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Research Article

Strong Convergence of an Implicit S-Iterative Process for Lipschitzian Hemicontractive Mappings

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We establish the strong convergence for the implicit S-iterative process associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

1. Introduction

Let H be a Hilbert space and let $T : H \to H$ be a mapping.

The mapping T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H. \tag{1.1}$$

If $L = 1$, then T is called *nonexpansive* and if $0 \leq L < 1$, then T is called *contractive*.

The mapping T is said to be *pseudocontractive* ([1, 2]) if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H, \tag{1.2}$$

and the mapping T is said to be *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H. \tag{1.3}$$

Let $F(T) := \{x \in H : Tx = x\}$ and the mapping T is called *hemicontractive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx\|^2, \quad \forall x \in H,\ x^* \in F(T). \tag{1.4}$$

It is easy to see the class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractive mappings. For the importance of fixed points of pseudocontractions the reader may consult [1].

In 1974, Ishikawa [3] proved the following result.

Theorem 1.1. *Let K be a compact convex subset of a Hilbert space H and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping.*

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions:

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$,

(ii) $\lim_{n\to\infty}\beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Another iteration scheme which has been studied extensively in connection with fixed points of pseudocontractive mappings.

In 2011, Sahu [4] and Sahu and Petruşel [5] introduced the S-iterative process as follows.

Let K be a nonempty convex subset of a normed space X and let $T : K \to K$ be a mapping. Then, for arbitrary $x_1 \in K$, the S-iterative process is defined by

$$\begin{aligned} x_{n+1} &= T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{aligned} \tag{1.6}$$

where $\{\beta_n\}$ is a real sequence in $[0,1]$.

In this paper, we establish the strong convergence for the implicit S-iterative process associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

2. Main Results

We need the follwing lemma.

Lemma 2.1 (see [6]). *For all $x,y \in H$ and $\lambda \in [0,1]$, the following well-known identity holds*

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Now we prove our main results.

Theorem 2.2. *Let K be a compact convex subset of a real Hilbert space H and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying*

$$\|x - Ty\| \leq \|Tx - Ty\|, \quad \forall x, y \in K. \tag{C}$$

Let $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying

(iv) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(v) $\sum_{n=1}^{\infty} \beta_n^2 < \infty$.

For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$\begin{aligned} x_n &= Ty_n, \\ y_n &= (1-\beta_n)x_{n-1} + \beta_n Tx_n, \quad n \geq 1. \end{aligned} \tag{2.2}$$

Then the sequence $\{x_n\}$ converges strongly to the fixed point x^* of T.

Proof. From Schauder's fixed point theorem, $F(T)$ is nonempty since K is a convex compact set and T is continuous, let $x^* \in F(T)$. Using the fact that T is hemicontractive we obtain

$$\|Tx_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \|x_n - Tx_n\|^2, \tag{2.3}$$

$$\|Ty_n - x^*\|^2 \leq \|y_n - x^*\|^2 + \|y_n - Ty_n\|^2. \tag{2.4}$$

Now by (v), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\beta_n \leq \min\left\{\frac{1}{3}, \frac{1}{L^2}\right\}, \tag{2.5}$$

which implies that

$$\frac{2\beta_n}{1-\beta_n} \leq 1. \tag{2.6}$$

With the help of (2.2), (2.3), and Lemma 2.1, we obtain the following estimates:

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &= \left\|(1-\beta_n)x_{n-1} + \beta_n T x_n - x^*\right\|^2 \\
&= \left\|(1-\beta_n)(x_{n-1} - x^*) + \beta_n (T x_n - x^*)\right\|^2 \\
&= (1-\beta_n)\|x_{n-1} - x^*\|^2 + \beta_n \|T x_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_{n-1} - T x_n\|^2 \\
&\leq (1-\beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\left(\|x_n - x^*\|^2 + \|x_n - T x_n\|^2\right) \\
&\quad - \beta_n(1-\beta_n)\|x_{n-1} - T x_n\|^2, \\
\left\|y_n - T y_n\right\|^2 &= \left\|(1-\beta_n)x_{n-1} + \beta_n T x_n - T y_n\right\|^2 \\
&= \left\|(1-\beta_n)(x_{n-1} - T y_n) + \beta_n (T x_n - T y_n)\right\|^2 \\
&= (1-\beta_n)\left\|x_{n-1} - T y_n\right\|^2 + \beta_n \left\|T x_n - T y_n\right\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_{n-1} - T x_n\|^2.
\end{aligned} \tag{2.7}$$

Substituting (2.7) in (2.4) we obtain

$$\begin{aligned}
\left\|T y_n - x^*\right\|^2 &\leq (1-\beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\left(\|x_n - x^*\|^2 + \|x_n - T x_n\|^2\right) \\
&\quad + (1-\beta_n)\left\|x_{n-1} - T y_n\right\|^2 + \beta_n\left\|T x_n - T y_n\right\|^2 \\
&\quad - 2\beta_n(1-\beta_n)\|x_{n-1} - T x_n\|^2.
\end{aligned} \tag{2.8}$$

Also with the help of condition (C) and (2.8), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|T y_n - x^*\right\|^2 \\
&\leq (1-\beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\left(\|x_n - x^*\|^2 + \|x_n - T x_n\|^2\right) \\
&\quad + (1-\beta_n)\left\|x_{n-1} - T y_n\right\|^2 + \beta_n\left\|T x_n - T y_n\right\|^2 \\
&\quad - 2\beta_n(1-\beta_n)\|x_{n-1} - T x_n\|^2 \\
&\leq (1-\beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|x_{n-1} - T y_n\right\|^2 \\
&\quad + 2\beta_n\left\|T x_n - T y_n\right\|^2 - 2\beta_n(1-\beta_n)\|x_{n-1} - T x_n\|^2,
\end{aligned} \tag{2.9}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|x_{n-1}-x^*\|^2+\left\|x_{n-1}-Ty_n\right\|^2\\&\quad+\frac{2\beta_n}{1-\beta_n}\left\|Tx_n-Ty_n\right\|^2-2\beta_n\|x_{n-1}-Tx_n\|^2\\&\le \|x_{n-1}-x^*\|^2+\left\|x_{n-1}-Ty_n\right\|^2+\left\|Tx_n-Ty_n\right\|^2\\&\quad-2\beta_n\|x_{n-1}-Tx_n\|^2,\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}\left\|x_{n-1}-Ty_n\right\|^2 &\le \left\|Tx_{n-1}-Ty_n\right\|^2\\&\le L^2\left\|x_{n-1}-y_n\right\|^2\\&= L^2\beta_n^2\|x_{n-1}-Tx_n\|^2,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\left\|Tx_n-Ty_n\right\|^2 &\le L^2\left\|x_n-y_n\right\|^2\\&\le L^2\left(\|x_n-x_{n-1}\|+\left\|x_{n-1}-y_n\right\|\right)^2\\&\le L^2\left(\|x_n-x_{n-1}\|+\beta_n\|x_{n-1}-Tx_n\|\right)^2\\&\le L^2\left(\|x_n-x_{n-1}\|+\beta_nM\right)^2,\\\|x_n-x_{n-1}\| &= \left\|x_{n-1}-Ty_n\right\|\\&\le \left\|Tx_{n-1}-Ty_n\right\|\\&\le L\left\|x_{n-1}-y_n\right\|\\&= L\beta_n\|x_{n-1}-Tx_n\|\\&\le L\beta_nM\end{aligned} \tag{2.12}$$

and consequently from (2.12), we obtain

$$\left\|Tx_n-Ty_n\right\|^2 \le L^2(1+L)^2M^2\beta_n^2. \tag{2.13}$$

Hence by (2.5), (2.10), (2.11), and (2.13), we have

$$\begin{aligned}\|x_n - x^*\|^2 &\leq \|x_{n-1} - x^*\|^2 + L^2\beta_n^2\|x_{n-1} - Tx_n\|^2 \\ &\quad + L^2(1+L)^2M^2\beta_n^2 - 2\beta_n\|x_{n-1} - Tx_n\|^2 \\ &= \|x_{n-1} - x^*\|^2 + L^2(1+L)^2M^2\beta_n^2 \\ &\quad - \beta_n\left(2 - L^2\beta_n\right)\|x_{n-1} - Tx_n\|^2 \\ &\leq \|x_{n-1} - x^*\|^2 + L^2(1+L)^2M^2\beta_n^2 - \beta_n\|x_{n-1} - Tx_n\|^2,\end{aligned} \tag{2.14}$$

which implies that

$$\beta_n\|x_{n-1} - Tx_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + L^2(1+L)^2M^2\beta_n^2, \tag{2.15}$$

so that

$$\frac{1}{2}\sum_{j=N}^{n}\beta_j\left\|x_{j-1} - Tx_j\right\|^2 \leq \|x_N - x^*\|^2 - \|x_n - x^*\|^2 + L^2(1+L)^2M^2\sum_{j=N}^{n}\beta_j^2. \tag{2.16}$$

Hence by conditions (iv) and (v), we get

$$\sum_{j=0}^{\infty}\left\|x_{j-1} - Tx_j\right\|^2 < \infty. \tag{2.17}$$

It implies that

$$\lim_{n\to\infty}\|x_{n-1} - Tx_n\| = 0. \tag{2.18}$$

Consider

$$\|x_n - Tx_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - Tx_n\|, \tag{2.19}$$

which implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{2.20}$$

The rest of the argument follows exactly as in the proof of Theorem of [3]. This completes the proof. □

Theorem 2.3. *Let K be a compact convex subset of a real Hilbert space H and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying the condition (C). Let $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying the conditions (iv) and (v).*

Assume that $P_K : H \to K$ be the projection operator of H onto K. Let $\{x_n\}$ be a sequence defined iteratively by

$$x_n = P_K(Ty_n),$$
$$y_n = P_K((1-\beta_n)x_{n-1} + \beta_n Tx_n), \quad n \geq 1. \tag{2.21}$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The operator P_K is nonexpansive (see, e.g., [2]). K is a Chebyshev subset of H so that, P_K is a single-valued mapping. Hence, we have the following estimate:

$$\begin{aligned}\|x_n - x^*\|^2 &= \|P_K(Ty_n) - P_K x^*\|^2 \\ &\leq \|Ty_n - x^*\|^2 \\ &\leq \|x_{n-1} - x^*\|^2 + L^2(1+L)^2 M^2 \beta_n^2 - \beta_n \|x_{n-1} - Tx_n\|^2.\end{aligned} \tag{2.22}$$

The set $K = K \cup T(K)$ is compact and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 2.2. This completes the proof. □

Remark 2.4. In main results, the condition (C) is not new and it is due to Liu et al. [7].

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