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Research Article

The General Iterative Methods for Asymptotically Nonexpansive Semigroups in Banach Spaces

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We introduce the general iterative methods for finding a common fixed point of asymptotically nonexpansive semigroups which is a unique solution of some variational inequalities. We prove the strong convergence theorems of such iterative scheme in a reflexive Banach space which admits a weakly continuous duality mapping. The main result extends various results existing in the current literature.

1. Introduction

Let E be a normed linear space. Let T be a self-mapping on E. Then T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $x, y \in E$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 1. \tag{1.1}$$

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [1] as an important generalization of the class of nonexpansive maps (i.e., mappings $T : E \to E$ such that $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in E$). We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in E : Tx = x\}$. A self-mapping $f : E \to E$ is a contraction on E if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in E. \tag{1.2}$$

We use Π_E to denote the collection of all contractions on E. That is, $\Pi_E = \{f : f$ is a contraction on $E\}$.

A family $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ of mappings of E into itself is called a *strongly continuous semigroup of Lipschitzian mappings* on E if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in E$;

(ii) $T(s+t) = T(s)T(t)$ for all $s, t \ge 0$;

(iii) for each $t > 0$, there exists a bounded measurable function $L_t : (0,\infty) \to [0,\infty)$ such that $\|T(t)x - T(t)y\| \le L_t\|x - y\|$, for all $x, y \in E$;

(iv) for all $x \in E$, the mapping $t \mapsto T(t)x$ is continuous.

A strongly continuous semigroup of Lipchitszian mappings $\mathcal{S}$ is called *strongly continuous semigroup of nonexpansive mappings* if $L_t = 1$ for all $t > 0$ and *strongly continuous semigroup of asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \le 1$. Note that for asymptotically nonexpansive semigroup $\mathcal{S}$, we can always assume that the Lipchitszian constant $\{L_t\}_{t>0}$ is such that $L_t \ge 1$ for each $t > 0$, L_t is nonincreasing in t, and $\lim_{n\to\infty} L_t = 1$; otherwise we replace L_t, for each $t > 0$, with $L_t := \max\{\sup_{s\ge t} L_s, 1\}$. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is,

$$F(\mathcal{S}) := \{x \in E : T(t)x = x,\ 0 \le t < \infty\} = \bigcap_{t\ge 0} F(T(t)). \tag{1.3}$$

$\mathcal{S}$ is called *uniformly asymptotically regular* on C [2, 3] if for all $h \ge 0$ and any bounded subset B of C,

$$\lim_{t\to\infty}\sup_{x\in B}\|T(h)T(t)x - T(t)x\| = 0, \tag{1.4}$$

and *almost uniformly asymtotically regular* on C [4] if

$$\lim_{t\to\infty}\sup_{x\in B}\left\|T(h)\frac{1}{t}\int_0^t T(s)x\,ds - \frac{1}{t}\int_0^t T(s)x\,ds\right\| = 0. \tag{1.5}$$

Let $u \in C$. Then, for each $t \in (0,1)$ and for a nonexpansive map T, there exists a unique point $x_t \in C$ satisfying the following condition:

$$x_t = (1-t)Tx_t + tu, \tag{1.6}$$

since the mapping $G_t(x) = (1-t)Tx + tu$ is a contraction. When H is a Hilbert space and T is a self-map, Browder [5] showed that $\{x_t\}$ converges strongly to an element of $F(T)$ which is nearest to u as $t \to 0^+$. This result was extended to more various general Banach space by Morales and Jung [6], Takahashi and Ueda [7], Reich [8], and a host of other authors.

Many authors (see, e.g., [9, 10]) have also shown the convergence of the path $x_n = (1-\alpha_n)T^n x_n + \alpha_n u$, in Banach spaces for asymptotically nonexpansive mapping self-map T under some conditions on α_n.

It is an interesting problem to extend the above results to a strongly continuous semigroup of nonexpansive mappings and a strongly continuous semigroup of asymptotically nonexpansive mappings.

Let $\mathcal{S}$ be a strongly continuous semigroup of nonexpansive self-mappings. In 1998 Shioji and Takahashi [11] introduced, in Hilbert space, the implicit iteration

$$u_n = (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n\,ds + \alpha_n u, \quad u \in C,\ n \geq 0, \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ is a sequence of positive real numbers divergent to ∞. Under certain restrictions to the sequence $\{\alpha_n\}$, Shioji and Takahashi proved strong convergence of (1.7) to a member of $F(\mathcal{S})$. Recently, Zegeye et al. [4] introduced the implicit (1.7) and the following explicit iteration process for a semigroup of asymptotically nonexpansive mappings:

$$u_{n+1} = (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n\,ds + \alpha_n u, \quad u \in C,\ n \geq 0, \tag{1.8}$$

where $t_n \in \mathbb{R}^+$ and $\alpha_n \in (0,1)$ in a reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose, in addition, that $\mathcal{S}$ is almost uniformly asymptotically regular. Then the implicit sequence (1.7) and explicit sequence (1.8) converge strongly to a point of $F(\mathcal{S})$.

On the other hand, by a gauge function φ we mean a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Let E^* be the dual space of E. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge function φ is defined by

$$J_\varphi(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|),\ \|f^*\| = \varphi(\|x\|) \right\}, \quad \forall x \in E. \tag{1.9}$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$ for all $x \neq 0$ (see [12]). Browder [12] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [12], we say that a Banach space E has *a weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single valued and continuous from the weak topology to the weak* topology; that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \geq 0, \tag{1.10}$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E, \tag{1.11}$$

where ∂ denotes the subdifferential in the sense of convex analysis.

In a Banach space E having a weakly continuous duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* [13] if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x) \rangle \geq \overline{\gamma}\|x\|\varphi(\|x\|), \tag{1.12}$$

$$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} \left| \langle (\alpha I - \beta A)x, J_\varphi(x) \rangle \right|, \quad \alpha \in [0,1], \ \beta \in [-1,1], \tag{1.13}$$

where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (1.12) reduces to

$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2 \quad \forall x \in H. \tag{1.14}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x \rangle - \langle x, b \rangle, \tag{1.15}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2009, motivated and inspired by Marino and Xu [14], Li et al. [15] introduced the following general iterative procedures for the approximation of common fixed points of a one-parameter nonexpansive semigroup $\{T(s) : s \geq 0\}$ on a nonempty closed convex subset C in a Hilbert space:

$$\begin{aligned} y_n &= \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n \, ds, \quad n \geq 0, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) x_n \, ds, \quad n \geq 0, \end{aligned} \tag{1.16}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are sequences in $[0,1]$ and $(0,\infty)$, respectively, A is a strongly positive bounded linear operator on H, and f is a contraction on H. And their convergence theorems can be proved under some appropriate control conditions on parameter $\{\alpha_n\}$ and $\{t_n\}$. Furthermore, by using these results, they obtained two mean ergodic theorems for nonexpansive mappings in a Hilbert space.

All of the above brings us to the following conjectures.

Question 1. Could we obtain strong convergence theorems for the general class of strongly continuous semigroup of asymptotically nonexpansive mappings in more general Banach spaces? such as a reflexive Banach space which admits a weakly continuous duality mapping J_φ, where $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function.

In this paper, inspired and motivated by Shioji and Takahashi [11], Zegeye et al. [4], Marino and Xu [14], Li et al. [15], and Wangkeeree et al. [13], we prove the strong convergence theorems of the iterative approximation methods (1.16) for the general class of the strongly continuous semigroup of asymptotically nonexpansive mappings in a reflexive Banach space which admits a weakly continuous duality mapping J_φ, where $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function and A is a strongly positive bounded linear operator on a Banach space E. The results in this paper generalize and improve some well-known results in Shioji and Takahashi [11], Li et al. [15], and many others.

2. Preliminaries

Throughout this paper, let E be a real Banach space and E^* be its dual space. We write $x_n \rightharpoonup x$ (resp., $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (resp., weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence. Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if, for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|(x+y)/2\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [16]). A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0}((\|x + ty\| - \|x\|)/t)$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$.

Now we collect some useful lemmas for proving the convergence result of this paper.

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [10].

Lemma 2.1 (see [10]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle. \tag{2.1}$$

In particular, for all $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle. \tag{2.2}$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$.*

Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E. \tag{2.3}$$

The next valuable lemma is proved for applying our main results.

Lemma 2.2 (see [13, Lemma 3.1]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ. Let A be a strong positive linear bounded operator on E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\overline{\gamma})$.*

In the following, we also need the following lemma that can be found in the existing literature [17, 18].

Lemma 2.3 (see [18, Lemma 2.1]). *Let $\{a_n\}$ be a sequence of a nonnegative real number satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\beta_n, \quad n \geq 0, \tag{2.4}$$

where $\{\gamma_n\} \subseteq (0,1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\beta_n \leq 0$. Then $\{a_n\}$ converges to zero, as $n \to \infty$.

3. Main Theorem

Theorem 3.1. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings on E with a sequence $\{L_t\} \subset [1,\infty)$ and $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$, let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \varphi(1)\overline{\gamma}/\alpha$, and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$, $t_n > 0$. Then the following holds.*

(i) *If $((1/t_n)\int_0^{t_n} L_s\,ds - 1)/\alpha_n < \varphi(1)\overline{\gamma} - \gamma\alpha$, for all $n \geq 0$, then there exists a sequence $\{y_n\} \subset E$ defined by*

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds, \quad n \geq 0. \tag{3.1}$$

(ii) *Suppose, in addition, that $\mathcal{S}$ is almost uniformly asymptotically regular and the real sequences $\{\alpha_n\}$ and $\{t_n\}$ satisfy the following conditions:*

(B1) $\lim_{n\to\infty} t_n = \infty$;
(B2) $\lim_{n\to\infty} \alpha_n = 0$;
(B3) $\lim_{n\to\infty}((1/t_n)\int_0^{t_n} L_s\,ds - 1)/\alpha_n = 0$.

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\widetilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\langle (A-\gamma f)\widetilde{x}, J_\varphi(\widetilde{x} - z)\rangle \leq 0, \quad z \in F(\mathcal{S}). \tag{3.2}$$

Proof. We first show the uniqueness of a solution of the variational inequality (3.2). Suppose both $\widetilde{x} \in F(\mathcal{S})$ and $x^* \in F(\mathcal{S})$ are solutions to (3.2), then

$$\begin{aligned} &\langle (A-\gamma f)\widetilde{x}, J_\varphi(\widetilde{x} - x^*)\rangle \leq 0, \\ &\langle (A-\gamma f)x^*, J_\varphi(x^* - \widetilde{x})\rangle \leq 0. \end{aligned} \tag{3.3}$$

Adding (3.3), we obtain

$$\langle (A-\gamma f)\widetilde{x} - (A-\gamma f)x^*, J_\varphi(\widetilde{x} - x^*)\rangle \leq 0. \tag{3.4}$$

Noticing that for any $x, y \in E$,

$$\begin{aligned}
&\langle (A-\gamma f)x-(A-\gamma f)y, J_\varphi(x-y)\rangle \\
&\quad = \langle A(x-y), J_\varphi(x-y)\rangle - \gamma\langle f(x)-f(y), J_\varphi(x-y)\rangle \\
&\quad \geq \overline{\gamma}\|x-y\|\varphi(\|x-y\|) - \gamma\|f(x)-f(y)\|\|J_\varphi(x-y)\| \\
&\quad \geq \overline{\gamma}\Phi(\|x-y\|) - \gamma\alpha\Phi(\|x-y\|) \\
&\quad = (\overline{\gamma}-\gamma\alpha)\Phi(\|x-y\|) \\
&\quad \geq (\overline{\gamma}\varphi(1)-\gamma\alpha)\Phi(\|x-y\|) \geq 0.
\end{aligned} \tag{3.5}$$

Therefore $\tilde{x} = x^*$ and the uniqueness is proved. Below, we use $\tilde{x}$ to denote the unique solution of (3.2).

Since $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without the loss of generality, that $\alpha_n < \varphi(1)\|A\|^{-1}$.

For each integer $n \geq 0$, define a mapping $G_n : E \to E$ by

$$G_n(y) = \alpha_n\gamma f(y) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y\,ds, \quad \forall y \in E. \tag{3.6}$$

We show that G_n is a contraction mapping. For any $x, y \in E$,

$$\begin{aligned}
&\|G_n(x)-G_n(y)\| \\
&\quad = \left\|\alpha_n\gamma f(x) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds - \alpha_n\gamma f(y) - (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y\,ds\right\| \\
&\quad \leq \|\alpha_n\gamma(f(x)-f(y))\| + \left\|(I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds - \frac{1}{t_n}\int_0^{t_n} T(s)y\,ds\right)\right\| \\
&\quad \leq \alpha_n\gamma\alpha\|x-y\| + \varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)\|x-y\| \\
&\quad = \left(\alpha_n\gamma\alpha + \varphi(1)\frac{1}{t_n}\int_0^{t_n} L_s\,ds - \varphi(1)\alpha_n\overline{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)\|x-y\| \\
&\quad \leq \left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds - \alpha_n\left(\varphi(1)\overline{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s\,ds - \gamma\alpha\right)\right)\|x-y\|.
\end{aligned} \tag{3.7}$$

Since $0 < ((1/t_n)\int_0^{t_n} L_s\,ds - 1)/\alpha_n < \varphi(1)\overline{\gamma} - \gamma\alpha$, we have

$$0 < \frac{(1/t_n)\int_0^{t_n} L_s\,ds - 1}{\alpha_n} < \varphi(1)\overline{\gamma} - \gamma\alpha \leq \varphi(1)\overline{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s\,ds - \gamma\alpha. \tag{3.8}$$

It then follows that $0 < ((1/t_n)\int_0^{t_n} L_s\,ds - \alpha_n(\varphi(1)\overline{\gamma}(1/t_n)\int_0^{t_n} L_s\,ds - \gamma\alpha)) < 1$. We have G_n is a contraction map with coefficient $((1/t_n)\int_0^{t_n} L_s\,ds - \alpha_n(\varphi(1)\overline{\gamma}(1/t_n)\int_0^{t_n} L_s\,ds - \gamma\alpha))$. Then, for each $n \geq 0$, there exists a unique $y_n \in E$ such that $G_n y_n = y_n$, that is,

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n\,ds, \quad n \geq 0. \tag{3.9}$$

Hence (i) is proved.

(ii) We first show that $\{y_n\}$ is bounded. Letting $p \in F(\mathcal{S})$ and using Lemma 2.2, we can calculate the following:

$$\begin{aligned}
\|y_n - p\| &= \left\|\alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n\,ds - p\right\| \\
&= \Bigg\|\alpha_n \gamma f(y_n) - \alpha_n \gamma f(p) + \alpha_n \gamma f(p) \\
&\quad + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n\,ds - (I - \alpha_n A)p - (\alpha_n A)p\Bigg\| \\
&\leq \alpha_n \gamma \|f(y_n) - f(p)\| + \alpha_n \|\gamma f(p) - A(p)\| \\
&\quad + \varphi(1)(1 - \alpha_n \overline{\gamma})\left\|\frac{1}{t_n}\int_0^{t_n} T(s) y_n\,ds - \frac{1}{t_n}\int_0^{t_n} T(s) p\,ds\right\| \\
&\leq \alpha_n \gamma\alpha \|y_n - p\| + \alpha_n \|\gamma f(p) - A(p)\| + \varphi(1)(1 - \alpha_n \overline{\gamma})\frac{1}{t_n}\int_0^{t_n} L_s ds \|y_n - p\| \\
&\leq \alpha_n \gamma\alpha \|y_n - p\| + \alpha_n \|\gamma f(p) - A(p)\| + \varphi(1)\frac{1}{t_n}\int_0^{t_n} L_s\,ds \|y_n - p\| \\
&\quad - \varphi(1)\alpha_n \overline{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s\,ds \|y_n - p\| \\
&\leq \alpha_n \gamma\alpha \|y_n - p\| + \alpha_n \|\gamma f(p) - A(p)\| + \frac{1}{t_n}\int_0^{t_n} L_s\,ds \|y_n - p\| \\
&\quad - \varphi(1)\alpha_n \overline{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s\,ds \|y_n - p\|.
\end{aligned} \tag{3.10}$$

Thus, we get that

$$\|y_n - p\| \leq \frac{\alpha_n \|\gamma f(p) - A(p)\|}{1 - \alpha_n \gamma\alpha - (1/t_n)\int_0^{t_n} L_s + \varphi(1)\alpha_n \overline{\gamma}(1/t_n)\int_0^{t_n} L_s\,ds}. \tag{3.11}$$

Calculating the right-hand side of the above inequality, we have

$$\begin{aligned}
&\frac{\alpha_n\|\gamma f(p)-A(p)\|}{1-\alpha_n\gamma\alpha-(1/t_n)\int_0^{t_n}L_s+\varphi(1)\alpha_n\overline{\gamma}(1/t_n)\int_0^{t_n}L_s\,ds+\varphi(1)\alpha_n\overline{\gamma}}\\
&\quad=\frac{\alpha_n\|\gamma f(p)-A(p)\|}{1-\alpha_n\gamma\alpha-(1/t_n)\int_0^{t_n}L_s+\varphi(1)\alpha_n\overline{\gamma}(1/t_n)\int_0^{t_n}L_s\,ds+\varphi(1)\alpha_n\overline{\gamma}-\varphi(1)\alpha_n\overline{\gamma}}\\
&\quad=\frac{\alpha_n\|\gamma f(p)-A(p)\|}{\varphi(1)\alpha_n\overline{\gamma}-\alpha_n\alpha\gamma+1-(1/t_n)\int_0^{t_n}L_s\,ds+\varphi(1)\alpha_n\overline{\gamma}(1/t_n)\int_0^{t_n}L_s\,ds-\varphi(1)\alpha_n\overline{\gamma}}\\
&\quad=\frac{\alpha_n\|\gamma f(p)-A(p)\|}{\alpha_n(\varphi(1)\overline{\gamma}-\alpha\gamma)-\left((1/t_n)\int_0^{t_n}L_s\,ds-1\right)+\varphi(1)\alpha_n\overline{\gamma}\left((1/t_n)\int_0^{t_n}L_s\,ds-1\right)}\\
&\quad=\frac{\alpha_n\|\gamma f(p)-A(p)\|}{\alpha_n(\varphi(1)\overline{\gamma}-\alpha\gamma)-(1-\varphi(1)\alpha_n\overline{\gamma})\left((1/t_n)\int_0^{t_n}L_s\,ds-1\right)}.
\end{aligned}\tag{3.12}$$

Thus, we get that

$$\|y_n-p\|\leq\frac{\|\gamma f(p)-A(p)\|}{(\varphi(1)\overline{\gamma}-\alpha\gamma)-(1-\alpha_n\overline{\gamma})d_n},\tag{3.13}$$

where $d_n=((1/t_n)\int_0^{t_n}L_s\,ds-1)/\alpha_n$. Thus, there exists $N>0$ such that $\|y_n-p\|\leq\|\gamma f(p)-A(p)\|/(\varphi(1)\overline{\gamma}-\gamma\alpha)$, for all $n\geq N$. Therefore, $\{y_n\}$ is bounded and hence $\{f(y_n)\}$ and $\{(1/t_n)\int_0^{t_n}T(s)y_n\,ds\}$ are also bounded.

Let $\delta_{t_n}(y_n)=(1/t_n)\int_0^{t_n}T(s)y_n\,ds$. Then, from (3.1), we get

$$\|y_n-\delta_{t_n}(y_n)\|=\alpha_n\|\gamma f(y_n)-A\delta_{t_n}(y_n)\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\tag{3.14}$$

Since $\mathcal{S}$ is almost uniformly asymptotically regular and (3.14), we have

$$\begin{aligned}
&\|y_n-T(h)y_n\|\\
&\quad\leq\|y_n-\delta_{t_n}(y_n)\|+\|\delta_{t_n}(y_n)-T(h)\delta_{t_n}(y_n)\|+\|T(h)\delta_{t_n}(y_n)-T(h)y_n\|\\
&\quad\leq\|y_n-\delta_{t_n}(y_n)\|+\|\delta_{t_n}(y_n)-T(h)\delta_{t_n}(y_n)\|+L_s\|\delta_{t_n}(y_n)-y_n\|\longrightarrow 0.
\end{aligned}\tag{3.15}$$

It follows from the reflexivity of E and the boundedness of sequence $\{y_n\}$ that there exists $\{y_{n_j}\}$ which is a subsequence of $\{y_n\}$ converging weakly to $w\in E$ as $j\to\infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.1 that

$$\limsup_{j\to\infty}\Phi\left(\|y_{n_j}-y\|\right)=\limsup_{j\to\infty}\Phi\left(\|y_{n_j}-w\|\right)+\Phi(\|y-w\|),\quad\forall x\in E.\tag{3.16}$$

Let

$$H(x) = \limsup_{j \to \infty} \Phi\left(\left\|y_{n_j} - y\right\|\right), \quad \forall y \in E. \tag{3.17}$$

It follows that

$$H(y) = H(w) + \Phi(\|y - w\|), \quad \forall y \in E. \tag{3.18}$$

For $h \geq 0$, from (3.15) we obtain

$$\begin{aligned} H(T(h)w) &= \limsup_{j \to \infty} \Phi\left(\left\|y_{n_j} - T(h)w\right\|\right) = \limsup_{j \to \infty} \Phi\left(\left\|T(h)y_{n_j} - T(h)w\right\|\right) \\ &\leq \limsup_{j \to \infty} \Phi\left(\left\|y_{n_j} - w\right\|\right) = H(w). \end{aligned} \tag{3.19}$$

On the other hand, however,

$$H(T(h)w) = H(w) + \Phi(\|T(h)w - w\|). \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\Phi(\|T(h)w - w\|) = H(T(h)w) - H(w) \leq 0. \tag{3.21}$$

This implies that $T(h)w = w$ for all $h \geq 0$, and so $w \in F(\mathcal{S})$. Next, we show that $y_{n_j} \to w$ as $j \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau$, for all $t \geq 0$, and $\varphi : [0, \infty) \to [0, \infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t). \tag{3.22}$$

Following Lemma 2.1, we have

$$\begin{aligned}
\Phi(\|y_n - w\|) &= \Phi\left(\left\|(I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds - (I-\alpha_n A)w \right.\right.\\
&\qquad \left.\left. + \alpha_n(\gamma f(y_n) - \gamma f(w) + \gamma f(w) - A(w))\right\|\right)\\
&\le \Phi\left(\left\|(I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds - (I-\alpha_n A)w + \alpha_n\gamma(f(y_n) - f(w))\right\|\right)\\
&\quad + \alpha_n\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle\\
&\le \Phi\left(\varphi(1)(1-\alpha_n\overline{\gamma})\left\|\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds - \frac{1}{t_n}\int_0^{t_n} T(s)w\,ds\right\| + \alpha_n\gamma\alpha\|y_n - w\|\right)\\
&\quad + \alpha_n\gamma\langle f(w) - f(w), J_\varphi(y_n - w)\rangle\\
&\le \Phi\left(\varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)\|y_n - w\| + \alpha_n\gamma\alpha\|y_n - w\|\right)\\
&\quad + \alpha_n\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle\\
&\le \Phi\left(\left[\varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right) + \alpha_n\gamma\alpha\right]\|y_n - w\|\right)\\
&\quad + \alpha_n\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle\\
&\le \left[\varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right) + \alpha_n\gamma\alpha\right]\Phi(\|y_n - w\|)\\
&\quad + \alpha_n\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle.
\end{aligned} \tag{3.23}$$

This implies that

$$\Phi(\|y_n - w\|) \le \frac{1}{1 - \varphi(1)(1-\alpha_n\overline{\gamma})\left((1/t_n)\int_0^{t_n} L_s\,ds\right) + \alpha_n\gamma\alpha}\alpha_n\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle, \tag{3.24}$$

also

$$\Phi(\|y_n - w\|) \le \frac{1}{(\varphi(1)\overline{\gamma} - \alpha\gamma) - (1-\alpha_n\overline{\gamma})d_n}\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle, \tag{3.25}$$

where $d_n = ((1/t_n)\int_0^{t_n} L_s\, ds - 1)/\alpha_n$. Now observing that $y_{n_j} \rightharpoonup w$ implies $J_\varphi(y_{n_j} - w) \rightharpoonup^* 0$, we conclude from the above inequality that

$$\Phi\left(\left\|y_{n_j} - w\right\|\right) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.26}$$

Hence $y_{n_j} \to w$ as $j \to \infty$. Next, we prove that w solves the variational inequality (3.2). For any $z \in F(\mathcal{S})$, we observe that

$$\begin{aligned}
&\left\langle \left(y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n\, ds\right) - \left(z - \frac{1}{t_n}\int_0^{t_n} T(s)z\, ds\right), J_\varphi(y_n - z)\right\rangle \\
&\quad = \langle y_n - z, J_\varphi(y_n - z)\rangle - \left\langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n\, ds - \frac{1}{t_n}\int_0^{t_n} T(s)z\, ds, J_\varphi(y_n - z)\right\rangle \\
&\quad \geq \Phi(\|y_n - z\|) - \left\|\frac{1}{t_n}\int_0^{t_n} T(s)y_n\, ds - \frac{1}{t_n}\int_0^{t_n} T(s)z ds\right\| \|J_\varphi(y_n - z)\| \\
&\quad \geq \Phi(\|y_n - z\|) - \frac{1}{t_n}\int_0^{t_n} L_s\, ds \|y_n - z\| \|J_\varphi(y_n - z)\| \\
&\quad = \Phi(\|y_n - z\|) - \frac{1}{t_n}\int_0^{t_n} L_s\, ds \Phi(\|y_n - z\|) \\
&\quad = \left(1 - \frac{1}{t_n}\int_0^{t_n} L_s\, ds\right)\Phi(\|y_n - z\|).
\end{aligned} \tag{3.27}$$

Since

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n\, ds, \tag{3.28}$$

we can derive that

$$(A - \gamma f)(y_n) = -\frac{1}{\alpha_n}\left(y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n\, ds\right) + \left(A(y_n) - A\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n\, ds\right)\right). \tag{3.29}$$

Since Φ is strictly increasing and $\|y_n - p\| \le M$ for some $M > 0$, we have $\Phi(\|y_n - p\|) \le \Phi(M)$. Thus

$$\begin{aligned}
&\langle (A-\gamma f)(y_n), J_\varphi(y_n - z)\rangle \\
&\quad = -\frac{1}{\alpha_n}\left\langle \left(y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds\right) - \left(z - \frac{1}{t_n}\int_0^{t_n} T(s)z\,ds\right), J_\varphi(y_n - z)\right\rangle \\
&\qquad + \left\langle A(y_n) - A\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds\right), J_\varphi(y_n - z)\right\rangle \\
&\quad \le \frac{\left((1/t_n)\int_0^{t_n} L_s\,ds - 1\right)}{\alpha_n}\Phi(\|y_n - z\|) + \left\langle A(y_n) - A\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds\right), J_\varphi(y_n - z)\right\rangle \\
&\quad \le \frac{\left((1/t_n)\int_0^{t_n} L_s\,ds - 1\right)}{\alpha_n}\Phi(M) + \left\langle A\left(y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds\right), J_\varphi(y_n - z)\right\rangle.
\end{aligned} \tag{3.30}$$

Notice that

$$y_{n_j} - \frac{1}{t_{n_j}}\int_0^{t_{n_j}} T(s)y_{n_j}\,ds \longrightarrow w - \frac{1}{t_{n_j}}\int_0^{t_{n_j}} T(s)w\,ds = w - w = 0. \tag{3.31}$$

Now using (B3) and replacing n with n_j in (3.30) and letting $j \to \infty$, we have

$$\langle (A-\gamma f)w, J_\varphi(w - z)\rangle \le 0. \tag{3.32}$$

So, $w \in F(\mathcal{S})$ is a solution of the variational inequality (3.2), and hence $w = \tilde{x}$ by the uniqueness. In a summary, we have shown that each cluster point of $\{y_n\}$ (at $n \to \infty$) equals $\tilde{x}$. Therefore, $y_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

If $A \equiv I$, the identity mapping on E, and $\gamma = 1$, then Theorem 3.1 reduces to the following corollary.

Corollary 3.2. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathcal{S} = \{T(s) : s \ge 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings on E with a sequence $\{L_t\} \subset [1,\infty)$ and $F(\mathcal{S}) \ne \emptyset$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$ and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$ and $t_n > 0$. Then the following holds.*

(i) *If $(1/t_n)\int_0^{t_n} L_s\,ds - 1 < \alpha_n(1-\alpha)$, for all $n \in \mathbb{N}$, then there exists a sequence $\{y_n\}$ defined by*

$$y_n = \alpha_n f(y_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,ds, \quad n \ge 0. \tag{3.33}$$

(ii) *Suppose, in addition, that* $\mathcal{S}$ *is almost uniformly asymptotically regular and the real sequences* $\{\alpha_n\}$ *and* $\{t_n\}$ *satisfy the following:*

(B1) $\lim_{n\to\infty} t_n = \infty$;

(B2) $\lim_{n\to\infty} \alpha_n = 0$;

(B3) $\lim_{n\to\infty}((1/t_n)\int_0^{t_n} L_s\, ds - 1)/\alpha_n = 0$.

Then $\{y_n\}$ *converges strongly as* $n \to \infty$ *to a common fixed point* $\tilde{x}$ *in* $F(\mathcal{S})$ *which solves the variational inequality:*

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle \le 0, \quad z \in F(\mathcal{S}). \tag{3.34}$$

If $E := H$ is a Hilbert space and $\mathcal{S} = \{T(s) : s \ge 0\}$ is a strongly continuous semigroup of nonexpansive mappings on H, then we have $L_t \equiv 1$ and Theorem 3.1 reduces to the following corollary.

Corollary 3.3 (see [15, Theorem 3.1]). *Let* H *be a real Hilbert space. Suppose that* $f : H \to H$ *is a contraction with coefficient* $\alpha \in (0,1)$ *and* $\mathcal{S} = \{T(s) : s \ge 0\}$ *a strongly continuous semigroup of nonexpansive mappings on H such that* $F(\mathcal{S}) \ne \emptyset$. *Let* A *be a strongly positive bounded linear operator with coefficient* $\overline{\gamma} > 0$ *and let* $\{\alpha_n\}$ *and* $\{t_n\}$ *be sequences of real numbers such that* $0 < \alpha_n < 1$, $t_n > 0$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$, *then for any* $0 < \gamma < \overline{\gamma}/\alpha$, *there is a unique* $\{y_n\}$ *in* H *such that*

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n\, ds, \quad n \ge 0 \tag{3.35}$$

and the iterative sequence $\{y_n\}$ *converges strongly as* $n \to \infty$ *to a common fixed point* $\tilde{x}$ *in* $F(\mathcal{S})$ *which solves the variational inequality:*

$$\langle (A-\gamma f)\tilde{x}, \tilde{x}-z\rangle \le 0, \quad z \in F(\mathcal{S}). \tag{3.36}$$

Theorem 3.4. *Let* E *be a reflexive strictly convex Banach space which admits a weakly continuous duality mapping* J_φ *with gauge* φ *such that* φ *is invariant on* $[0,1]$. *Let* $\mathcal{S} = \{T(s) : s \ge 0\}$ *be a strongly continuous semigroup of asymptotically nonexpansive mappings on* E *with a sequence* $\{L_t\} \subset [1,\infty)$ *and* $F(\mathcal{S}) \ne \emptyset$. *Let* $f \in \Pi_E$ *with coefficient* $\alpha \in (0,1)$; *let* A *be a strongly positive bounded linear operator with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \varphi(1)\overline{\gamma}/\alpha$. *For any* $x_0 \in C$, *let the sequence* $\{x_n\}$ *be defined by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) x_n\, ds, \quad n \ge 0. \tag{3.37}$$

Suppose, in addition, that $\mathcal{S}$ *is almost uniformly asymptotically regular. Let* $\{\alpha_n\}$ *and* $\{t_n\}$ *be sequences of real numbers such that* $0 < \alpha_n < 1$, $t_n > 0$,

(C1) $\lim_{n\to\infty} t_n = \infty$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty}((1/t_n)\int_0^{t_n} L_s\, ds - 1)/\alpha_n = 0$.

Then $\{x_n\}$ *converges strongly as* $n \to \infty$ *to a common fixed point* $\tilde{x}$ *in* $F(\mathcal{S})$ *which solves the variational inequality* (3.2).

Proof. First we show that $\{x_n\}$ is bounded. By condition (C3) and given $0 < \varepsilon < \varphi(1)\overline{\gamma} - \alpha\gamma$ there exists $N > 0$ such that $((1/t_n)\int_0^{t_n} L_s\,ds - 1)/\alpha_n < \varepsilon$ for all $n \geq N$. Thus

$$(1-\varphi(1)\overline{\gamma}\alpha_n)\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds - 1\right) \leq \frac{1}{t_n}\int_0^{t_n} L_s\,ds - 1 < \varepsilon\alpha_n, \tag{3.38}$$

for all $n \geq N$. Since $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without the loss of generality, that $\alpha_n < \varphi(1)\|A\|^{-1}$.

Claim that $\|x_n - p\| \leq M$, $n \geq 0$, where $M := \max\{\|x_0 - p\|, \ldots, \|x_N - p\|, \|f(p) - p\|/(\varphi(1)\overline{\gamma} - \alpha\gamma - \varepsilon)\}$.

Let $p \in F(\mathcal{S})$. Then from (3.56) we get that

$$\begin{aligned}
\|x_{n+1}-p\| &= \left\|\alpha_n\gamma f(x_n) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - p\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - \alpha_n A(p) - (I-\alpha_n A)p\right\| \\
&\leq \|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(p) + \alpha_n\gamma f(p) - \alpha_n A(p)\| \\
&\quad + \left\|(I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - \frac{1}{t_n}\int_0^{t_n} T(s)p\,ds\right)\right\| \\
&\leq \|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(p) + \alpha_n\gamma f(p) - \alpha_n A(p)\| \\
&\quad + \|I-\alpha_n A\|\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - \frac{1}{t_n}\int_0^{t_n} T(s)p\,ds\right\| \\
&\leq \alpha_n\|\gamma f(p) - Ap\| + \alpha_n\alpha\gamma\|x_n - p\| + \varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)\|x_n - p\| \\
&= \alpha_n\|\gamma f(p) - Ap\| + \left(\alpha_n\alpha\gamma + \varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)\right)\|x_n - p\| \\
&\leq \alpha_n\|\gamma f(p) - Ap\| + \left(\alpha_n\alpha\gamma + \frac{1}{t_n}\int_0^{t_n} L_s\,ds - \varphi(1)\alpha_n\overline{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)\|x_n - p\| \\
&\leq \alpha_n\|\gamma f(p) - Ap\| \\
&\quad + \left(1 + (1-\varphi(1)\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds - 1\right) - \alpha_n(\varphi(1)\overline{\gamma} - \alpha\gamma)\right)\|x_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n \|f(p)-p\| + (1-\alpha_n(\varphi(1)\overline{\gamma}-\alpha\gamma)+\varepsilon\alpha_n)\|x_n-p\| \\
&= \alpha_n \|f(p)-p\| + (1-\alpha_n(\varphi(1)\overline{\gamma}-\alpha\gamma-\varepsilon))\|x_n-p\| \\
&\le \max\left\{ \frac{\|f(p)-p\|}{(\varphi(1)\overline{\gamma}-\alpha\gamma-\varepsilon)}, \|x_n-p\| \right\}.
\end{aligned}
\tag{3.39}
$$

By induction,

$$
\|x_n-p\| \le \max\left\{ \frac{\|f(p)-p\|}{(\varphi(1)\overline{\gamma}-\alpha\gamma-\varepsilon)}, \|x_N-p\| \right\}, \quad \forall n \ge N, \tag{3.40}
$$

and hence $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{(1/t_n)\int_0^{t_n} T(s)x_n\,ds\}$. Let $\delta_{t_n}(x_n) := (1/t_n)\int_0^{t_n} T(s)x_n\,ds$. Then, since $\alpha_n \to 0$ as $n \to \infty$, we obtain that

$$
\|x_{n+1}-\delta_{t_n}(x_n)\| = \alpha_n \|\gamma f(x_n) - A\delta_{t_n}(x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.41}
$$

For any $h > 0$, we have

$$
\begin{aligned}
\|T(h)x_{n+1}-x_{n+1}\| &\le \|T(h)x_{n+1}-T(h)\delta_{t_n}(x_n)\| + \|T(h)\delta_{t_n}(x_n)-\delta_{t_n}(x_n)\| + \|\delta_{t_n}(x_n)-x_{n+1}\| \\
&\le L_h\|x_{n+1}-\delta_{t_n}(x_n)\| + \|T(h)\delta_{t_n}(x_n)-\delta_{t_n}(x_n)\| + \|\delta_{t_n}(x_n)-x_{n+1}\|,
\end{aligned}
\tag{3.42}
$$

it follows from (3.41) and $\mathcal{S}$ is almost uniformly asymptotically regular that

$$
\|T(h)x_{n+1}-x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.43}
$$

Next, we prove that

$$
\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle \le 0. \tag{3.44}
$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$
\lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k}-\tilde{x})\rangle = \limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle. \tag{3.45}
$$

It follows from the reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in E$ as $i \to \infty$. Since J_φ is weakly continuous, we have by Lemma 2.1 that

$$
\limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}}-x\right\|\right) = \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}}-w\right\|\right) + \Phi(\|x-w\|), \quad \forall x \in E. \tag{3.46}
$$

Let

$$H(x) = \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - x\right\|\right), \quad \forall x \in E. \tag{3.47}$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \quad \forall x \in E. \tag{3.48}$$

From (3.43), for each $h > 0$, we obtain

$$\begin{aligned} H(T(h)w) &= \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - T(h)w\right\|\right) = \limsup_{i\to\infty} \Phi\left(\left\|T(h)x_{n_{k_i}} - T(h)w\right\|\right) \\ &\le \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - w\right\|\right) = H(w). \end{aligned} \tag{3.49}$$

On the other hand, however,

$$H(T(h)w) = H(w) + \Phi(\|T(h)w - w\|). \tag{3.50}$$

It follows from (3.49) and (3.50) that

$$\Phi(\|T(h)w - w\|) = H(T(h)w) - H(w) \le 0. \tag{3.51}$$

This implies that $T(h)w = w$ for all $h > 0$, and so $w \in F(\mathcal{S})$. Since the duality map J_φ is single valued and weakly continuous, we get that

$$\begin{aligned} \limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle &= \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle \\ &= \lim_{i\to\infty} \left\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi\left(x_{n_{k_i}} - \tilde{x}\right)\right\rangle \\ &= \langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - w)\rangle \le 0 \end{aligned} \tag{3.52}$$

as required. Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$. It follows from Lemma 2.1(i) that

$$\begin{aligned} \Phi(\|x_{n+1} - \tilde{x}\|) = \Phi\Bigg(\Bigg\|&(I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - (I - \alpha_n A)\tilde{x} \\ &+ \alpha_n(\gamma f(x_n) - \gamma f(\tilde{x}) + \gamma f(\tilde{x}) - A(\tilde{x}))\Bigg\|\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\leq \Phi\left(\left\|(I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds-(I-\alpha_n A)\tilde{x}+\alpha_n\gamma\big(f(x_n)-f(\tilde{x})\big)\right\|\right)\\
&\quad+\alpha_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(y_n-\tilde{x})\rangle\\
&\leq \Phi\left(\varphi(1)(1-\alpha_n\overline{\gamma})\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds-\frac{1}{t_n}\int_0^{t_n} T(s)\tilde{x}ds\right\|+\alpha_n\gamma\alpha\|x_n-\tilde{x}\|\right)\\
&\quad+\alpha_n\gamma\langle f(\tilde{x})-f(\tilde{x}),J_\varphi(y_n-\tilde{x})\rangle\\
&\leq \Phi\left(\varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)\|x_n-\tilde{x}\|+\alpha_n\gamma\alpha\|x_n-\tilde{x}\|\right)\\
&\quad+\alpha_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle\\
&\leq \Phi\left(\left[\varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)+\alpha_n\gamma\alpha\right]\|x_n-\tilde{x}\|\right)\\
&\quad+\alpha_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle\\
&\leq \left[\varphi(1)(1-\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds\right)+\alpha_n\gamma\alpha\right]\Phi(\|x_n-\tilde{x}\|)\\
&\quad+\alpha_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle\\
&\leq \left[(1-\varphi(1)\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds-1\right)+1-\alpha_n\big(\varphi(1)\overline{\gamma}-\gamma\alpha\big)\right]\Phi(\|x_n-\tilde{x}\|)\\
&\quad+\alpha_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle\\
&\leq \big(1-\alpha_n\big(\varphi(1)\overline{\gamma}-\gamma\alpha\big)\big)\Phi(\|x_nX-\tilde{x}\|)+(1-\varphi(1)\alpha_n\overline{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s\,ds-1\right)M\\
&\quad+\alpha_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle,
\end{aligned}
\tag{3.53}
$$

where $M>0$ such that $\Phi(\|x_n-\tilde{x}\|)\leq M$. Put

$$
\begin{gathered}
s_n=\alpha_n\big(\varphi(1)\overline{\gamma}-\gamma\alpha\big),\\
t_n=\left(\frac{1-\varphi(1)\alpha_n\overline{\gamma}}{\varphi(1)\overline{\gamma}-\gamma\alpha}\right)\left(\frac{(1/t_n)\int_0^{t_n} L_s\,ds-1}{\alpha_n}\right)M+\frac{1}{(\varphi(1)\overline{\gamma}-\gamma\alpha)}\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle.
\end{gathered}
\tag{3.54}
$$

Then (3.53) is reduced to

$$
\Phi(\|x_{n+1}-\tilde{x}\|)\leq(1-s_n)\Phi(\|x_n-\tilde{x}\|)+s_nt_n. \tag{3.55}
$$

Applying Lemma 2.3 to (3.55), we conclude that $\Phi(\|x_{n+1} - \tilde{x}\|) \to 0$ as $n \to \infty$; that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

If $A \equiv I$, the identity mapping on E, and $\gamma = 1$, then Theorem 3.4 reduces to the following corollary.

Corollary 3.5. *Let E be a reflexive strictly convex Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings from C into C with a sequence $\{L_t\} \subset [1,\infty)$, $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$ and the sequence $\{x_n\}$ be defined by $x_0 \in C$,*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds, \quad n \geq 0. \tag{3.56}$$

Suppose, in addition, that $\mathcal{S}$ is almost uniformly asymptotically regular. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$, $t_n > 0$,

(C1) $\lim_{n\to\infty} t_n = \infty$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty}((1/t_n)\int_0^{t_n} L_s\,ds - 1)/\alpha_n = 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle \leq 0, \quad z \in F(\mathcal{S}). \tag{3.57}$$

If $E := H$ is a Hilbert space and $\mathcal{S} = \{T(s) : s \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on H, then we have $L_t \equiv 1$ and Theorem 3.4 reduces to the following corollary.

Corollary 3.6 (see [15, Theorem 3.2]). *Let H be a real Hilbert space. Suppose that $f : H \to H$ is a contraction with coefficient $\alpha \in (0,1)$ and $\mathcal{S} = \{T(s) : s \geq 0\}$ a strongly continuous semigroup of nonexpansive mappings on H such that $F(\mathcal{S}) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$ and let the sequence $\{x_n\}$ be defined by $x_0 \in C$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds, \quad n \geq 0. \tag{3.58}$$

Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$, $t_n > 0$,

(C1) $\lim_{n\to\infty} t_n = \infty$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\langle (A-\gamma f)\tilde{x}, \tilde{x}-z\rangle \leq 0, \quad z \in F(\mathcal{S}). \tag{3.59}$$

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