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Research Article

Admissibility for Nonuniform (μ, ν) Contraction and Dichotomy

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The relation between the notions of nonuniform asymptotic stability and admissibility is considered. Using appropriate Lyapunov norms, it is showed that if any of their associated $\mathcal{L}^p$ spaces, with $p \in (1, \infty]$, is admissible for a given evolution process, then this process is a nonuniform (μ, ν) contraction and dichotomy. A collection of admissible Banach spaces for any given nonuniform (μ, ν) contraction and dichotomy is provided.

1. Introduction

The study of the admissibility property has a fairly long history, and it goes back to the pioneering work of Perron [1] in 1930. Perron concerned originally the existence of bounded solutions of the equation

$$x' = A(t)x + f(t) \tag{1.1}$$

in $\mathbb{R}_n$ for any bounded continuous perturbation $f : \mathbb{R}_0^+ \to \mathbb{R}_n$. This property can be used to deduce the stability or the conditional stability under sufficiently small perturbations of a given linear equation:

$$x' = A(t)x. \tag{1.2}$$

His result served as a starting point for many works on the qualitative theory of the solutions of differential equations. Moreover, a simple consequence of one of the main results in that paper stated explicitly in [2, Theorem 1] is probably the first step in the literature concerning the study of the relation between admissibility and the notions of stability and conditional stability. We refer the reader to [2] for details. Relevant results concerning the extension of Perron's problem in the more general framework of the infinite-dimensional Banach spaces with bounded $A(t)$ were obtained by Daleckij and Krein [3], Massera and Schäffer [4], and the work of Levitan and Zhikov [5] for certain cases of unbounded $A(t)$.

Over the last decades an increasing interest can be seen in the study of the asymptotic behavior of evolution equations in abstract spaces. In [6, 7], Latushkin et al. studied the dichotomy of linear skew-product semiflows defined on compact spaces. Using the so-called evolution semigroup, they expressed its dichotomy in terms of hyperbolicity of a family of weighted shift operators. In [8–10], Preda et al. considered related problems in the particular case of uniform exponential behavior. A large class of Schäffer spaces, which were introduced by Schäffer in [11] (see also [4]), acted as admissible spaces for the case of uniform exponential dichotomies. It is worth noting here the works by Huy [12–16] in the study of the existence of an exponential dichotomy for evolution equations.

In the case of nonuniform exponential dichotomies, Preda and Megan [17] obtained related results also for the class of Schäffer spaces, but using a notion of dichotomy which is different from the original one motivated by ergodic theory and the nonuniform hyperbolicity theory, as detailed, for example, in [18, 19]. In the more recent work [20], the authors consider the same weaker notion of exponential dichotomy and obtain sharper relations between admissibility and stability for perturbations and solutions in C_0. Important contributions in this aspect have been made by Barreira et al. [2, 18, 19, 21–25]. Particularly, in [22], Barreira and Valls showed an equivalence between the admissibility of their associated $\mathcal{L}^p$ spaces ($p \in (1, \infty]$) and the nonuniform exponential stability of certain evolution families by using appropriate adapted norms. They also establish a collection of admissible Banach spaces for any given nonuniform exponential dichotomy in [2]. Recently, Preda et al. [26] studied the connection between the (non)uniform exponential dichotomy of a non(uniform) exponentially bounded, strongly continuous evolution family and the admissibility of some function spaces, which extended those results established in [2, 22].

In the present paper, inspired by Barreira and Valls [2, 22], we give a characterization of nonuniform asymptotic stability in terms of admissibility property. We consider a more general type of dichotomy which is called (μ, ν) dichotomy in [21], also proposed in [27]. In this dichotomy, not only the usual exponential behavior is replaced by an arbitrary, which may correspond, for example, to situations when the Lyapunov exponents are all infinity or are all zero, but also different growth rates for the uniform and nonuniform parts of the dichotomy are considered. It extended exponential dichotomy in various ways. In [21], it has also been showed that there is a large class of equations exhibiting this behavior. We emphasize that the characterization in our paper is a very general one; it includes as particular cases many interesting situations among them we can mention some results in previous references. To some extent, our results have a certain significance to study the theory of nonuniform hyperbolicity.

2. Admissibility for Nonuniform (μ, ν) Contractions

We first concentrate on the simpler case of admissibility for nonuniform (μ, ν) contractions, leaving the more elaborate case of admissibility for nonuniform (μ, ν) dichotomies for the second part of the paper. This allows us to present the results and their proofs without some accessory technicalities. After the introduction of some basic notions, using appropriate adapted Lyapunov norms, we show that the admissibility with respect to some space $\mathcal{L}^p$ with $p \in (1, \infty]$ is sufficient for an evolution process to be a nonuniform (μ, ν) contraction.

2.1. Basic Notions

We say that an increasing function $\mu : \mathbb{R}^+ \to [1, +\infty)$ is a growth rate if

$$\mu(0) = 1, \qquad \lim_{t \to +\infty} \mu(t) = +\infty. \tag{2.1}$$

We say that a family of linear operators $T(t, s)$, $t \geq s \geq 0$ in a Banach space X is an evolution process if:

(1) $T(t, t) = Id$ and $T(t, \tau)T(\tau, s) = T(t, s)$, $t, \tau, s > 0$;

(2) $(t, s, x) \mapsto T(t, s)x$ is continuous for $t \geq s \geq 0$ and $x \in X$.

In this section, we also assume that

(3) there exist $\omega \geq 0$, $D > 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\|T(t, s)\| \leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{2.2}$$

We consider the new norms

$$\|x\|'_t = \sup\left\{ \|T(\sigma, t)x\| \left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t \right\}, \quad x \in X,\ t \in \mathbb{R}_0^+. \tag{2.3}$$

These satisfy

$$\|x\| \leq \|x\|'_t \leq D\nu^{\varepsilon}(t)\|x\|, \quad x \in X,\ t \in \mathbb{R}_0^+. \tag{2.4}$$

Moreover, with respect to these norms the evolution process has the following bounded growth property.

Proposition 2.1. *If T is an evolution process, then*

$$\|T(t, s)x\|'_t \leq \left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \|x\|'_s \tag{2.5}$$

for every $t \geq s \geq 0$ and $x \in X$.

Proof. We have

$$\begin{aligned} \|T(t, s)x\|'_t &= \sup\left\{ \|T(\sigma, t)x\| \left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t \right\} \\ &\leq \left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \sup\left\{ \|T(\sigma, t)x\| \left(\frac{\mu(\sigma)}{\mu(s)}\right)^{-\omega}, \sigma \geq s \right\} \\ &= \left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \|x\|'_s \end{aligned} \tag{2.6}$$

which yields the desired inequality. □

Definition 2.2. We say that an evolution process T is a nonuniform (μ, ν) contraction in $\mathbb{R}_0^+$ if there exist some constants α, $D > 0$, $\varepsilon \geq 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\|T(t,s)\| \leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} \nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{2.7}$$

When $\varepsilon = 0$, we say that (1.2) has a uniform (μ, ν) contraction or simply a (μ, ν) contraction.

In the following, we introduce several Banach spaces that are used throughout the paper. We first set

$$L^p = \left\{ f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \ \text{ Lebesgue-measurable} : \|f\|_p < \infty \right\} \tag{2.8}$$

for each $p \in [1, \infty)$, and

$$L^\infty = \left\{ f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \ \text{ Lebesgue-measurable} : \|f\|_\infty < \infty \right\} \tag{2.9}$$

Respectively, with the norms

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p\right)^{1/p}, \qquad \|f\|_\infty = \operatorname*{ess\,sup}_{t \in \mathbb{R}_0^+} |f(t)|. \tag{2.10}$$

Then for each $p \in [1, \infty]$ the set $\mathcal{L}^p$ of the equivalence classes $[f]$ of functions $g \in L^p$ such that $g = f$ Lebesgue-almost everywhere is a Banach space (again with the norms in (2.10)).

For each Banach space $E = \mathcal{L}^p$, with $p \in [1, \infty]$, we set

$$E(X) = \left\{ f : \mathbb{R}_0^+ \longrightarrow X \ \text{Bochner-measurable} : t \longmapsto \|f(t)\|_t' \in E \right\} \tag{2.11}$$

using the norms $\|\cdot\|_t'$ in (2.3), and we endow $E(X) = \mathcal{L}^p(X)$ with the norm

$$\|f\|_p' = \|F\|_p, \quad \text{where } F(t) = \|f(t)\|_t'. \tag{2.12}$$

Repeating arguments in the proof of Theorem 3 in [22], we obtain the following statement.

Lemma 2.3. *For each $p \in [1, \infty]$ and $E = \mathcal{L}^p$, the set $E(X)$ is a Banach space with the norm in* (2.12)*, and the convergence in $E(X)$ implies the pointwise convergence Lebesgue-almost everywhere.*

Definition 2.4. We say that a Banach space E is admissible for the evolution process T if for each $f \in E(X)$ the function $x_f : \mathbb{R}_0^+ \to X$ defined by

$$x_f(t) = \int_0^\infty T(t,\tau) f(\tau) d\tau \tag{2.13}$$

is in $\mathcal{L}^\infty$ (see (2.11)).

By Lemma 2.3 we know that $\mathcal{L}^\infty$ is a Banach space with the norm

$$\|g\|'_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+}\|g(t)\|'_t. \tag{2.14}$$

Lemma 2.5. *There exists $K > 0$ such that*

$$\|x_f\|'_\infty \le K\|f\|'_p \quad \text{for every } f \in E(X). \tag{2.15}$$

Proof. We define a linear operator $G : E(X) \to \mathcal{L}^\infty(X)$ by $Gf = x_f$. We use the closed graph theorem to show that G is bounded. For this, let us take a sequence $(f_n)_{n\in\mathbb{N}} \subset E(X)$ and $f \in E(X)$ such that $f_n \to f$ in $E(X)$ when $n \to \infty$ and also $h \in \mathcal{L}^\infty(X)$ such that $Gf_n \to h$ in $\mathcal{L}^\infty(X)$ when $n \to \infty$. We need to show that $Gf = h$ Lebesgue-almost everywhere. For each $t \ge 0$ and $n \in N$ we have

$$\begin{aligned}
\|(Gf_n)(t) - (Gf)(t)\|'_t &= \sup\left\{\left\|\int_0^t T(\sigma,t)T(t,\tau)(f_n(\tau)-f(\tau))d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \ge t\right\} \\
&= \sup\left\{\left\|\int_0^t T(\sigma,\tau)(f_n(\tau)-f(\tau))d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \ge t\right\} \\
&\le \sup\left\{\int_0^t \|T(\sigma,\tau)(f_n(\tau)-f(\tau))\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} d\tau : \sigma \ge t\right\} \\
&= \sup\left\{\int_0^t \|T(\sigma,\tau)(f_n(\tau)-f(\tau))\|\left(\frac{\mu(\sigma)}{\mu(\tau)}\right)^{-\omega}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\omega} d\tau : \sigma \ge t\right\} \\
&= \int_0^t \|f_n(\tau)-f(\tau)\|'_\tau\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\omega} d\tau \\
&\le \mu(t)^\omega \int_0^t \|f_n(\tau)-f(\tau)\|'_\tau d\tau.
\end{aligned} \tag{2.16}$$

According to Hölder's inequality, there exists $\alpha = \alpha([0,t])$ such that

$$\|(Gf_n)(t) - (Gf)(t)\|'_t \le \mu(t)^\omega \int_0^t \|f_n(\tau)-f(\tau)\|'_\tau d\tau \le \mu(t)^\omega \alpha \|(f_n(\tau)-f(\tau))\|'_p. \tag{2.17}$$

Therefore, for each $t \ge 0$, letting $n \to \infty$ we find that $(Gf_n)(t) \to (Gf)(t)$. This shows that $Gf = h$ Lebesgue-almost everywhere, and by the closed graph theorem, we conclude that G is a bounded operator. This completes the proof of the lemma. □

2.2. Criterion for Nonuniform (μ,ν) *Contraction*

Theorem 2.6. *If for some $p \in (1,\infty]$ the space $E = \mathcal{L}^p$ is admissible for the evolution process T, then T is a nonuniform (μ,ν) contraction.*

Proof. We follow arguments in [22]. Given $x \in X$ and $t_0 \geq 0$, we define a function $f : \mathbb{R}_0^+ \to X$ by

$$f(t) = \begin{cases} T(t,t_0)x, & t \in [t_0, t_0+1] \\ 0, & t \in \mathbb{R}_0^+ \setminus [t_0, t_0+1]. \end{cases} \tag{2.18}$$

We note that

$$\left\| f(t) \right\|_t' \leq \|T(t,t_0)x\|_t' \chi_{[t_0,t_0+1]}(t). \tag{2.19}$$

Then, for each $t \in [t_0, t_0+1]$ and $x \in X$, we have

$$\begin{aligned} \|T(t,t_0)x\|_t' &= \sup\left\{ \|T(\sigma,t)T(t,t_0)x\| \left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t \right\} \\ &\leq \left(\frac{\mu(t)}{\mu(t_0)}\right)^{\omega} \sup\left\{ \|T(\sigma,t_0)x\| \left(\frac{\mu(\sigma)}{\mu(t_0)}\right)^{-\omega}, \sigma \geq t_0 \right\} \\ &= \left(\frac{\mu(t)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \\ &\leq \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}'. \end{aligned} \tag{2.20}$$

Therefore,

$$\left\| f(t) \right\|_p' \leq \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \left\| \chi_{[t_0,t_0+1]}(t) \right\|_p = \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \tag{2.21}$$

and in particular $f \in E(X)$. On the other hand, according to (2.13) and (2.18), we have

$$x_f(t) = \int_{t_0}^{t_0+1} T(t,\tau)T(\tau,t_0)x\,d\tau = T(t,t_0)x \tag{2.22}$$

for all $t \geq t_0 + 1$, which implies that

$$\|T(t,t_0)x\|_t' = \left\| x_f \right\|_t' \leq \left\| x_f \right\|_\infty'. \tag{2.23}$$

By Lemma 2.5 and (2.21)–(2.23), we obtain

$$\|T(t,t_0)x\|_t' \leq \left\| x_f \right\|_\infty' \leq K \left\| f \right\|_p' \leq K \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \tag{2.24}$$

for all $t \geq t_0 + 1$, $t_0 \geq 0$, and $x \in X$. We claim that

$$\|T(t,t_0)\|' := \sup_{x \neq 0} \frac{\|T(t,t_0)x\|_t'}{\|x\|_{t_0}'} \leq L, \quad L = \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \max\{K, 1\} \tag{2.25}$$

for all $t \geq t_0$. Indeed, for $t \geq t_0 + 1$ inequality (2.25) follows from (2.24), and for $t \in [t_0, t_0 + 1]$ the inequality follows from (2.20).

Now given $x \in X$, $t_0 \geq 0$, and $\delta > 0$, we define a function $g : \mathbb{R}_0^+ \to X$ by

$$g(t) = \begin{cases} T(t,t_0)x, & t \in [t_0, t_0 + \delta] \\ 0, & t \in \mathbb{R}_0^+ \setminus [t_0, t_0 + \delta]. \end{cases} \tag{2.26}$$

It follows from (2.25) that

$$\|g(t)\|_t' \leq \|T(t,t_0)x\|_t' \leq L\|x\|_{t_0}' \tag{2.27}$$

and thus,

$$g \in E(X), \qquad \|g\|_p' \leq L\delta^{1/p}\|x\|_{t_0}'. \tag{2.28}$$

On the other hand, writing $y = T(t_0 + \delta, t_0)x$,

$$\begin{aligned}
\frac{\delta^2}{2}\|y\|_{t_0+\delta}' &= \left\|\int_{t_0}^{t_0+\delta} (\tau - t_0) y \, d\tau\right\|_{t_0+\delta}' \\
&= \sup\left\{\left\|T(\sigma, t_0+\delta)\int_{t_0}^{t_0+\delta}(\tau - t_0)y\,d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} : \sigma \geq t_0 + \delta\right\} \\
&= \sup\left\{\left\|\int_{t_0}^{t_0+\delta}(\tau - t_0)T(\sigma, t_0)x\,d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} : \sigma \geq t_0 + \delta\right\} \\
&\leq \sup\left\{\int_{t_0}^{t_0+\delta}(\tau - t_0)\|T(\sigma, t_0)x\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} d\tau : \sigma \geq t_0 + \delta\right\} \\
&= \int_{t_0}^{t_0+\delta}(\tau - t_0)\sup\left\{\|T(\sigma, t_0+\delta)y\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} : \sigma \geq t_0 + \delta\right\} d\tau \\
&= \int_{t_0}^{t_0+\delta}(\tau - t_0)\|y\|_{t_0+\delta}'\,d\tau \\
&= \int_{t_0}^{t_0+\delta}(\tau - t_0)\|T(t_0+\delta, \tau)T(\tau, t_0)x\|_{t_0}'\,d\tau
\end{aligned} \tag{2.29}$$

Since

$$x_g(t) = \int_0^t T(t,\tau)g(\tau)d\tau = \begin{cases} 0, & t \in [0, t_0] \\ (t,t_0)T(t,t_0)x, & t \in [t_0, t_0+\delta], \\ \delta T(t,t_0)x, & t \in [t_0+\delta, \infty), \end{cases} \tag{2.30}$$

it follows from Lemma 2.5, (2.25), and (2.28) that

$$\begin{aligned} \frac{\delta^2}{2}\|T(t_0+\delta,t_0)x\|'_{t_0+\delta} &\leq L\int_{t_0}^{t_0+\delta}(\tau-t_0)\|T(\tau,t_0)x\|'_\tau d\tau \\ &= L\int_{t_0}^{t_0+\delta}\left\|x_g(\tau)\right\|'_\tau d\tau \leq L\delta\left\|x_g\right\|'_\infty \\ &\leq KL\delta\left\|g\right\|'_p \leq KL^2\delta^{(p+1)/p}\|x\|'_{t_0} \end{aligned} \tag{2.31}$$

for all $t_0 \geq 0$, $\delta > 0$, and $x \in X$; we thus obtain

$$\frac{\delta^2}{2}\|T(t_0+\delta,t_0)x\|'_{t_0+\delta} \leq KL^2\delta^{(p+1)/p}\|x\|'_{t_0} \tag{2.32}$$

so

$$\|T(t_0+\delta,t_0)\|' \leq 2KL^2\delta^{(1-p)/p} \tag{2.33}$$

for all $t_0 \geq 0$ and $\delta > 0$. Since $(1-p)/p < 0$ for $p \in (1,\infty]$, there exists $\delta_0 > 0$ sufficiently large such that

$$K_0 := 2KL^2\delta^{(1-p)/p} < 1. \tag{2.34}$$

Setting $n = [(\ln\mu(t) - \ln\mu(t_0))/\delta_0]$ for each $t \geq t_0$, we have

$$T(t,t_0) = T(t,t_0+n\delta_0)T(t_0+n\delta_0,t_0). \tag{2.35}$$

By (2.25) and (2.33) we obtain

$$\begin{aligned} \|T(t,t_0)\|' &\leq L\|T(t_0+n\delta_0,t_0)\|' \\ &\leq L\prod_{i=0}^{n-1}\|T(t_0+(i+1)\delta_0,t_0+i\delta_0)\|' \leq LK_0^n \end{aligned} \tag{2.36}$$

for $t \geq t_0$. By (2.34) and

$$n = \left[\frac{\ln\mu(t)-\ln\mu(t_0)}{\delta_0}\right] \geq \frac{\ln\mu(t)-\ln\mu(t_0)}{\delta_0} - 1 \tag{2.37}$$

this implies that

$$\|T(t,t_0)\|' \leq d\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha}, \tag{2.38}$$

where

$$d = \frac{L}{K_0}, \qquad \alpha = -\frac{1}{\delta_0}\ln K_0. \tag{2.39}$$

We note that $d, \alpha > 0$. Since

$$\|T(t,t_0)x\|_t' \geq \|T(t,t_0)x\|, \tag{2.40}$$

and by (2.4),

$$\|x\|_{t_0}' = \sup\left\{\|T(\sigma,t_0)x\|\left(\frac{\mu(\sigma)}{\mu(t_0)}\right)^{-\omega} : \sigma \geq t_0\right\} \leq D\nu^{\varepsilon}(t_0)\|x\|. \tag{2.41}$$

It follows from (2.38) that

$$\|T(t,t_0)\| = \sup_{x\neq 0}\frac{\|T(t,t_0)x\|}{\|x\|} \leq D\nu^{\varepsilon}(t_0)\sup_{x\neq 0}\frac{\|T(t,t_0)x\|_t'}{\|x\|_{t_0}'} \leq dD\nu^{\varepsilon}(t_0)\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha} \tag{2.42}$$

for any $t \geq t_0$. Therefore, the evolution process T is a nonuniform (μ,ν) contraction with α and $\widetilde{D} = dD$. This concludes the proof of Theorem 2.6. □

2.3. Admissible Spaces for Nonuniform (μ,ν) *Contractions*

We consider the spaces

$$L_D^p = \left\{f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \;\text{ Lebesgue-measurable} : \|f\|_{p,D} < \infty\right\} \tag{2.43}$$

for each $p \in [1,\infty)$, and

$$L_D^\infty = \left\{f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \;\text{ Lebesgue-measurable} : \|f\|_{\infty,D} < \infty\right\}, \tag{2.44}$$

respectively, with the norms

$$\begin{gathered}\|f\|_{p,D} = \left(\int_0^\infty |f(t)|^p\left(D\nu^{\varepsilon}(t)\left(\frac{\mu(t)}{\mu'(t)}\right)^{1/q}\right)^p\right)^{1/p}, \\ \|f\|_{\infty,D} = \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+}\left(|f(t)|D\nu^{\varepsilon}(t)\frac{\mu(t)}{\mu'(t)}\right).\end{gathered} \tag{2.45}$$

In a similar manner to that in Lemma 2.3 these normed spaces induce Banach spaces $\mathcal{L}_D^p$ and $\mathcal{L}_D^p(X)$ for each $p \in [1,\infty]$, the last one with norm

$$\|f\|'_{p,D} = \|F\|_{p,D}, \quad \text{where } F(t) = \|f(t)\|'_t. \tag{2.46}$$

Theorem 2.7. *If the evolution process T is a nonuniform (μ,ν) contraction, then for any $p \in [1,\infty]$ the space $\mathcal{L}_D^p$ is admissible for T.*

Proof. We first take $f \in \mathcal{L}_D^\infty$. Then

$$\begin{aligned}
\|x_f(t)\|'_t &= \sup\left\{\left\|\int_0^t T(\sigma,t)T(t,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \geq t\right\}\\
&= \sup\left\{\left\|\int_0^t T(\sigma,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \geq t\right\}\\
&\leq \sup\left\{\int_0^t \|T(\sigma,\tau)\| \cdot \|f(\tau)\|d\tau : \sigma \geq t\right\}\\
&\leq \sup\left\{\int_0^t D\nu^\tau(\tau)\left(\frac{\mu(\sigma)}{\mu(\tau)}\right)^{-\alpha}\|f(\tau)\|'_\tau d\tau : \sigma \geq t\right\}\\
&\leq \sup\left\{\int_0^t D\nu^\tau(\tau)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\|f(\tau)\|'_\tau d\tau : \sigma \geq t\right\}\\
&\leq \|f\|'_{\infty,D}\int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&\leq \|f\|'_{\infty,D}\frac{1-\mu(t)^{-\alpha}}{\alpha} \leq \frac{1}{\alpha}\|f\|'_{\infty,D}.
\end{aligned} \tag{2.47}$$

Therefore,

$$\|x_f\|'_\infty = \sup_{t\geq 0}\|x_f(t)\|'_t \leq \sup_{t\geq 0}\frac{1}{\alpha}\|f\|'_{\infty,D} < \infty \tag{2.48}$$

and $\mathcal{L}_D^\infty$ is admissible for T.

Now we take $f \in \mathcal{L}_D^p(X)$ for some $p \in [1,\infty)$. Using Hölder's inequality we obtain

$$\begin{aligned}
\|x_f(t)\|_t' &= \sup\left\{\left\|\int_0^t T(\sigma,t)T(t,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \geq t\right\} \\
&\leq \sup\left\{\int_0^t D\nu^\tau(t)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\|f(\tau)\|_\tau' d\tau : \sigma \geq t\right\} \\
&\leq \|f\|_{p,D}'\left(\int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha q}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\right)^{1/q} \\
&\leq \|f\|_{p,D}'\left(\frac{1-\mu(t)^{-\alpha q}}{\alpha q}\right)^{1/q} \leq \frac{1}{(\alpha q)^{1/q}}\|f\|_{p,D}',
\end{aligned} \tag{2.49}$$

where $1/p + 1/q = 1$. We conclude that $\mathcal{L}_D^p$ is also admissible for T. □

3. Admissibility for Nonuniform (μ,ν) Dichotomies

We consider in this second part admissibility for nonuniform (μ,ν) dichotomies. It generalizes the usual notion of exponential dichotomy in several ways: besides introducing a nonuniform term, causing that any conditional stability may be nonuniform, we consider rates that may not be exponential as well as different rates in the uniform and nonuniform parts. After introducing some basic notions, we show that the admissibility with respect to some space $\mathcal{L}^p$ with $p \in (1,\infty]$ is sufficient for an evolution process to be a nonuniform (μ,ν) dichotomy. When compared to the case of contractions, this creates substantial complications. We also provide a collection of admissible Banach spaces for any given nonuniform (μ,ν) dichotomy.

3.1. Basic Notions

We consider an evolution process $T(t,s)$, $t \geq s \geq 0$ satisfied 1, 2 in Section 2.

We also consider a function $P : \mathbb{R}_0^+ \to B(X)$, where $B(X)$ is the set of bounded linear operators in X, such that

(1) $P(t)^2 = P(t)$, for every $t \geq 0$;

(2) $(t,x) \mapsto P(t)x$ is continuous in $\mathbb{R}_0^+ \times X$.

We will refer to P as a projection function. Given an evolution process T, we say that a projection function P is compatible with T if:

(1) $T(t,s)p(s) = P(t)T(t,s)$, for every $t,\tau,s > 0$;

(2) the map

$$T(t,\sigma) \mid \ker P(\sigma) : \ker P(\sigma) \longrightarrow \ker P(t) \tag{3.1}$$

is invertible for every $t \geq s \geq 0$.

We also assume that

(3) there exist $D > 0$, $\omega \geq 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\|T(t,s)P(s)\| \leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{\omega}\nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{3.2}$$

$$\|T(t,s)Q(s)\| \leq D\left(\frac{\mu(s)}{\mu(t)}\right)^{\omega}\nu^{\varepsilon}(s), \quad s \geq t \geq 0. \tag{3.3}$$

We note that due to the invertibility assumption in condition (1.2), condition (3.3) is simply a version of (3.2) when time goes backwards.

We always consider in the paper an evolution process T together with a projection function P which is compatible with T (and which satisfies (3.2) and (3.3)). We write

$$U(t,s) = T(t,s)P(s), \qquad V(t,s) = T(t,s)Q(s), \tag{3.4}$$

where $Q(t) = Id - P(t)$ for each $t \geq 0$. we consider the new norms

$$\begin{aligned} \|x\|_t' = \ &\sup\left\{\|U(\sigma,t)x\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t\right\} \\ &+ \sup\left\{\|V(\sigma,t)x\|\left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t\right\}. \end{aligned} \tag{3.5}$$

for each $x \in X$ and $t \in \mathbb{R}_0^+$, where $V(\sigma,t)$ denotes the inverse of the map in (3.1). We have

$$\|x\|_t' \geq \|P(t)x\| + \|Q(t)x\| \geq \|P(t)x + Q(t)x\| = \|x\|. \tag{3.6}$$

Moreover, by (3.2) and (3.3),

$$\|x\|_t' \leq 2D\nu^{\varepsilon}(t)\|x\|, \quad x \in X, \ t \in \mathbb{R}_0^+. \tag{3.7}$$

Definition 3.1. We say that an evolution process T is a nonuniform (μ,ν) dichotomy in $\mathbb{R}^+$ if there exist a projection function $P : \mathbb{R}_0^+ \to B(X)$ compatible with T, some constants α, $D > 0$, $\varepsilon \geq 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu^{\varepsilon}(s), \quad t \geq s \geq 0. \\ \|T(t,s)Q(s)\| &\leq D\left(\frac{\mu(s)}{\mu(t)}\right)^{-\beta}\nu^{\varepsilon}(s), \quad s \geq t \geq 0, \end{aligned} \tag{3.8}$$

When $\varepsilon = 0$, we say that (1.2) has a uniform (μ,ν) dichotomy or simply a (μ,ν) dichotomy.

In the following, we still consider several spaces L^p, L^∞, respectively, with the norms (2.10), which induce Banach spaces $\mathcal{L}^p$ for each $p \in [1,\infty]$. We also set $E(X)$ as in (2.11) but using the norms $\|\cdot\|_t'$ in (3.5), we endow $E(X) = \mathcal{L}^p(X)$ with the norm in (2.12).

We also obtain easily the same statement in Lemma 2.3 for the set $E(X)$ using the norms $\|\cdot\|_t'$ in (3.5).

Definition 3.2. We say that a Banach space E is admissible for the evolution process T if for each $f \in E(X)$

(1) the function

$$\mathbb{R}_0^+ \ni \tau \longmapsto \begin{cases} V(t,\tau)f(\tau), & \tau \geq t, \\ 0, & 0 \leq \tau < t \end{cases} \tag{3.9}$$

is in $\mathcal{L}^1(X)$ for each $t \geq 0$;

(2) the function $x_f : \mathbb{R}_0^+ \to X$ defined by

$$x_f(t) = \int_0^t U(t,\tau)f(\tau)d\tau - \int_t^\infty V(t,\tau)f(\tau)d\tau \tag{3.10}$$

is in $\mathcal{L}^\infty(X)$.

We note that since $\|Q(t)x\| \leq \|Q(t)x\|_t'$ for every $x \in X$, and $t \geq 0$, any function in $\mathcal{L}^1(X)$ is also integrable in R_0^+, and thus the first condition ensures that the function x_f is well defined. By Lemma 2.3 we know that $\mathcal{L}^\infty(X)$ is a Banach space with the norm

$$\|g\|_\infty' = \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+} \|g(t)\|_t'. \tag{3.11}$$

Lemma 3.3. *If for some $p \in [1,\infty]$ the space $E = \mathcal{L}^p$ is admissible for the evolution process T, then there exists $K > 0$ such that*

$$\|x_f\|_\infty' \leq K\|f\|_p' \quad \text{for every } f \in E(X). \tag{3.12}$$

Proof. We follow arguments in [2]. For each $t \geq 0$, we define a map $H_t : E(X) \to \mathcal{L}^1(X)$ by

$$(H_t f)(\tau) = \begin{cases} V(t,\tau)f(\tau), & \tau \geq t, \\ 0, & 0 \leq \tau < t. \end{cases} \tag{3.13}$$

Clearly, H_t is linear. We use the closed graph theorem to show that H_t is bounded. For this, let us take a sequence $(f_n)_{n\in N} \subset E(X)$ and $f \in E(X)$ such that $f_n \to f$ in $E(X)$ when $n \to \infty$, and also $g \in \mathcal{L}^1(X)$ such that $H_t f_n \to g$ in $\mathcal{L}^1(X)$ when $n \to \infty$. We need to show that

$H_t f = g$ Lebesgue-almost everywhere. By Lemma 2.3, the sequence f_n converges pointwise Lebesgue-almost everywhere. Therefore,

$$(H_t f_n)(\tau) = V(t,\tau) f_n(\tau) \longrightarrow V(t,\tau) f(\tau) = (H_t f)(\tau) \tag{3.14}$$

when $n \to \infty$, for Lebesgue-almost every $\tau \in [t,+\infty)$. On the other hand, since $H_t f_n \to g$ in $\mathcal{L}^1(X)$ when $n \to \infty$, we also have $(H_t f_n)(\tau) \to g(\tau)$ when $n \to \infty$, for Lebesgue-almost every $\tau \in [t,+\infty)$. This shows that $H_t f = g$ Lebesgue-almost everywhere, and H_t is bounded for each $t \geq 0$.

We define a linear operator $G : E(X) \to \mathcal{L}^\infty(X)$ by $Gf = x_f$. We use again the closed graph theorem to show that G is bounded. For this, let us take a sequence $(f_n)_{n\in N} \subset E(X)$ and $f \in E(X)$ such that $f_n \to f$ in $E(X)$ when $n \to \infty$, and also $h \in \mathcal{L}^\infty(X)$ such that $Gf_n \to h$ in $\mathcal{L}^\infty(X)$ when $n \to \infty$. We write

$$\begin{aligned} (G_1 f)(t) &= P(t)(Gf)(t) = \int_0^t U(t,\tau) f(\tau) d\tau, \\ (G_2 f)(t) &= Q(t)(Gf)(t) = -\int_t^\infty V(t,\tau) f(\tau) d\tau. \end{aligned} \tag{3.15}$$

Using the similar proof of Lemma 2.5, for each $t \geq 0$ and $n \in N$ we have

$$\begin{aligned} \left\| (G_1 f_n)(t) - (G_1 f)(t) \right\|_t' &= \sup\left\{ \left\| \int_0^t U(\sigma,t) U(t,\tau)(f_n(\tau) - f(\tau)) d\tau \right\| \left(\frac{\mu(\sigma)}{\mu(t)} \right)^{-\omega} : \sigma \geq t \right\} \\ &\leq \mu(t)^\omega \int_0^t \left\| f_n(\tau) - f(\tau) \right\|_\tau' d\tau. \end{aligned} \tag{3.16}$$

According to Hölder's inequality, there exists $\alpha = \alpha([0,t])$ such that

$$\left\| (G_1 f_n)(t) - (G_1 f)(t) \right\|_t' \leq \mu(t)^\omega \alpha \left\| (f_n(\tau) - f(\tau)) \right\|_p'. \tag{3.17}$$

Furthermore, we have

$$\begin{aligned} \left\| (G_2 f_n)(t) - (G_2 f)(t) \right\|_t' &\leq \sup\left\{ \int_t^\infty \left\| V(\sigma,t) V(t,\tau)(f_n(\tau) - f(\tau)) \right\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} d\tau : 0 \leq \sigma \leq t \right\} \\ &= \int_t^\infty \sup\left\{ \left\| V(\sigma,t) H_t(f_n - f)(\tau) \right\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} : 0 \leq \sigma \leq t \right\} d\tau \\ &= \int_t^\infty \left\| H_t(f_n - f)(\tau) \right\|_t' d\tau = \left\| H_t(f_n - f) \right\|_t^1. \end{aligned} \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\left\|(Gf_n)(t)-(Gf)(t)\right\|_t' \le \mu(t)^{\omega}\alpha\left\|f_n-f\right\|_p' + \left\|H_t(f_n-f)\right\|_t^1. \tag{3.19}$$

Therefore, for each $t \ge 0$, letting $n \to \infty$ we find that $(Gf_n)(t) \to (Gf)(t)$. This shows that $Gf = h$ Lebesgue-almost everywhere, and by the closed graph theorem, we conclude that G is a bounded operator. This completes the proof of the lemma. □

3.2. *Criterion for Nonuniform* (μ,ν) *Dichotomy*

Theorem 3.4. *If for some $p \in (1,\infty]$ the space $E = \mathcal{L}^p$ is admissible for the evolution process T, then T is a nonuniform (μ,ν) dichotomy.*

Proof. We first consider the space $P_{t_0} = \operatorname{Im} P(t_0)$. Given $x \in P_{t_0}$ and $t_0 \ge 0$, repeating argument of the proof in Theorem 2.6, except limiting $T(t,t_0)$ on P_{t_0}, we obtain

$$\|T(t,t_0) \mid P_{t_0}\|' \le d\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha}, \tag{3.20}$$

where

$$d = \frac{L}{K_0}, \qquad \alpha = -\frac{1}{\delta_0}\ln K_0. \tag{3.21}$$

We note that $d,\alpha > 0$. For each $x \in P_{t_0}$, we have

$$\|T(t,t_0)x\|_t' \ge \|T(t,t_0)x\|, \tag{3.22}$$

and by (3.2),

$$\|x\|_{t_0}' = \sup\left\{\|U(\sigma,t_0)x\|\left(\frac{\mu(\sigma)}{\mu(t_0)}\right)^{-\omega} : \sigma \ge t_0\right\} \le D\nu^{\varepsilon}(t_0)\|x\|. \tag{3.23}$$

It follows from (3.5) and (3.20) that

$$\begin{aligned}
\|T(t,t_0) \mid P_{t_0}\| &= \sup_{x\in P_{t_0}\setminus\{0\}} \frac{\|T(t,t_0)x\|}{\|x\|} \\
&\le D\nu^{\varepsilon}(t_0) \sup_{x\in P_{t_0}\setminus\{0\}} \frac{\|T(t,t_0)x\|_t'}{\|x\|_{t_0}'} \\
&\le dD\nu^{\varepsilon}(t_0)\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha}
\end{aligned} \tag{3.24}$$

for any $t \ge t_0$.

Now we consider the space $Q_{t_0} = \operatorname{Ker} P(t_0)$. Given $x \in Q_{t_0}$ and $t_0 \geq 0$, we define a function $f : \mathbb{R}_0^+ \to X$ by

$$f(t) = \begin{cases} V(t,t_0)x, & t \in \mathbb{R}_0^+ \bigcap [t_0 - 1, t_0] \\ 0, & t \in \mathbb{R}_0^+ \setminus [t_0 - 1, t_0]. \end{cases} \tag{3.25}$$

Clearly, $f(t) \in Q_t$ for every $t \geq 0$. Moreover, for each $t \in [0, t_0 - 1]$ (this interval may be empty), we have

$$\begin{aligned} x_f(t) &= -\int_{t_0-1}^{t_0} V(t,\tau)V(\tau,t_0)x d\tau \\ &= -\int_{t_0-1}^{t_0} V(t,t_0)x d\tau = -V(t,t_0)x, \end{aligned} \tag{3.26}$$

and it follows from Lemma 3.3 that

$$\|V(t,t_0)x\|_t' = \left\|x_f\right\|_\infty' \leq K\left\|f\right\|_p' \tag{3.27}$$

for $t \in [0, t_0 - 1]$.

On the other hand, for each $t \in [t_0 - 1, t_0]$, we have

$$\begin{aligned} \left\|f(t)\right\|_t' &= \sup\left\{ \|V(\sigma,t)f(t)\| \left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t \right\} \\ &\leq \sup\left\{ \|V(\sigma,t)V(t,t_0)x\| \left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t \right\} \\ &\leq \left(\frac{\mu(t_0)}{\mu(t)}\right)^{\omega} \sup\left\{ \|V(\sigma,t_0)x\| \left(\frac{\mu(t_0)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t_0 \right\} \\ &\leq \left(\frac{\mu(t_0)}{\mu(t_0-1)}\right)^{\omega} \|x\|_{t_0}'. \end{aligned} \tag{3.28}$$

So

$$\left\|f\right\|_p' = \int_{\mathbb{R}_0^+ \bigcap [t_0-1,t_0]} \left\|f(t)\right\|_t' dt \leq \left(\frac{\mu(t_0)}{\mu(t_0-1)}\right)^{\omega} \|x\|_{t_0}' \tag{3.29}$$

and in particular $f \in E(X)$. We thus have

$$\|V(t,t_0)x\|_t' \leq K\left\|f\right\|_p' \leq K\left(\frac{\mu(t_0)}{\mu(t_0-1)}\right)^{\omega} \|x\|_{t_0}' \tag{3.30}$$

for every $t \in [0, t_0 - 1]$, and $x \in Q_{t_0}$. This implies that

$$\|V(t,t_0)\|' := \sup_{x \in Q_{t_0} \setminus \{0\}} \frac{\|V(t,t_0)x\|'_t}{\|x\|'_{t_0}} \leq L, \quad L = \left(\frac{\mu(t_0)}{\mu(t_0-1)}\right)^{\omega} \max\{K, 1\} \tag{3.31}$$

for all $t \leq t_0$. Indeed, for $t \in [0, t_0 - 1]$ inequality (3.31) follows from (3.30), and for $t \in \mathbb{R}_0^+ \bigcap [t_0 - 1, t_0]$ the inequality follows from (3.28).

Now given $x \in Q_{t_0}$, $t_0 \geq 0$, and $\delta > 0$, we define a function $g : \mathbb{R}_0^+ \to X$ by

$$g(t) = \begin{cases} V(t,t_0)x, & t \in \mathbb{R}_0^+ \bigcap [t_0 - \delta, t_0] \\ 0, & t \in \mathbb{R}_0^+ [t_0 - \delta, t_0]. \end{cases} \tag{3.32}$$

It follows from (3.31) that

$$\left\|g(t)\right\|'_t \leq \|V(t,t_0)x\|'_t \leq L\|x\|'_{t_0}, \tag{3.33}$$

and thus,

$$g \in E(X), \qquad \left\|g\right\|'_p \leq L\delta^{1/p}\|x\|'_{t_0}. \tag{3.34}$$

On the other hand, in a similar manner to that in (2.29),

$$\begin{aligned} \frac{\delta^2}{2}\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} &= \left\|\int_{t_0-\delta}^{t_0} (\tau - t_0)V(t_0-\delta,t_0)x\,d\tau\right\|'_{t_0-\delta} \\ &\leq \int_{t_0-\delta}^{t_0} (t_0-\tau)\sup\left\{\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} : 0 \leq \sigma \leq t_0-\delta\right\}d\tau \\ &= \int_{t_0-\delta}^{t_0} (t_0-\tau)\|V(t_0-\delta,t_0)x\|'_{t_0}d\tau \\ &= \int_{t_0-\delta}^{t_0} (t_0-\tau)\|V(t_0-\delta,\tau)V(\tau,t_0)x\|'_{t_0}d\tau. \end{aligned} \tag{3.35}$$

Since

$$x_g(t) = -\int_t^{\infty} V(t,\tau)g(\tau)d\tau = \begin{cases} 0, & t \in [t_0, \infty) \\ (t-t_0)V(t,t_0)x, & t \in [t_0-\delta, t_0]. \\ -\delta V(t,t_0)x, & t \in [0, t_0-\delta]. \end{cases} \tag{3.36}$$

It follows from Lemma 3.3, (3.31), and (3.34) that

$$\begin{aligned}
\frac{\delta^2}{2}\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} &\le L\int_{t_0-\delta}^{t_0}(t_0-\tau)\|V(\tau,t_0)x\|'_{\tau}d\tau \\
&= L\int_{t_0-\delta}^{t_0}\left\|x_g(\tau)\right\|'_{\tau}d\tau \le L\delta\left\|x_g\right\|'_{\infty} \\
&\le KL\delta\left\|g\right\|'_{p} \le KL^2\delta^{(p+1)/p}\|x\|'_{t_0}
\end{aligned} \tag{3.37}$$

for all $t_0 \ge 0$, $\delta > 0$, and $x \in Q_{t_0}$; we thus obtain

$$\frac{\delta^2}{2}\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} \le KL^2\delta^{(p+1)/p}\|x\|'_{t_0} \tag{3.38}$$

so

$$\|V(t_0-\delta,t_0)\|' \le 2KL^2\delta^{(1-p)/p} \tag{3.39}$$

for all $t_0 \ge 0$ and $\delta > 0$. Taking the same δ_0 as before, and setting $n = [(\ln\mu(t_0) - \ln\mu(t))/\delta_0]$ for each $t \le t_0$, we have

$$V(t,t_0) = V(t,t_0-n\delta_0)V(t_0-n\delta_0,t_0). \tag{3.40}$$

By (3.31) and (3.39) we obtain

$$\begin{aligned}
\|V(t,t_0)\|' &\le L\|V(t_0-n\delta_0,t_0)\|' \\
&\le L\prod_{i=0}^{n-1}\|V(t_0-(i+1)\delta_0,t_0-i\delta_0)\|' \le LK_0^n
\end{aligned} \tag{3.41}$$

for $t \le t_0$, where $K_0 := 2KL^2\delta_0^{(1-p)/p} < 1$. Since

$$n = \left[\frac{\ln\mu(t_0) - \ln\mu(t)}{\delta_0}\right] \ge \frac{\ln\mu(t_0) - \ln\mu(t)}{\delta_0} - 1, \tag{3.42}$$

this implies that

$$\|V(t,t_0)\|' \le d\left(\frac{\mu(t)}{\mu(t_0)}\right)^{\alpha}, \tag{3.43}$$

where

$$d = \frac{L}{K_0}, \qquad \alpha = -\frac{1}{\delta_0}\ln K_0. \tag{3.44}$$

We note that $d, \alpha > 0$. By (3.6)

$$\|V(t,t_0)x\|_t' \geq \|V(t,t_0)x\|, \tag{3.45}$$

and by (3.3),

$$\|x\|_{t_0}' = \sup\left\{\|V(\sigma,t_0)x\|\left(\frac{\mu(t_0)}{\mu(\sigma)}\right)^{-\omega} : 0 \leq \sigma \leq t_0\right\} \leq D\nu^{\varepsilon}(t_0)\|x\| \tag{3.46}$$

for $x \in Q_{t_0}$. It follows from (3.43) that

$$\begin{aligned} \|V(t,t_0) \mid Q_{t_0}\| &= \sup_{x\in Q_{t_0}\setminus\{0\}} \frac{\|V(t,t_0)x\|}{\|x\|} \\ &\leq D\nu^{\varepsilon}(t_0) \sup_{x\in Q_{t_0}\setminus\{0\}} \frac{\|V(t,t_0)x\|_t'}{\|x\|_{t_0}'} \\ &\leq dD\nu^{\varepsilon}(t_0)\left(\frac{\mu(t)}{\mu(t_0)}\right)^{\alpha} \end{aligned} \tag{3.47}$$

for any $t \leq t_0$. To show that T is a nonuniform exponential dichotomy, we note that setting $t = s$ in (3.2) and (3.3) yields

$$\|P(s)\| \leq D\nu^{\varepsilon}(s), \qquad \|Q(s)\| \leq D\nu^{\varepsilon}(s) \tag{3.48}$$

for every $s \geq 0$. Together with (3.24) and (3.47) this implies that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq \|T(t,s) \mid P_s\|\|P(s)\| \\ &\leq dD^2\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu^{2\varepsilon}(s), \quad t \geq s \geq 0. \\ \|T(t,s)Q(s)\| &\leq \|T(t,s) \mid Q_s\|\|P(s)\| \\ &\leq dD^2\left(\frac{\mu(s)}{\mu(t)}\right)^{-\alpha}\nu^{2\varepsilon}(s), \quad s \geq t \geq 0. \end{aligned} \tag{3.49}$$

This shows that T is a nonuniform (μ, ν) dichotomy with α, α, dD^2 and 2ε. □

3.3. *Admissible Spaces for a Nonuniform* (μ, ν) *Dichotomy*

We consider the spaces

$$L_D^p = \left\{f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \;\text{ Lebesgue-measurable} : \|f\|_{p,D} < \infty\right\} \tag{3.50}$$

for each $p \in [1,\infty)$, and

$$L_D^\infty = \left\{ f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \ \text{ Lebesgue-measurable} : \|f\|_{\infty,D} < \infty \right\}, \tag{3.51}$$

respectively, with the norms

$$\begin{gathered} \|f\|_{p,D} = \left(\int_0^\infty |f(t)|^p \left(D\nu^\varepsilon(t) \left(\frac{\mu(t)}{\mu'(t)} \right)^{1/q} \right)^p \right)^{1/p}, \\ \|f\|_{\infty,D} = \operatorname*{ess\,sup}_{t \in \mathbb{R}_0^+} \left(|f(t)| D\nu^\varepsilon(t) \frac{\mu(t)}{\mu'(t)} \right). \end{gathered} \tag{3.52}$$

In a similar manner to Lemma 2.3 these normed spaces induce Banach spaces $\mathcal{L}_D^p$ and $\mathcal{L}_D^p(X)$ for each $p \in [1,\infty]$, the last one with norm

$$\|f\|'_{p,D} = \|F\|_{p,D}, \quad \text{where } F(t) = \|f(t)\|'_t. \tag{3.53}$$

Theorem 3.5. *If the evolution process T is a nonuniform (μ,ν) dichotomy, then for any $p \in [1,\infty]$ the space $\mathcal{L}_D^p$ is admissible for T.*

Proof. We first take $f \in L_D^\infty$. Then

$$\begin{aligned} \|x_f(t)\|'_t = {} & \sup\left\{ \left\| \int_0^t U(\sigma,t)U(t,\tau)f(\tau)d\tau \right\| \left(\frac{\mu(\sigma)}{\mu(t)} \right)^{-\omega} : \sigma \geq t \right\} \\ & + \sup\left\{ \left\| \int_t^\infty V(\sigma,t)V(t,\tau)f(\tau)d\tau \right\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} : 0 \leq \sigma \leq t \right\} \\ \leq {} & \sup\left\{ \int_0^t \|U(\sigma,\tau)\| \cdot \|f(\tau)\| \left(\frac{\mu(\sigma)}{\mu(t)} \right)^{-\omega} d\tau : \sigma \geq t \right\} \\ & + \sup\left\{ \int_t^\infty \|V(\sigma,\tau)\| \cdot \|f(\tau)\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} d\tau : 0 \leq \sigma \leq t \right\} \\ \leq {} & \sup\left\{ \int_0^t D\nu^\varepsilon(\tau) \left(\frac{\mu(\sigma)}{\mu(\tau)} \right)^{-\alpha} \|f(\tau)\| d\tau : \sigma \geq t \right\} \\ & + \sup\left\{ \int_t^\infty D\nu^\varepsilon(\tau) \left(\frac{\mu(\sigma)}{\mu(\tau)} \right)^{\beta} \|f(\tau)\| d\tau : 0 \leq \sigma \leq t \right\} \\ \leq {} & \sup\left\{ \int_0^t D\nu^\varepsilon(\tau) \left(\frac{\mu(t)}{\mu(\tau)} \right)^{-\alpha} \|f(\tau)\|'_\tau d\tau : \sigma \geq t \right\} \\ & + \sup\left\{ \int_t^\infty D\nu^\varepsilon(\tau) \left(\frac{\mu(t)}{\mu(\tau)} \right)^{\beta} \|f(\tau)\|'_\tau d\tau : 0 \leq \sigma \leq t \right\} \end{aligned}$$

$$\begin{aligned}
&\le \|f\|'_{\infty,D} \int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \frac{\mu'(\tau)}{\mu(\tau)} d\tau + \|f\|'_{\infty,D} \int_t^\infty \left(\frac{\mu(t)}{\mu(\tau)}\right)^{\beta} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\
&= \frac{1}{\alpha}\|f\|'_{\infty,D}\left(1-\mu(t)^{-\alpha}\right) + \frac{1}{\beta}\|f\|'_{\infty,D} \\
&\le \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\|f\|'_{\infty,D}.
\end{aligned} \tag{3.54}$$

Therefore,

$$\|x_f\|'_\infty = \sup_{t\ge 0}\|x_f(t)\|'_t \le \sup_{t\ge 0}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\|f\|'_{\infty,D} < \infty \tag{3.55}$$

and $\mathcal{L}_D^\infty$ is admissible for T.

Now we take $f \in \mathcal{L}_D^p(X)$ for some $p \in [1,\infty)$. Using Hölder's inequality we obtain

$$\begin{aligned}
\|x_f(t)\|'_t = {} & \sup\left\{\left\|\int_0^t U(\sigma,t)U(t,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \ge t\right\} \\
&+ \sup\left\{\left\|\int_t^\infty V(\sigma,t)V(t,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega} : 0 \le \sigma \le t\right\} \\
\le {} & \sup\left\{\int_0^t D\nu^\varepsilon(\tau)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\|f(\tau)\|'_\tau d\tau : \sigma \ge t\right\} \\
&+ \sup\left\{\int_t^\infty D\nu^\varepsilon(\tau)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\beta}\|f(\tau)\|'_\tau d\tau : 0 \le \sigma \le t\right\} \\
\le {} & \|f\|'_{p,D}\left(\int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha q}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\right)^{1/q} \\
&+ \|f\|'_{p,D}\left(\int_t^\infty \left(\frac{\mu(t)}{\mu(\tau)}\right)^{\beta q}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\right)^{1/q} \\
= {} & \|f\|'_{p,D}\left(\frac{1-\mu(t)^{-\alpha q}}{\alpha q}\right)^{1/q} + \left(\frac{1}{\beta q}\right)^{1/q}\|f\|'_{p,D} \\
\le {} & \left(\frac{1}{(\alpha q)^{1/q}} + \frac{1}{(\beta q)^{1/q}}\right)\|f\|'_{p,D},
\end{aligned} \tag{3.56}$$

where $1/p + 1/q = 1$. We conclude that $\mathcal{L}_D^p$ is also admissible for T. □

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