boundedness of the solutions of system (1.4). Positivity means that the species is persistent and boundedness implies a natural restriction, which shows our model (1.4) is reasonable.

Theorem 2.1. *The solutions of system* (1.4) *with initial conditions* (1.5) *are positive for all* $t \geq 0$*.*

Proof. It is easy to see that $x_m(t)$ and $y(t)$ are positive for all $t \geq 0$.

We only show $x_i(t) > 0$.

Since

$$x_m(t) > 0, \tag{2.1}$$

hence

$$\dot{x}_i(t) > -d_1 x_i(t) - be^{-d_1\tau}x_m(t-\tau). \tag{2.2}$$

Considering the equation

$$\dot{u}(t) = -d_1 u(t) - be^{-d_1\tau}x_m(t-\tau),$$
$$u(0) = x_i(0), \tag{2.3}$$

we have $x_i(t) > u(t)$ on $t \in (0, \tau]$, and from variation-of-constants formula, we get

$$u(t) = e^{-d_1 t}\left[x_i(0) - \int_0^t be^{d_1(s-\tau)}x_m(s-\tau)ds\right], \tag{2.4}$$

hence,

$$\begin{aligned}
u(\tau) &= e^{-d_1\tau}\left[x_i(0) - \int_0^{\tau} be^{d_1(s-\tau)}x_m(s-\tau)ds\right] \\
&= e^{-d_1\tau}\left[x_i(0) - \int_{-\tau}^{0} be^{d_1\theta}x_m(\theta)d\theta\right] \\
&= 0.
\end{aligned} \tag{2.5}$$

Thus, $x_i(t) > 0$ on $t \in [0,\tau]$. By induction, we can show that $x_i(t) > 0$ for all $t \geq 0$. The proof is complete. □

Theorem 2.2. *Every solution of system* (1.4) *with initial conditions* (1.5) *are bounded for all* $t \geq 0$, *and all of these solutions are ultimately bounded.*

Proof. Let $V(t) = a_2x_i(t) + a_2x_m(t) + a_1y(t)$, calculating the derivative of $V(t)$ with respect to t along the positive solution of system (1.4), we have

$$\begin{aligned}
\dot{V}(t) &= a_2\dot{x}_i(t) + a_2\dot{x}_m(t) + a_1\dot{y}(t) \\
&= a_2\left[bx_m(t) - d_1x_i(t) - d_2x_m(t) - \beta x_m^2(t) - \frac{a_1x_m(t)y(t)}{1+cx_m(t)}\right] \\
&\quad + a_1\left[-dy(t) + \frac{a_2x_m(t)y(t)}{1+cx_m(t)}\right] \\
&= (a_2b - a_2d_2)x_m(t) - a_2d_1x_i(t) - a_1dy(t) - a_2\beta x_m^2(t).
\end{aligned} \tag{2.6}$$

For a small positive constant $s \leq \min\{d_1, d\}$,

$$\begin{aligned}
\dot{V}(t) + sV(t) &= (a_2b_1 - a_2d_2 + a_2s)x_m(t) - (a_2d_1 - a_2s)x_i(t) \\
&\quad - (a_1d - a_1s)y(t) - a_2\beta x_m^2(t) \\
&< a_2(b - d_2 + s)x_2(t) - a_2\beta x_2^2(t).
\end{aligned} \tag{2.7}$$

Hence there exists a positive constant K, such that

$$\dot{V}(t) + sV(t) \leq K, \tag{2.8}$$

thus we get

$$V(t) \leq \left(V(0) - \frac{K}{s}\right)e^{-st} + \frac{K}{s}. \tag{2.9}$$

Therefore, $V(t)$ is ultimately bounded, that is, each solution $z(t) = (x_i(t), x_m(t), y(t))$ of system (1.4) is ultimately bounded. The proof is complete. □

3. Equilibria Analysis

Note that, in the system (1.4), the equation for the variable x_i has a form

$$\dot{x}_i(t) = bx_m(t) - d_1x_i(t) - be^{-d_1\tau}x_m(t-\tau) := -d_1x_1(t) + f(x_m(t), x_m(t-\tau)). \tag{3.1}$$

If $x_m(t)$ is bounded and $x_m(t) \to x^*$ as $t \to \infty$, then $x_i(t) \to f(x^*, x^*)/d_1$ as $t \to \infty$; that is, the asymptotic behavior of $x_i(t)$ is dependent on that of $x_m(t)$. Therefore, in following sections we just need to study the asymptotic behavior of the following system:

$$\begin{aligned} \dot{x}(t) &= be^{-d_1\tau}x(t-\tau) - d_2x(t) - \beta x^2(t) - \frac{a_1x(t)y(t)}{1+cx(t)}, \\ \dot{y}(t) &= \frac{a_2x(t)y(t)}{1+cx(t)} - dy(t). \end{aligned} \tag{3.2}$$

By a straightforward calculation, we can get two nonnegative equilibria: $E_0 = (0,0)$ and $E_1 = (x_1, 0) = ((be^{-d_1\tau} - d_2)/\beta, 0)$, E_1 exists if and only if

(H_1): $be^{-d_1\tau} > d_2$ holds.

Further, if

(H_2): $a_2 - cd > 0$ and $(be^{-d_1\tau} - d_2)\ (a_2 - cd) - \beta d > 0$ are satisfied, then system (3.2) has a unique positive equilibrium $E = (x^*, y^*) = (d/(a_2 - cd), (a_2[(be^{-d_1\tau} - d_2)(a_2 - cd) - \beta \mathrm{d}])/a_1(a_2 - cd)^2)$.

Remark 3.1. (H_1) is equivalent to $\tau \in [0, \tau_1)$, where $\tau_1 = (1/d_1)\ln(b/d_2)$. From ($H_2$), we have $\tau \in [0, \tau_2)$ and $\tau_2 = (1/d_1)\ln(b(a_2 - cd)/(\beta d + d_2(a_2 - cd)))$; hence, if E is a positive equilibrium, there must be

$$(b - d_2)(a_2 - cd) > \beta d, \tag{3.3}$$

$$\tau_1 > \tau_2. \tag{3.4}$$

In some of the subsequent sections, we always assume that (3.3) is valid; otherwise, there is no positive equilibrium of system (3.2).

3.1. $E_0 = (0,0)$

First, we analyze the stability of equilibrium E_0, and the associated characteristic equation has the form

$$\left(\lambda + d_2 - be^{-\lambda\tau - d_1\tau}\right)(\lambda + d) = 0. \tag{3.5}$$

Theorem 3.2. (1) *If* $d_2 > b$, *then the equilibrium* E_0 *of system* (3.2) *is uniformly asymptotically stable.*

(2) *If* $d_2 < b$, *then the equilibrium* E_0 *of system* (3.2) *is*

(i) *unstable for all* $\tau \in [0, \tau_1)$ *(see Remark 3.1),*

(ii) *stable, but not asymptotically stable when* $\tau = \tau_1$,

(iii) *asymptotically stable for all* $\tau \in (\tau_1, +\infty)$.

Proof. $\lambda = -d$ has always negative real part. Denote $G(\lambda, \tau) = \lambda + d_2 - be^{-d_1\tau}e^{-\lambda\tau}$.

(1) Let $\tau = 0$ in $G(\lambda, \tau)$, then we have that

$$G(\lambda, 0) = \lambda + d_2 - b = 0, \tag{3.6}$$

if $d_2 > b$, then

$$\lambda = b - d_2 < 0, \tag{3.7}$$

that is, the trivial solution E_0 of system (3.2) is stable when there is no delay τ. If the increasing of τ leads to instability of system (3.2), then there is an $\omega > 0$ such that $G(\omega i, \tau) = 0$ for some $\tau > 0$. Note that

$$\begin{aligned} d_2 &= be^{-d_1\tau}\cos\omega\tau, \\ \omega &= -be^{-d_1\tau}\sin\omega\tau. \end{aligned} \tag{3.8}$$

Hence,

$$\omega^2 = \left(be^{-d_1\tau}\right)^2 - d_2^2. \tag{3.9}$$

But, from hypothesis $d_2 > b$, we know that (3.9) has no real root. Hence, the equilibrium E_0 of system (3.2) is uniformly asymptotic stable for any delay $\tau > 0$.

(2) If $d_2 < b$, then the trivial solution E_0 of system (3.2) is unstable when there is no delay τ.

(i) When $\tau \in [0, \tau_1)$, we know that (3.9) has a positive root $\omega(\tau)$. For $\tau \in [0, \tau_1)$, let $\theta(\tau) \in (0, 2\pi)$ be defined by

$$\begin{aligned} \cos\theta(\tau) &= \frac{d_2}{be^{-d_1\tau}}, \\ \sin\theta(\tau) &= -\frac{\omega(\tau)}{be^{-d_1\tau}}, \end{aligned} \tag{3.10}$$

where, jointly with (3.9), we define the following maps:

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad n \in N. \tag{3.11}$$

If $S_n(\tau) = 0$ has no roots in $[0, \tau_1)$ when $n = 0$, then we know that the equilibrium E_0 of system (3.2) is unstable through using geometric stability switch criteria [1]. Otherwise, let $\tau^* = \min\{\tau : \tau \in [0, \tau_1), S_0(\tau) = 0\}$, then

$$\operatorname{sign}\left\{ \frac{d\operatorname{Re}\lambda}{d\tau}\bigg|_{\lambda=\omega(\tau^*)i} \right\} = \operatorname{sign}\left\{ \frac{dS_n(\tau)}{d\tau}\bigg|_{\tau=\tau^*} \right\} > 0 \quad (\text{because of } S_0(0) < 0), \tag{3.12}$$

which straightforwards implies (i).

(ii) When $\tau = \tau_1$, we have $G(\lambda, \tau_1) = \lambda + d_2 - d_2 e^{-\lambda\tau_1}$, and $\lambda = 0$ is always a simple eigenvalue. Suppose that $G(\lambda, \tau_1) = 0$ has a root $\lambda = u + iv$, where $u > 0$, then

$$(u + d_2)^2 + v^2 = d_2^2 e^{-2u\tau_1} \le d_2^2, \tag{3.13}$$

a contradiction. Hence, (ii) is true.

(iii) When $\tau \in (\tau_1, +\infty)$, (3.9) has no positive root, which means that the stability of E_0 does not change as τ increases. Let $\tau_1^0 = \tau_1 + \varepsilon$, where ε is a positive constant; we consider the following characteristic equation with delay-independent coefficients:

$$\lambda + d_2 - be^{-d_1\tau_1^0} e^{-\lambda\tau} = 0, \quad d_2 > be^{-d_1\tau_1^0}. \tag{3.14}$$

By Theorem 3.4.1 [2, page 66] we know that (3.14) has only negative real-part eigenvalues; thus, (iii) is valid. The proof is complete. □

Remark 3.3. From Theorem 3.2 we know that when the birth rate of prey species is less than death rate of itself, then the prey species tend to be extinct; therefore, the predator species reduces to zero since there is no enough foods, which accords with the basic evolution law of biology. We also see that time delay can stabilize an unstable system and keep this state to τ large enough. This phenomenon cannot appear in the case that the coefficients of a corresponding characteristic equation are independent of the delay.

3.2. $E_1 = (x_1, 0)$

Now, we analyze the stability of E_1. The characteristic equation of system (3.2) at E_1 takes the form

$$\left(\lambda - d_2 + 2be^{-d_1\tau} - be^{-\lambda\tau - d_1\tau}\right)\left(\lambda + d - \frac{a_2 x_1}{1 + cx_1}\right) = 0. \tag{3.15}$$

Denote $H_1(\lambda, \tau) = \lambda - d_2 + 2be^{-d_1\tau} - be^{-\lambda\tau - d_1\tau}$ and $H_2(\lambda) = \lambda + d - a_2(x_1)/(1 + cx_1)$.

Obviously, the root of equation $H_2(\lambda) = 0$ is positive when $\tau \in [0, \tau_2)$ and negative when $\tau \in (\tau_2, \tau_1)$; hence, we only need to consider $H_1(\lambda, \tau) = 0$. By the same method with the proof of Theorem 3.2, we know that $H_1(\lambda, \tau) = 0$ has only negative real-part characteristic roots when $\tau \in [0, \tau_1)$.

Summarizing the discussion above, we get the following conclusion.

Theorem 3.4. *If* (H_1) *is satisfied, then the equilibrium* E_1 *of system* (3.2) *is*

(1) *unstable when* $\tau \in [0, \tau_2)$ *(see Remark 3.1),*

(2) *stable, but not asymptotically stable when* $\tau = \tau_2$*,*

(3) *asymptotically stable when* $\tau \in (\tau_2, \tau_1)$*.*

3.3. $E = (x^*, y^*)$

Finally, we analyze the stability of positive equilibrium E of system (3.2). The linearizing equation of system (3.2) at E is

$$\begin{aligned} \dot{x}(t) &= be^{-d_1\tau}x(t-\tau) - \left[d_2 + \frac{a_1 y^*}{(1+cx^*)^2} + 2\beta x^*\right]x(t) - \frac{a_1 x^*}{1+cx^*}y(t), \\ \dot{y}(t) &= \frac{a_2 y^*}{(1+cx^*)^2}x(t) + \left(\frac{a_2 x^*}{1+cx^*} - d_2\right)y(t), \end{aligned} \tag{3.16}$$

hence, the characteristic equation takes the form

$$D(\lambda, \tau) = \lambda^2 + p(\tau)\lambda + q(\tau) + r(\tau)\lambda e^{-\lambda\tau} = 0, \tag{3.17}$$

where

$$\begin{aligned} p(\tau) &= d_2 + \frac{a_1 y^*}{(1+cx^*)^2} + 2\beta x^*, \\ q(\tau) &= \frac{a_1 d y^*}{(1+cx^*)^2}, \\ r(\tau) &= -d_2 - \beta x^* - \frac{a_1 y^*}{1+cx^*}. \end{aligned} \tag{3.18}$$

Note that these are dependent on time delay τ since y^* includes τ. In order to apply the geometric criterion of Beretta and Kuang [1], we rewrite $D(\lambda, \tau) = 0$ into

$$D(\lambda, \tau) = P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0, \tag{3.19}$$

where

$$\begin{aligned} P(\lambda, \tau) &= \lambda^2 + p(\tau)\lambda + q(\tau), \\ Q(\lambda, \tau) &= r(\tau)\lambda. \end{aligned} \tag{3.20}$$

For convenience, let $f(\tau) = \beta a_2 - c((be^{-d_1\tau} - d_2)(a_2 - cd) - \beta d)$, $\tau \in [0, \tau_2)$, then $f(\tau)$ is strictly monotone increasing function in τ. If $f(0) > 0$, hence, $f(\tau) > 0$ for all $\tau \in [0, \tau_2)$. Otherwise, that is, if $f(0) < 0$, since $f(\tau_2) = \beta a_2 > 0$, there is a unique $\tau_3 \in (0, \tau_2)$ such that $f(\tau_3) = 0$.

Lemma 3.5. (1) *If* (H_2) *and* (H_3)*:* $\beta a_2 - c[(b-d_2)(a_2-cd)-\beta d] > 0$ *hold, then* $q(\tau) > 0$ *and* $p(\tau)+r(\tau) > 0$ *for all* $\tau \in [0,\tau_2)$*.*

(2) *If* (H_2) *and* (H_4)*:* $\beta a_2 - c[(b-d_2)(a_2-cd)-\beta d] < 0$ *hold, then* $q(\tau) > 0$ *for all* $\tau \in [0,\tau_2)$*,* $p(\tau)+r(\tau) < 0$ *when* $\tau \in [0,\tau_3)$ *and* $p(\tau)+r(\tau) > 0$ *when* $\tau \in (\tau_3,\tau_2)$*.*

Proof. Obviously, $q(\tau) > 0$. From (3.18), we have

$$\begin{aligned} p(\tau)+r(\tau) &= \beta x^* - \frac{c a_1 y^* x^*}{(1+cx^*)^2} \\ &= \frac{x^*}{(1+cx^*)^2}\left[\beta(1+cx^*)^2 - c a_1 y^*\right] \\ &= \frac{x^*}{(1+cx^*)(a_2-cd)}\left[\beta a_2 - c\left(\left(be^{-d_1\tau}-d_2\right)(a_2-cd)-\beta d\right)\right]. \end{aligned} \tag{3.21}$$

(1) If $f(0) > 0$, then $f(\tau) > 0$ for all $\tau \in [0,\tau_2)$; hence, $p(\tau)+r(\tau) > 0$ for all $\tau \in [0,\tau_2)$.

(2) If $f(0) < 0$, then $f(\tau) < 0$ when $\tau \in [0,\tau_3)$ and $f(\tau) > 0$ when $\tau \in (\tau_3,\tau_2)$; hence, (2) is true. The proof is complete. □

Step 1. When $\tau = 0$, (3.17) becomes

$$D(\lambda,0) = \lambda^2 + \left(p(0)+r(0)\right)\lambda + q(0) = 0, \tag{3.22}$$

by Routh-Hurwitz Criterion we know that $D(\lambda,0)=0$ has only negative real-part eigenvalue if (H_3) holds and has positive real-part eigenvalue if (H_4) holds. Hence, we obtain the following theorem.

Theorem 3.6. (1) *If* (H_2) *and* (H_3) *are satisfied, then the equilibrium* E *of system* (3.2) *is asymptotically stable when* $\tau = 0$*.*

(2) *If* (H_2) *and* (H_4) *are satisfied, then the equilibrium* E *of system* (3.2) *is unstable when* $\tau = 0$*.*

Step 2. Suppose $D(\omega i,\tau) = 0$ for some $\tau \in [0,\tau_2)$ and $\omega > 0$, we get

$$\begin{aligned} \sin\omega\tau &= \frac{(\omega^2-q)}{\omega r}, \\ \cos\omega\tau &= \frac{-p}{r}. \end{aligned} \tag{3.23}$$

Hence,

$$\begin{aligned} F(\omega,\tau) &= |P(\omega i,\tau)|^2 - |Q(\omega i,\tau)|^2 \\ &= \omega^4 + A(\tau)\omega^2 + B(\tau) \\ &= 0, \end{aligned} \tag{3.24}$$

where

$$A(\tau) = p^2 - 2q - r^2, \qquad B(\tau) = q^2. \tag{3.25}$$

Now, we take hypotheses (H_3) or (H_4) into account, respectively.

Case 1 ($\beta a_2 - c[(b - d_2)(a_2 - cd) - \beta d] > 0$). In this case, since

$$\begin{aligned} \triangle(\tau) &= A(\tau)^2 - 4B(\tau) \\ &= \left(p^2 - 2q - r^2\right)^2 - 4q^2 \\ &= \left(p^2 - 4q - r^2\right)(p - r)(p + r), \\ B(\tau) &= q^2(\tau) > 0, \qquad p - r > 0, \qquad p + r > 0. \end{aligned} \tag{3.26}$$

Thus, if $A(\tau) \geq 0$, then (3.24) has no positive real root, and if $A(\tau) < 0$, then $\triangle(\tau) < 0$ and (3.24) has no real root, that is, the stability of equilibrium E cannot change for all $\tau \in [0, \tau_2)$. Summarizing the discussion above, we obtain the following theorem.

Theorem 3.7. *Assume that* (H_2) *and* (H_3) *are satisfied, then the equilibrium* E *of system* (3.2) *is asymptotically stable for all* $\tau \in [0, \tau_2)$.

Case 2 ($\beta a_2 - c[(b - d_2)(a_2 - cd) - \beta d] < 0$). In this case, when $\tau \in (\tau_3, \tau_2)$, by the similar analysis we also obtain that the stability of equilibrium E cannot change for all $\tau \in (\tau_3, \tau_2)$. Let $\tau_0 = \tau_3 + \varepsilon$, where ε is a positive constant; we consider the following characteristic equation with delay-independent coefficients:

$$D(\lambda, \tau) = \lambda^2 + p(\tau_0)\lambda + q(\tau_0) + r(\tau_0)\lambda e^{-\lambda\tau} = 0, \tag{3.27}$$

where $p(\tau_0) + r(\tau_0) > 0$ and $q(\tau_0) > 0$. By Theorem 3.4.1 [2, page 66] we know that (3.27) has only negative real-part eigenvalues, then the equilibrium E is asymptotically stable for all $\tau \in (\tau_3, \tau_2)$.

When $\tau \in [0, \tau_3)$, since $p(\tau) + r(\tau) < 0$, hence, $A(\tau) < 0$, $B(\tau) > 0$, and $\triangle(\tau) > 0$; we easily know that (3.24) has two positive real roots $\omega_+(\tau)$ and $\omega_-(\tau)$, where

$$\omega_\pm(\tau) = \sqrt{\frac{-A(\tau) \pm \sqrt{A^2(\tau) - 4B(\tau)}}{2}}. \tag{3.28}$$

For $\tau \in [0, \tau_3)$, let $\theta(\tau) \in (0, 2\pi)$ be defined by

$$\begin{aligned} \sin\theta_\pm(\tau) &= \frac{(\omega_\pm^2 - q)}{r\omega_\pm}, \\ \cos\theta_\pm(\tau) &= \frac{-p}{r}, \end{aligned} \tag{3.29}$$

where, jointly with (3.23), we define the following maps:

$$S_n^{\pm}(\tau) = \tau - \frac{\theta_{\pm}(\tau) + 2n\pi}{\omega_{\pm}(\tau)}, \quad n \in N. \tag{3.30}$$

Obviously, $\sin\theta_+(\tau) = (\omega_+^2 - q)/r\omega_+ < 0$, $\sin\theta_-(\tau) = (\omega_-^2 - q)/(r\omega_-) > 0$, and $\cos\theta_{\pm}(\tau) = -p/r > 0$, which imply

$$\begin{gathered} \theta_+(\tau) = 2\pi - \arccos\frac{-p}{r}, \\ \theta_-(\tau) = \arccos\frac{-p}{r}. \end{gathered} \tag{3.31}$$

Furthermore, we note that $S_0^-(0) < 0$ and $\lim_{\tau\to\tau_3} S_0^-(\tau) = \tau_3 > 0$ (because of $\lim_{\tau\to\tau_3}\theta_-(\tau) = 0$); thus, equation $S_0^-(\tau) = 0$ has at least one root in $[0, \tau_3)$; let $\tau^* = \min\{\tau : \tau \in [0, \tau_3), S_0^-(\tau) = 0\}$, then

$$\delta(\tau^*) := \operatorname{sign}\left\{ \left.\frac{d\operatorname{Re}\lambda}{d\tau}\right|_{\lambda=\omega_-(\tau^*)i} \right\} = -\operatorname{sign}\left\{ \left.\frac{dS_0^-(\tau)}{d\tau}\right|_{\tau=\tau^*} \right\} < 0, \tag{3.32}$$

that is, the pair of simple conjugate pure imaginary roots crosses the imaginary axis from left to right. Summarizing the discussion above, we obtain the following result.

Theorem 3.8. *Assume that (H_2) and (H_4) are satisfied; if equation $S_0^+(0) = 0$ has no root in $[0, \tau^*)$, then the equilibrium E of system* (3.2) *is unstable for all $\tau \in [0, \tau^*)$ and becomes asymptotically stable for τ staying in some right neighborhood of τ^*; hence, system* (3.2) *undergoes Hopf bifurcation when $\tau = \tau^*$. The equilibrium E of system* (3.2) *is asymptotically stable for $\tau \in (\tau_3, \tau_2)$.*

4. Global Stability

In this section, we discuss the global stability of boundary equilibria. In some of the subsequent analysis, we will need the following result.

Lemma 4.1 (see [16]). *Consider the following equation:*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t) \tag{4.1}$$

($a, b, c, \tau > 0$, $x(t) > 0$ *for* $-\tau \le t \le 0$), *then one has the following.*

(i) *If $a > b$, then* $\lim_{t\to\infty} x(t) = (a-b)/c$.

(ii) *If $a < b$, then* $\lim_{t\to\infty} x(t) = 0$.

Theorem 4.2. *Assume that $be^{-d_1\tau} < d_2$, then the equilibrium E_0 of system* (3.2) *is globally asymptotically stable.*

Proof. We know that if $be^{-d_1\tau} < d_2$, then E_0 is locally asymptotically stable. Now we show

$$\lim_{t\to\infty}(x, y) = (0, 0). \tag{4.2}$$

From the first equation of system (3.2), we get

$$\dot{x}(t) \leq be^{-d_1\tau}x(t-\tau) - d_2x(t) - \beta x^2(t). \tag{4.3}$$

By Lemma 4.1 and comparing principle, we obtain

$$\lim_{t\to\infty} x = 0, \tag{4.4}$$

that is, for the $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$0 < x(t) < \varepsilon, \quad t > T_1, \tag{4.5}$$

which jointly with the second equation of system (3.2) leads to

$$\dot{y}(t) < (\varepsilon a_2 - d)y(t). \tag{4.6}$$

Since the arbitrariness of $\varepsilon > 0$, we have

$$\lim_{t\to\infty} y = 0. \tag{4.7}$$

The proof is complete. □

Theorem 4.3. *Assume that (H_1) is satisfied, then the equilibrium E_1 of system* (3.2) *is globally asymptotically stable for $\tau \in (\tau_2, \tau_1)$.*

Proof. We need to show attractivity. From the first equation of system (3.2), we get

$$\dot{x}(t) \leq be^{-d_1\tau}x(t-\tau) - d_2x(t) - \beta x^2(t). \tag{4.8}$$

By Lemma 4.1 and comparing principle, we obtain

$$\limsup_{t\to\infty} x(t) \leq \frac{be^{-d_1\tau} - d_2}{\beta} = x_1, \tag{4.9}$$

that is, for the $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$0 < x(t) < x_1 + \varepsilon, \quad \text{for } t > T_1, \tag{4.10}$$

which jointly with the second equation of system (3.2) leads to

$$\dot{y}(t) < \left[\frac{a_2(x_1+\varepsilon)}{1+c(x_1+\varepsilon)} - d\right] y(t). \tag{4.11}$$

On the other hand, for $\tau \in (\tau_2, \tau_1)$, we have

$$\frac{a_2 x_1}{1+cx_1} - d = \frac{(a_2 - cd)x_1 - d}{1+cx_1} < 0. \tag{4.12}$$

Thus, for $\varepsilon > 0$ is small enough, there is

$$\frac{a_2(x_1+\varepsilon)}{1+c(x_1+\varepsilon)} - d < 0, \tag{4.13}$$

which leads to

$$\lim_{t\to\infty} y(t) = 0, \tag{4.14}$$

that is, for the $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$0 < y(t) < \varepsilon, \quad \text{for } t > T_2, \tag{4.15}$$

hence, we have

$$\dot{x}(t) \geq be^{-d_1\tau}x(t-\tau) - (d_2+\varepsilon)x(t) - \beta x^2(t), \tag{4.16}$$

by hypothesis (H_1), Lemma 4.1, and the arbitrariness of $\varepsilon > 0$, there is

$$\liminf_{t\to\infty} x(t) \geq \frac{be^{-d_1\tau} - d_2}{\beta} = x_1. \tag{4.17}$$

The proof is complete. □

5. Numerical Simulation

In this section, we present some numerical simulations of system (3.2) to illustrate our theoretical analysis.

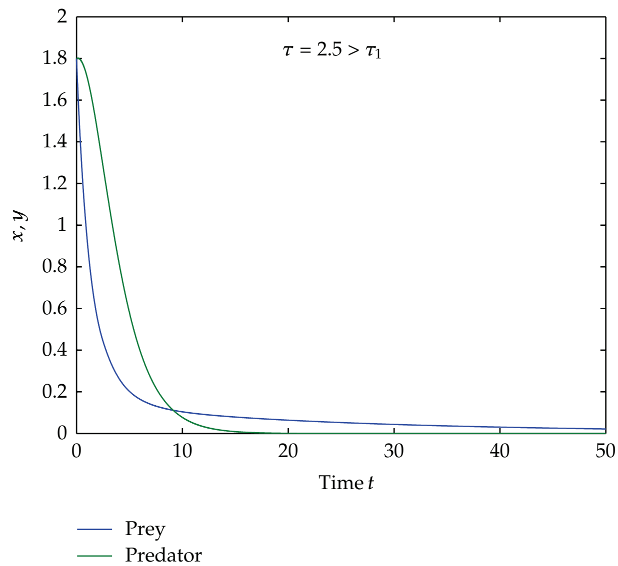

Figure 1: The equilibrium E_0 is globally stable when $\tau = 2.5$.

Example 5.1. Let $b = a_2 = 0.8,\ d = a_1 = 0.5,\ d_1 = c = 1,\ d_2 = 0.1$, and $\beta = 0.2$, and we consider the following system:

$$\dot{x}(t) = 0.8e^{-0.5\tau}x(t-\tau) - 0.1x(t) - 0.2x^2(t) - \frac{0.5x(t)y(t)}{1+x(t)},$$
$$\dot{y}(t) = \frac{0.8x(t)y(t)}{1+x(t)} - 0.5y(t). \tag{5.1}$$

By calculating, we have that $\tau_1 \doteq 2.0794,\ \tau_2 \doteq 0.6131$, and hypotheses (H_2) and (H_3) hold. By the discussion above, we obtain following corollary.

Corollary 5.2. *For system* (5.1)*, when* $\tau \in [0, 0.6131)$*,* E *is asymptotically stable,* E_1 *is globally stable when* $\tau \in (0.6131, 2.0794)$*, and* E_0 *is globally stable when* $\tau \in (2.0794, +\infty)$.

First, we take $\tau = 2.5$ and initial value is that

$$x(t) = y(t) = 1.8, \quad t \in [-\tau, 0]. \tag{5.2}$$

Figure 1 illustrates that prey and predator converge asymptotically to zero, when $\tau = 1$ and initial value is that $x(t) = y(t) = 5,\ t \in [-\tau, 0]$.

Figure 2 illustrates that prey and predator converge asymptotically to E_1, where $x_1 \doteq 0.9715$.

Finally, we take $\tau = 0.001$; system (5.1) has positive equilibrium $E \doteq (1.6667, 1.9513)$.

Figure 3 shows that the positive solutions converge asymptotically to the steady-state E.

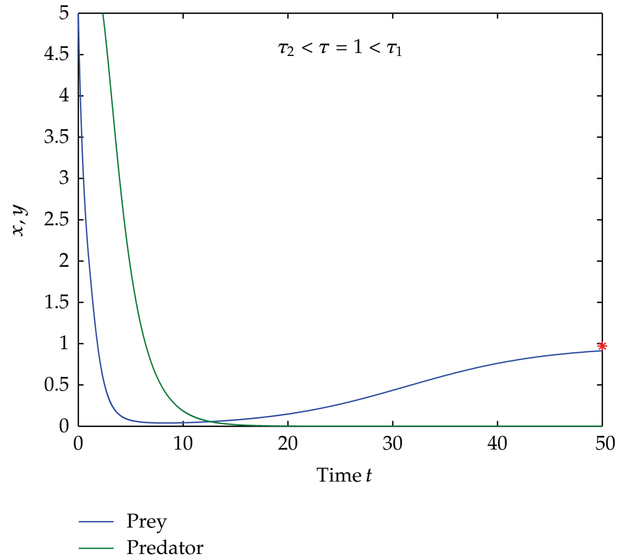

Figure 2: The equilibrium E_1 is globally stable when $\tau = 1$.

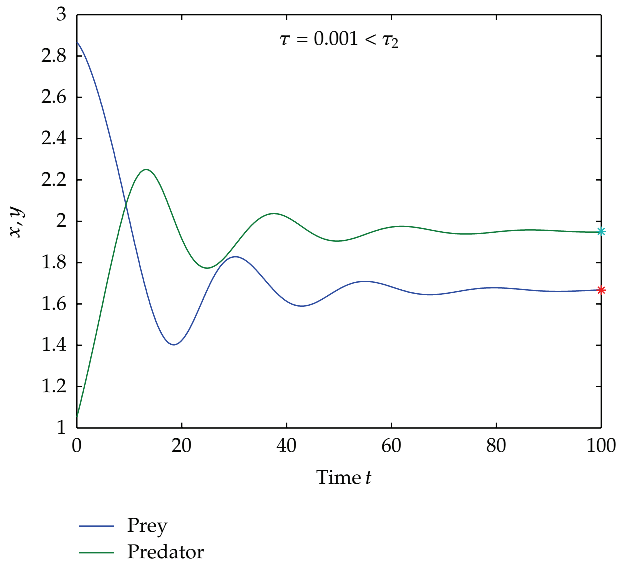

Figure 3: The equilibrium E is asymptotically stable when $\tau = 0.001$.

Example 5.3. In order to illustrate Theorem 3.8, we choose a set of parameters as $b = 0.8$, $d = 0.2$, $a_1 = 0.5$, $a_2 = 3$, $d_1 = 0.05$, $c = 5$, $d_2 = 0.1$, and $\beta = 1$, then system (3.2) reduces to

$$\begin{aligned} \dot{x}(t) &= 0.8e^{-0.05\tau}x(t-\tau) - 0.1x(t) - x^2(t) - \frac{0.5x(t)y(t)}{1+5x(t)}, \\ \dot{y}(t) &= \frac{3x(t)y(t)}{1+5x(t)} - 0.2y(t). \end{aligned} \tag{5.3}$$

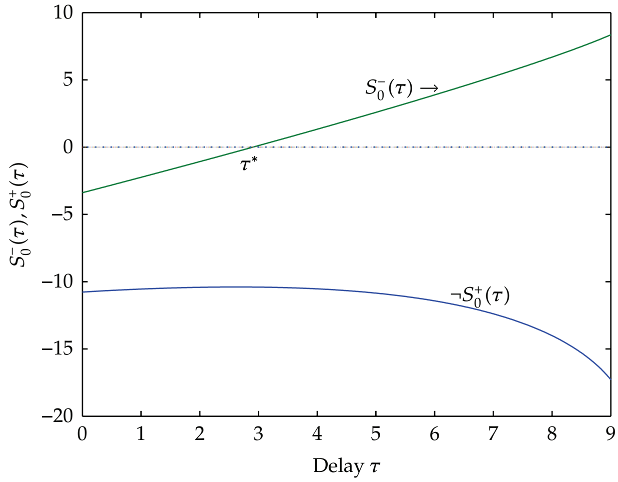

Figure 4: The graphs of $S_0^+(\tau)$ and $S_0^-(\tau)$.

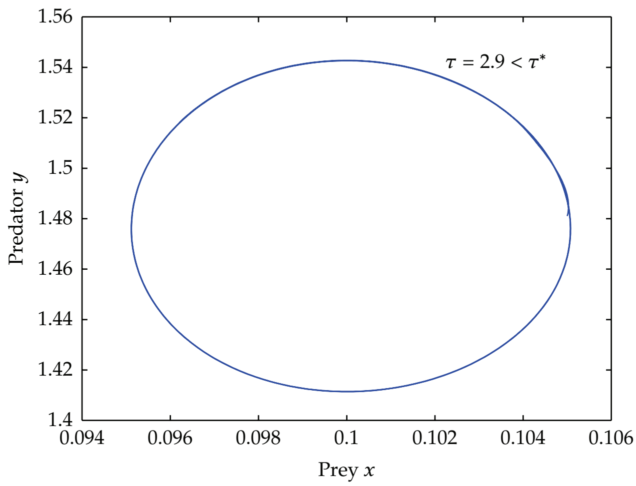

Figure 5: The bifurcating periodic solutions from E when $\tau = 2.9 < \tau^*$.

By calculating, we have that $\tau_2 \doteq 27.7259$, $\tau_3 \doteq 9.4001$, and hypotheses (H_2) and (H_4) hold.

Figure 4 shows that equation $S_0^+(\tau) = 0$ has no root and equation $S_0^-(\tau) = 0$ has a unique root τ^*, where $\tau^* \doteq 2.9104$, which leads to the following corollary.

Corollary 5.4. *The equilibrium E of system* (5.3) *is unstable for all $\tau \in [0, \tau^*)$ and becomes asymptotically stable for τ staying in some right neighborhood of τ^*; hence, system* (5.3) *undergoes Hopf bifurcation when $\tau = \tau^*$.*

By Corollary 5.4, we know that the positive equilibrium is unstable and positive solutions are oscillatory when $\tau = 2.9 < \tau^*$, the bifurcating periodic solutions exist at least for the value of τ slightly larger than the critical value τ^*, the computer simulation is given in Figure 5. We continue to choose $\tau = 3 > \tau^*$, by Corollary 5.4 we know that the positive equilibrium becomes asymptotically stable, and the computer simulation is depicted in

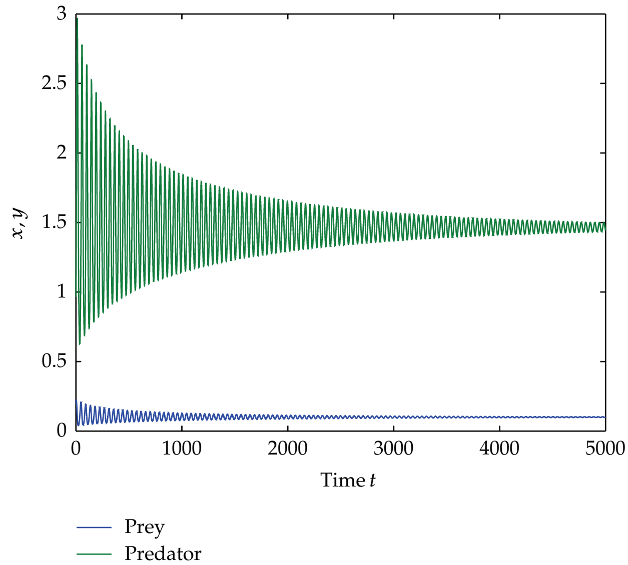

Figure 6: The positive equilibrium is stable when $\tau = 3 > \tau^*$.

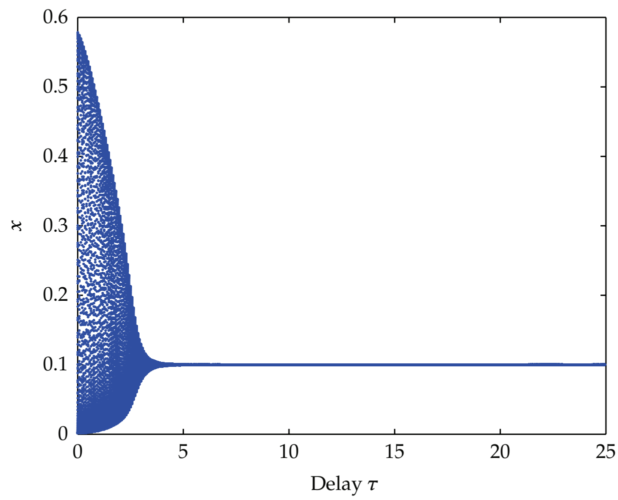

Figure 7: The relationship between ultimate oscillation range of mature prey and delay τ.

Figure 6. In Figures 7 and 8, we plot the relationship between delay τ and ultimate amplitudes of $x(t)$ and $y(t)$, which indicate the fact that the positive equilibrium E is from instability to stability as τ increases.

6. Discussion

In this paper, a model which describes the Holling type predator-prey system with stage structure for prey is proposed. Sufficient conditions which ensure the stability of equilibria and the existence of Hopf bifurcation are obtained. Numerical results show the feasibility of our results.

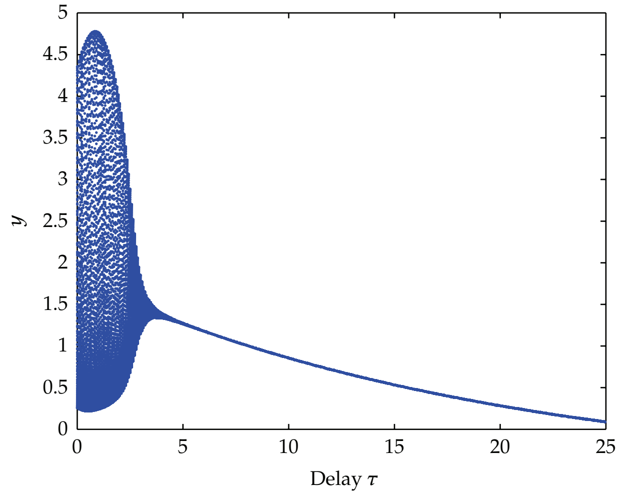

Figure 8: The relationship between ultimate oscillation range of predator and delay τ.

By the analysis, we find the system (1.4) has complex dynamics behavior. Our results indicate that the stability switches of the positive equilibrium may occur as time delay τ increases and delay τ can stabilize system (1.4). Under condition (H_4), the positive equilibrium is stable only if delay τ is larger, consequently, y^* becomes smaller, that is, predator coexists with prey in lower population densities (with respect to delay τ), which is different from [22]. In [22], predator coexists with prey only if the population densities of prey are larger and those of predator are lower (with respect to delay τ). Moreover, Liu and Zhang [6] investigated system (1.2) when functional response is Beddington-DeAngelis type, by monotone dynamic theories, they obtained that the positive equilibrium is globally asymptotically stable provided that k_2 (describing the magnitude of interference among predators) is more than a positive constant; that is, their results cannot directly apply to study system (1.4). Furthermore, they did not consider d_2, the death rate of mature prey population, which play an important role on stability of the equilibria and existence of Hopf bifurcation. Hypothesis (H_3) means that $d_2 > d_2^0$, where $d_2^0 = b - (\beta(a_2 + cd)/c(a_2 - cd))$, and Hypothesis (H_4) means that $d_2 < d_2^0$. Hence, from Theorem 3.7 and Theorem 3.8, we know that d_2 can also stabilize system (1.4).

There are still many interesting and challenging mathematical questions that need to be studied for system (1.4). For example, we do not discuss the global stability of positive equilibrium. We will leave this for future work.

Acknowledgments

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