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Research Article

Periodic Solutions of a Type of Liénard Higher Order Delay Functional Differential Equation with Complex Deviating Argument

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The author has studied the existence of periodic solutions of a type of higher order delay functional differential equations with neutral type by using the theory of coincidence degree, and some new sufficient conditions for the existence of periodic solutions have been obtained.

1. Introduction and Lemma

With the rapid development of modern science and technology, functional differential equation with time delay has been widely applied in many areas such as bioengineering, systems analysis, and dynamics. Functional differential equation with complex deviating argument is an important type of the above function. Because the property of the solution to this kind of equation is impossibly estimated, so the literature on the functional differential equation with complex argument is relatively rare [1]. In recent years, with the maturity of the theory of nonlinear functional analysis and algebraic topology, we have the powerful tools of the study on the functional differential equation with complex deviating argument, so it is possible to study the above equation. Furthermore, the study on the periodic solutions of functional differential equation is always one of the most important subject that people concerned for its widespread use. Many results of the study of Duffing-typed functional differential equation and Liénard-typed functional differential equation have been obtained, for example, the literatures [2–18]. Hitherto, the literature of the discussion of higher order functional differential equations has not been found a lot [19]. In this paper I have studied and derived some sufficient conditions that guarantee the existence of periodic solutions for a type of higher order functional differential equations with complex deviating argument as the following:

$$\sum_{i=1}^{m} a_i x^{(i)}(t) + f(x(t))\dot{x}(t) + \beta(t)g(x(x(t))) = p(t) \quad (a_i \neq 0), \tag{$*$}$$

and some new results have been obtained.

In order to establish the existence of T-periodic solutions of $(*)$, we make some preparations.

Definition 1.1. Let X, Y are Banach spaces, and let Ω be an open and bounded subset in X, and let $L : \mathrm{Dom}(L) \subseteq X \to Y$ be linear mapping; the mapping L will be called a Fredholm mapping of index zero if dim ker L = codim Im $L < +\alpha$ and Im L is closed in Y.

Definition 1.2. Let $P : X \to \ker(L)$, let $Q : Y \to Y/\operatorname{Im}(L)$ be projectors, and let $N : \overline{\Omega} \to Y$ be nonlinear mapping; the mapping N will be called L-compact on $\overline{\Omega}$ if $QN : \overline{\Omega} \to Y/\operatorname{Im}(L)$ and $(L|_{\ker(P)})^{-1}(I - Q)N : \overline{\Omega} \to X$ are compact.

Lemma 1.3 (see [20]). *Let X, Y be Banach spaces; $L : D_L \subset X \to Y$ is a Fredholm mapping of index zero $P : X \to X$; $Q : X \to Y$ are continuous mapping projectors; Ω is an open bounded set in X; $N : \overline{\Omega} \times [0,1] \to Y$ is L-Compact on $\overline{\Omega}$, furthermore suppose that:*

(a) $Lx \neq \lambda N(x,\lambda)$, *for all* $x \in D_L \cap \partial\Omega$, $\lambda \in (0,1)$;

(b) $QN(x,0) \neq 0$, *for all* $x \in \ker(L) \cap \partial\Omega$;

(c) $\deg(QN(x,0),\ \ker(L) \cap \Omega, 0) \neq 0$,

then the equation $Lx = N(x,1)$ has at least one solution on $\overline{\Omega}$, where deg *is Brouwer degree.*

2. Main Results and Proof of Theorems

Theorem 2.1. *Suppose that f, β, g, p are continuous for their variables, respectively, $p(t+T) = p(t)$, $\beta(t+T) = \beta(t) > 0$, $\int_0^T p(t)dt = 0$, and furthermore suppose that*

(a) $\exists A > 0$, *for all* $x \in \mathbb{R}$, *when* $|x| > A$, *such that* $xg(x) > 0$;

(b) $\exists M > 0$, *for all* $x \in \mathbb{R}$, *such that* $|g(x)| \leq M$;

(c) $f_1 = \sup_{x\in\mathbb{R}}|f(x)| < (a_m - k(T^{m-1} - T^{m-2} - \cdots - T))/T^{m-1}$,

where $k = \max\{|a_i|\}$, $i = 1, 2, \ldots, m-1$ and $a_m > k(T^{m-1} + T^{m-2} + \cdots + T)$, then $(*)$ has at least one T-periodic solution.

Proof of Theorem 2.1. In order to use continuation theorem to obtain T-periodic solution of $(*)$, we firstly make some required preparations. Let

$$X = \left\{x \in c^{m-1}(\mathbb{R},\mathbb{R}) \mid x(t+T) = x(t)\right\}, \qquad Y = \{y \in C(\mathbb{R},\mathbb{R}) \mid y(t+T) = y(t)\}, \tag{2.1}$$

and the norm of X and Y is $\|x\| = \max_{0\leq i\leq m-1}\{|x^{(i)}|_\infty\}$, $|x^{(i)}|_\infty = \max_{t\in\mathbb{R}}\{|x^{(i)}(t)|\}$, $i = 1, 2, \ldots, m-1$, and $\|y\| = \max_{t\in\mathbb{R}}\{|y(t)|\}$, respectively; then the X and Y with this norm are Banach spaces.

Firstly, we study the priori bound of T-periodic solution of following equation:

$$\sum_{i=1}^{m} a_i x^{(i)}(t) + \lambda f(x(t))\dot{x}(t) + \lambda\beta(t)g(x(x(t))) = \lambda^2 p(t). \tag{2.2}$$

Suppose that $x = x(t) \in X$ is an arbitrary T-periodic solution of (2.2), put $x(t)$ into, (2.2) and then integrate both sides of (2.2) on $[0, T]$, so

$$\int_0^T \beta(t)g(x(x(t)))dt = 0. \tag{2.3}$$

For the continuity of β, g, x, there must exist a number $t_0 \in [0, T]$ such that

$$\beta(t_0)g(x(x(t_0))) = 0, \tag{2.4}$$

that is,

$$g(x(x(t_0))) = 0. \tag{2.5}$$

For the condition (a) of Theorem 2.1, we have

$$|x(x(t_0))| \leq A. \tag{2.6}$$

Let

$$x(t_0) = nT - t_1, \quad n \in N, \ t_1 \in [0, T], \tag{2.7}$$

so

$$|x(t_1)| = |x(x(t_0))| \leq A. \tag{2.8}$$

In view of

$$\forall t \in [0, T], \quad x(t) = x(t_1) + \int_{t_1}^{t} \dot{x}(s)ds, \tag{2.9}$$

we have

$$|x(t)| = \left| x(t_1) + \int_{t_1}^{t} \dot{x}(s)ds \right| \leq A + \int_{t_1}^{t} |\dot{x}(s)|ds \leq A + \int_0^T |\dot{x}(t)|dt, \tag{2.10}$$

that is,

$$\left|x^{(0)}\right|_{\infty} = |x|_{\infty} \leq A + \int_0^T |\dot{x}(t)|dt. \tag{2.11}$$

Noting $x(t) = x(t+T)$, so there must exist the number $\xi_i \in [0,T]$ such that $x^{(i)}(\xi_i) = 0$, where $i = 1,2,3,\ldots,m-1$.

For all $t \in [0,T]$,

$$x^{(i)}(t) = x^{(i)}(\xi_i) + \int_{\xi_i}^{t} x^{(i+1)}(s)ds = \int_{\xi_i}^{t} x^{(i+1)}(s)ds, \tag{2.12}$$

we have

$$\begin{aligned}\left|x^{(i)}(t)\right| &= \left|\int_{\xi_i}^{t} x^{(i+1)}(s)ds\right| \leq \int_0^T \left|x^{(i+1)}(t)\right|dt \leq T\int_0^T \left|x^{(i+2)}(t)\right|dt \\ &\leq T^2 \cdot \int_0^T \left|x^{(i+3)}(t)\right|dt \leq \cdots \leq T^{m-(i+1)}\int_0^T \left|x^{(i+m-i)}(t)\right|dt = T^{m-(i+1)}\int_0^T \left|x^{(m)}(t)\right|dt,\end{aligned} \tag{2.13}$$

that is,

$$\left|x^{(i)}\right|_{\infty} \leq T^{m-(i+1)}\int_0^T \left|x^{(m)}(t)\right|dt, \quad i = 1,2,\ldots,m-1. \tag{2.14}$$

Combining (2.11), (2.14), we get

$$\left|x^{(0)}\right|_{\infty} = |x|_{\infty} \leq A + T^{m-1}\int_0^T \left|x^{(m)}(t)\right|dt. \tag{2.15}$$

By (2.2), we get

$$\begin{aligned}\int_0^T |a_m x^m(t)|dt \leq& \int_0^T \left|\lambda f(x(t))\dot{x}(t)\right|dt + \int_0^T \left|\lambda\beta(t)g(x(x(t)))\right|dt + \int_0^T \left|\lambda^2 p(t)\right|dt \\ &+ \int_0^T |a_1\dot{x}(t)|dt + \int_0^T |a_2\ddot{x}(t)|dt + \cdots + \int_0^T \left|a_{m-3}x^{(m-3)}(t)\right|dt \\ &+ \int_0^T \left|a_{m-2}x^{(m-2)}(t)\right|dt + \cdots + \int_0^T \left|a_{m-1}x^{(m-1)}(t)\right|dt,\end{aligned} \tag{2.16}$$

where $\beta_1 = \max_{t\in\mathbb{R}} \beta(t)$, $p_1 = \max_{t\in\mathbb{R}}\{|p(t)|\}$, and $k = \max\{|a_i|\}$, $i = 1,2,3,\ldots,m-1$.

Noting (2.14) and the conditions (b), (c) of Theorem 2.1, we have

$$\begin{aligned}\int_0^T \left|a_m x^{(m)}(t)\right|dt \le f_1 T\cdot T^{m-2}\int_0^T\left|x^{(m)}(t)\right|dt + \beta_1 TM + p_1 T \\ + kT\cdot T^{m-(1+1)}\int_0^T\left|x^{(m)}(t)\right|dt + kT\cdot T^{m-(2+1)}\int_0^T\left|x^{(m)}(t)\right|dt \\ + \cdots + kT\cdot T^{m-(m-1+1)}\int_0^T\left|x^{(m)}(t)\right|dt,\end{aligned} \tag{2.17}$$

so

$$a_m\int_0^T\left|x^{(m)}(t)\right|dt \le \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1T^{m-1}\right)\int_0^T\left|x^{(m)}(t)\right|dt + \beta_1 TM + p_1 T, \tag{2.18}$$

where $a_m > T^{m-1}\sum_{i=1}^m f_i + kT^{m-1} + kT^{m-2} + \cdots + kT$.

Let

$$\frac{\beta_1 TM + p_1 T}{a_0 - \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1}\right)} \triangleq A_1, \tag{2.19}$$

that is,

$$\int_0^T\left|x^{(m)}(t)\right|dt \le A_1. \tag{2.20}$$

Noting (2.14), (2.15), and (2.20), we have

$$\begin{aligned}\left|x^{(0)}\right|_\infty = |x|_\infty \le A + T^{m-1}A_1 \triangleq \omega_0, \\ \left|x^{(i)}\right|_\infty \le T^{m-(i+1)}A_1 \triangleq \omega_i, \quad i = 1,2,\ldots,m-1.\end{aligned} \tag{2.21}$$

Let $\omega = \max_{0\le i\le m}\{\omega_i + 1\}$, and let $\Omega = \{x \mid x\in X : \|x\| < \omega\}$; then Ω is an open and bounded set in X.

Let

$$\begin{aligned}L : D_L \subset X \longrightarrow Y : x \longrightarrow Lx = \sum_{i=1}^m a_i x^{(i)}(t), \\ N : X\times I \longrightarrow Y : x \longrightarrow N(x,\lambda) = -f(x(t))\dot{x}(t) - \beta(t)g(x(x(t))) + \lambda p(t);\end{aligned} \tag{2.22}$$

then the corresponding equation of $Lx = \lambda N(x,\lambda)$ is (2.2).

Now, we define projection operators as follows;

$$P : X \longrightarrow \ker(L) : x \longrightarrow Px = \frac{1}{T}\int_0^T x(t)dt,$$
$$Q : Y \longrightarrow \frac{Y}{\operatorname{Im}(L)} : y \longrightarrow Qy = \frac{1}{T}\int_0^T y(t)dt. \tag{2.23}$$

Obviously, P, Q are continuous operators, $\operatorname{Im}(P) = \mathbb{R} = \ker(L)$, $\ker(Q) = \operatorname{Im}(L)$, and it is easy to prove that L is a Fredholm mapping of index zero and is L-Compact on $\overline{\Omega}$.

From the above discussion and the construction of Ω, we know that for all $x \in D_L \cap \partial\Omega$, $\lambda \in (0,1)$, $Lx \neq \lambda N(x,\lambda)$, therefore the condition (a) of Lemma 1.3 holds.

For arbitrary $x \in \ker(L) \cap \partial\Omega$, $\|x\| = \omega$, by the definition of Q, N, we have

$$\begin{aligned} QN(x,0) &= \frac{1}{T}\int_0^T \left[-f(x(t))\dot{x}(t) - \beta(t)g(x(x(t)))\right]dt \\ &= -\frac{1}{T}\int_0^T \beta(t)g(x(x(t)))dt, \end{aligned} \tag{2.24}$$

so

$$\begin{aligned} xQN(x,0) &= -\frac{1}{T}x\int_0^T \beta(t)g(x(x(t)))dt \\ &= -\frac{1}{T}xg(x)\int_0^T \beta(t)dt \neq 0, \end{aligned} \tag{2.25}$$

therefore the condition (b) of Lemma 1.3 holds.

Making a transformation.

$$H(x,\mu) = -\mu x + (1-\mu)QN(x,0), \quad \forall x \in \partial\Omega \cap \ker(L),\ \mu \in [0,1], \tag{2.26}$$

we have

$$\begin{aligned} xH(x,\mu) &= -\mu x^2 + x(1-\mu)QN(x,0) \\ &= -\mu x^2 - (1-\mu)\frac{1}{T}g(x)x\int_0^T \beta(t)dt < 0. \end{aligned} \tag{2.27}$$

So $xH(x,\mu) \neq 0$, that is, $H(x,\mu) \neq 0$ is a homotopy, $\deg(QN(x,0), \ker(L) \cap \Omega, 0) = \deg(-I, \ker(L) \cap \Omega, 0) = \deg(-I, \mathbb{R} \cap \Omega, 0) \neq 0$, where I is an identity mapping, and the condition (c) of Lemma 1.3 holds.

From above all, the requirements of Lemma 1.3 are all satisfied, so (∗) has at least one T-periodic solution under the condition of Theorem 2.1, so the proof of Theorem 2.1 is completed. □

Remark 2.2. In Theorem 2.1, if $\beta(t) < 0$ and the condition (a) of Theorem 2.1 is when $|x| > A$, $xg(x) < 0$, and the rest are unchangeable, then $(*)$ has at least one T-periodic solution.

If the $g(x)$ is not a bounded function, we have the following theorem.

Theorem 2.3. *Suppose that f, β, g, p are continuous for their variables, respectively, $p(t+T) = p(t)$, $\beta(t+T) = \beta(t) > 0$, $\int_0^T p(t)dt = 0$, and furthermore suppose following:*

(a) $\exists A > 0$, *for all* $x \in \mathbb{R}$, *when* $|x| > A$, *such that* $xg(x) > 0$;

(b) $\exists M > 0$, *for all* $x \in \mathbb{R}$, *such that* $|g(x)| \leq M|x| + c$;

(c) $f_1 = \sup_{x\in\mathbb{R}}|f(x)| < (a_m - kT^{m-1} - kT^{m-2} - \cdots - kT - \beta_1 T^m)/T^{m-1}$,

where $k = \max\{|a_i|\}$, $i = 1, 2, \ldots, m-1$, *and* $a_m > kT^{m-1} + kT^{m-2} + \cdots + kT + \beta_1 T^m$, *then* $(*)$ *has at least one T-periodic solution.*

Proof of Theorem 2.3. Banach spaces X, Y and the mappings L, P, Q, and N are the same to Theorem 2.1, and their property are equal to Theorem 2.1, then the corresponding equation of $Lx = \lambda N(x, \lambda)$ is

$$\sum_{i=1}^{m} a_i x^{(i)}(t) + \lambda f(x(t))\dot{x}(t) + \lambda\beta(t) g(x(x(t))) = \lambda^2 p(t). \tag{2.28}$$

It is similar to Theorem 2.1, there must exist a number $t_1 \in [0, T]$, such that

$$|x(t_1)| \leq A, \tag{2.29}$$

and it is easy to obtain

$$\begin{gathered} \left|x^{(i)}\right|_\infty \leq T^{m-(i+1)} \int_0^T \left|x^{(m)}(t)\right| dt, \quad i = 1, 2, \ldots, m-1, \\ \left|x^{(0)}\right|_\infty = |x|_\infty \leq A + T^{m-1} \int_0^T \left|x^{(m)}(t)\right| dt. \end{gathered} \tag{2.30}$$

Noting (2.28), (2.30) and the conditions (b), (c) of Theorem 2.3, we have

$$\begin{aligned} \int_0^T \left|a_m x^{(m)}\right| dt \leq & \int_0^T \left|\lambda f(x(t))\dot{x}(t)\right| dt + \int_0^T \left|\lambda\beta(t) g(x(x(t)))\right| dt + \int_0^T \left|\lambda^2 p(t)\right| dt \\ & + kT \cdot T^{m-(1+1)} \int_0^T |x^m(t)| dt + kT \cdot T^{m-(2+1)} \int_0^T |x^m(t)| dt \\ & + \cdots + kT \cdot T^{m-(m-1+1)} \int_0^T |x^m(t)| dt \end{aligned}$$

$$\begin{aligned}
&\leq f_1 T \cdot T^{m-2} \int_0^T |x^m(t)| dt + \beta_1 T[M|x(x(t))| + c] + p_1 T \\
&\quad + kT \cdot T^{m-(1+1)} \int_0^T |x^m(t)| dt + kT \cdot T^{m-(2+1)} \int_0^T |x^m(t)| dt \\
&\quad + \cdots + kT \cdot T^{m-(m-1+1)} \int_0^T |x^m(t)| dt.
\end{aligned} \tag{2.31}$$

So

$$\begin{aligned}
a_m \int_0^T |x^m(t)| dt &\leq \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} \right) \int_0^T |x^m(t)| dt \\
&\quad + \beta_1 Tc + \beta_1 TM |x|_\alpha + p_1 T \\
&\leq \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} \right) \int_0^T |x^m(t)| dt \\
&\quad + \beta_1 Tc + \beta_1 TM \left(A + T^{m-1} \int_0^T \left| x^{(m)}(t) \right| dt \right) + p_1 T \\
&\leq \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} + \beta_1 T^m \right) \int_0^T |x^m(t)| dt \\
&\quad + \beta_1 Tc + \beta_1 TMA + p_1 T,
\end{aligned} \tag{2.32}$$

where $k = \max\{|a_i|\},\ i = 1, 2, 3, \ldots, m-1$, and $a_m > kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} + \beta_1 T^m$.

Let

$$\frac{\beta_1 Tc + \beta_1 TMA + p_1 T}{a_m - \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} + \beta_1 T^m \right)} \triangleq A_1, \tag{2.33}$$

that is,

$$\int_0^T |x^m(t)| dt \leq A_1. \tag{2.34}$$

Noting (2.30) and (2.34), we have

$$\begin{aligned}
\left| x^{(0)} \right|_\infty = |x|_\infty &\leq A + T^{m-1} A_1 \triangleq \omega_0, \\
\left| x^{(i)} \right|_\infty &\leq T^{m-(i+1)} A_1 \triangleq \omega_i, \quad i = 1, 2, \ldots, m-1.
\end{aligned} \tag{2.35}$$

Let $\omega = \max_{0 \leq i \leq m}\{\omega_i + 1\}$, and we take $\Omega = \{x \mid x \in X : \|x\| < \omega\}$; then Ω is an open and bounded set in X.

Similarly to Theorem 2.1, we prove easily that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$ and the conditions (a), (b), and (c) of Lemma 1.3 hold.

From above all, the requirements of Lemma 1.3 are all satisfied, so $(*)$ has at least one T-periodic solution under the condition of Theorem 2.3, so far the proof of Theorem 2.3 is completed. □

Remark 2.4. In Theorem 2.3, if $\beta(t) < 0$ and the condition (a) of Theorem 2.3 is when $|x| > A$, $xg(x) < 0$, and the rest are unchangeable, then $(*)$ has at least one T-periodic solution.

If the $\int_0^T p(t)dt \neq 0$, we have the following theorem.

Theorem 2.5. *Suppose that f, β, g, p are continuous for their variables, respectively, $\beta(t+T) = \beta(t) > 0$, and meet the condition (a) of Theorem 2.1 and furthermore suppose as follows:*

(a) $\lim_{|x|\to+\alpha}|g(x)| = +\alpha$;

(b) $\exists a, b, c > 0$, *such that* $|g(x)| \leq ag(x) + b|x| + c$;

(c) $f_1 = \sup_{x\in\mathbb{R}}|f(x)|\ (a_m - kT^{m-1} - kT^{m-2} - \cdots - kT - f_1T^{m-1} - b\beta_1T^m)/T^{m-1}$,

where $k = \max\{|a_i|\}$, $i = 1, 2, \ldots, m-1$, *and* $a_m > kT^{m-1} + kT^{m-2} + \cdots + kT + f_1T^{m-1} + b\beta_1T^m$, *then $(*)$ has at least one T-periodic solution.*

Proof of Theorem 2.5. Banach spaces X, Y and the mappings L, P, Q, and N are the same to Theorem 2.1, and their property are equal to Theorem 2.1, then the corresponding equation of $Lx = \lambda N(x, \lambda)$ is

$$\sum_{i=1}^{m} a_i x^{(i)}(t) + \lambda f(x(t))\dot{x}(t) + \lambda\beta(t)g(x(x(t))) = \lambda^2 p(t). \tag{2.36}$$

Suppose that $x = x(t) \in X$ is an arbitrary T-periodic solution of (2.36), put $x(t)$ into (2.36), and then integrate both sides of (2.36) on $[0, T]$, so

$$\int_0^T \beta(t)g(x(x(t)))dt = \int_0^T \lambda p(t)dt. \tag{2.37}$$

For the continuity of β, g, x, there must exist a number $t_1 \in [0, T]$, such that

$$g(x(x(t_1))) = \frac{\lambda\int_0^T p(t)dt}{\int_0^T \beta(t)dt}. \tag{2.38}$$

Combing the condition (a) of Theorem 2.5, there must exist $A_1 > 0$, such that

$$|x(x(t_1))| \leq A_1. \tag{2.39}$$

Similarly to Theorem 2.1, we have

$$\left|x^{(i)}\right|_{\infty} \le T^{m-(i+1)} \int_0^T \left|x^{(m)}(t)\right| dt, \quad i = 1, 2, \dots, m-1, \tag{2.40}$$

$$\left|x^{(0)}\right|_{\infty} = |x|_{\infty} \le A_1 + T^{m-1} \int_0^T \left|x^{(m)}(t)\right| \, dt. \tag{2.41}$$

By (2.36), (2.37), (2.39), and (2.41) and the conditions (b), (c) of Theorem 2.5, we have

$$\begin{aligned}
\int_0^T \left|a_m x^{(m)}\right| dt \le{}& \int_0^T \left|\lambda f(x(t))\dot{x}(t)\right| dt + \int_0^T \left|\lambda \beta(t) g(x(x(t)))\right| dt + \int_0^T \left|\lambda^2 p(t)\right| dt \\
&+ kT \cdot T^{m-(1+1)} \int_0^T |x^m(t)| dt + kT \cdot T^{m-(2+1)} \int_0^T |x^m(t)| dt \\
&+ \cdots + kt \cdot T^{m-(m-1+1)} \int_0^T |x^m(t)| dt \\
\le{}& f_1 T |\dot{x}|_{\alpha} + \left(kT^{m-1} + kT^{m-2} + \cdots + a_{m-1}T\right) \int_0^T |x^m(t)| dt \\
&+ \int_0^T a\beta(t) g(x(x(t))) dt + \int_0^T b\beta(t) [|x(x(t))| + c] dt + p_1 T \\
\le{}& f_1 T T^{m-2} \int_0^T \left|x^{(m)}(t)\right| dt + \left(kT^{m-1} + kT^{m-2} + \cdots + kT\right) \int_0^T |x^m(t)| dt \\
&+ a\beta_1 T p_1 + b\beta_1 T |x|_{\alpha} + b\beta_1 T c + p_1 T \le f_1 T^{m-1} \int_0^T \left|x^{(m)}(t)\right| dt \\
&+ \left(kT^{m-1} + kT^{m-2} + \cdots + kT\right) \int_0^T |x^m(t)| dt \\
&+ b\beta_1 T \left(A_1 + T^{m-1} \int_0^T \left|x^{(m)}(t)\right| dt \right) + a\beta_1 T p_1 + b\beta_1 T c + p_1 T \\
\le{}& \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} + b\beta_1 T^m\right) \int_0^T |x^m(t)| dt \\
&+ b\beta_1 T A_1 + a\beta_1 T p_1 + b\beta_1 T c + p_1 T.
\end{aligned} \tag{2.42}$$

So

$$\begin{aligned}
&\left[a_m - \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} + b\beta_1 T^m\right)\right] \int_0^T |x^m(t)| dt \\
&\quad \le b\beta_1 T A_1 + a\beta_1 T p_1 + b\beta_1 T c + p_1 T.
\end{aligned} \tag{2.43}$$

Let

$$\frac{b\beta_1 T A_1 + a\beta_1 T p_1 + b\beta_1 T c + p_1 T}{a_m - \left(kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} + b\beta_1 T^m\right)} \triangleq A_2, \tag{2.44}$$

that is,

$$\int_0^T |x^m(t)| dt \le A_2. \tag{2.45}$$

Noting (2.40), (2.41), and (2.45), we have

$$\begin{aligned} \left|x^{(0)}\right|_\infty &= |x|_\infty \le A + T^{m-1} A_2 \triangleq \omega_0, \\ \left|x^{(i)}\right|_\infty &\le T^{m-(i+1)} A_2 \triangleq \omega_i, \quad i = 1, 2, \ldots, m-1. \end{aligned} \tag{2.46}$$

For condition (a), there exist $M_0 > 0$ and $A_0 > 0$, such that $|x| > M_0$, $|g(x)| > A_0$; let $\omega = \max_{0 \le i \le m}\{\omega_i + 1, M_0\}$, and we take $\Omega = \{x \mid x \in X : \|x\| < \omega\}$; then Ω is an open and bounded set in X.

Similarly to Theorem 2.1, we prove easily that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$ and the conditions (a), (b), and (c) of Lemma 1.3 hold.

From above all, the requirements of Lemma 1.3 are all satisfied, so $(*)$ has at least one T-periodic solution under the condition of Theorem 2.5, so the proof of Theorem 2.5 is completed. □

Remark 2.6. In Theorem 2.5, if $\beta(t) < 0$ and the condition (a) of Theorem 2.1 is when $|x| > A$, $xg(x) < 0$, and the rest are unchangeable, then $(*)$ has at least one T-periodic solution.

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References

[1] Z. Zheng, *Theory of Functional Diffrential Equation*, Anhui Education Press, Hefei, China, 1994.

[2] H. Wang and X. Suo, "Periodic solutions of a type of second order functional differential equation with complex deviating argument," *Journal of Hebei Normal University*, vol. 28, no. 6.

[3] X. Liu, M. Jia, and W. Ge, "Periodic solutions to a type of Duffing equation with complex deviating argument," *Applied Mathematics A*, vol. 181, pp. 51–56, 2003.

[4] Z. G. Xiang, C. M. Liu, and X. K. Huang, "Periodic solutions of Liénard delay equations," *Journal of Jishou University*, vol. 19, no. 4, pp. 35–40, 1998.

[5] E. Pascale and R. Iannacci, *Periodic Solution of a GenerAlized Linard Equation with Delay*, vol. 1017 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.

[6] B. Liu and L. Huang, "Existence and uniqueness of periodic solutions for a kind of first order neutral functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 121–132, 2006.

[7] G. Wang and J. Yan, "Existence of periodic solution for first order nonlinear neutral delay equations," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 14, no. 2, pp. 189–194, 2001.

[8] S. Lu and W. Ge, "Existence of periodic solutions for a kind of second-order neutral functional differential equation," *Applied Mathematics and Computation*, vol. 157, no. 2, pp. 433–448, 2004.

[9] S. Lu, "Existence of periodic solutions to a p-Laplacian Liénard differential equation with a deviating argument," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 68, no. 6, pp. 1453–1461, 2008.

[10] G. Fan and Y. Li, "Existence of positive periodic solutions for a periodic logistic equation," *Applied Mathematics and Computation*, vol. 139, no. 2-3, pp. 311–321, 2003.

[11] X. Yang, "Multiple periodic solutions for a class of second order differential equations," *Applied Mathematics Letters*, vol. 18, no. 1, pp. 91–99, 2005.

[12] C. Huang, Y. He, L. Huang, and W. Tan, "New results on the periodic solutions for a kind of Rayleigh equation with two deviating arguments," *Mathematical and Computer Modelling*, vol. 46, no. 5-6, pp. 604–611, 2007.

[13] Z. Zhang and Z. Wang, "Periodic solutions of the third order functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 1, pp. 115–134, 2004.

[14] J. Zhou, S. Sun, and Z. Liu, "Periodic solutions of forced Liénard-type equations," *Applied Mathematics and Computation*, vol. 161, no. 2, pp. 656–666, 2005.

[15] Y. Chen, "Periodic solutions of a delayed periodic logistic equation," *Applied Mathematics Letters*, vol. 16, no. 7, pp. 1047–1051, 2003.

[16] J. K. Hale and J. Mawhin, "Coincidence degree and periodic solutions of neutral equations," *Journal of Differential Equations*, vol. 15, pp. 295–307, 1974.

[17] B.-W. Liu and L.-H. Huang, "Periodic solutions for a class of forced Liénard-type equations," *Acta Mathematicae Applicatae Sinica*, vol. 21, no. 1, pp. 81–92, 2005.

[18] J. Mawhin, "Periodic solutions of some vector retarded functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 45, pp. 588–603, 1974.

[19] S. Z. Chen, "Existence of periodic solutions for a higher-order functional differential equation," *Pure and Applied Mathematics*, vol. 22, no. 1, pp. 108–110, 2006.

[20] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, vol. 568 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.