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Research Article

A Kantorovich Type of Szasz Operators Including Brenke-Type Polynomials

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We give a Kantorovich variant of a generalization of Szasz operators defined by means of the Brenke-type polynomials and obtain convergence properties of these operators by using Korovkin's theorem. We also present the order of convergence with the help of a classical approach, the second modulus of continuity, and Peetre's K-functional. Furthermore, an example of Kantorovich type of the operators including Gould-Hopper polynomials is presented and Voronovskaya-type result is given for these operators including Gould-Hopper polynomials.

1. Introduction

The Szasz operators (also called Szasz-Mirakyan operators) which are defined by [1]

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $n \in \mathbb{N}$, $x \geq 0$, and $f \in C[0,\infty)$ have an important role in the approximation theory, and their approximation properties have been investigated by many researchers.

In [2], Jakimovski and Leviatan proposed a generalization of Szasz operators by means of the Appell polynomials $p_k(x)$ which have the generating functions of the form:

$$g(t)e^{tx} = \sum_{k=0}^{\infty}p_k(x)t^k, \tag{1.2}$$

where $g(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 \neq 0)$ is an analytic function in the disc $|z| < R$, $(R > 1)$ and $g(1) \neq 0$. Under the assumption that $p_k(x) \geq 0$ for $x \in [0, \infty)$, Jakimovski and Leviatan [2], defined the following linear positive operators:

$$P_n(f; x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{1.3}$$

After that, Ismail [3] defined another generalization of Szasz operators involving the operators (1.1) and (1.3) by means of Sheffer polynomials. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 \neq 0)$ and $H(z) = \sum_{k=1}^{\infty} h_k z^k (h_1 \neq 0)$ be analytic functions in the disc $|z| < R$, $(R > 1)$. Here, a_k and h_k are real. The Sheffer polynomials $p_k(x)$ are generated by

$$A(t) e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x) t^k. \tag{1.4}$$

With the help of these polynomials, Ismail constructed the following linear positive operators:

$$T_n(f; x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N} \tag{1.5}$$

under the assumptions

(i) for $x \in [0, \infty)$, $p_k(x) \geq 0$,

(ii) $A(1) \neq 0$ and $H'(1) = 1$.

Later, Varma et al. [4] defined another generalization of Szasz operators by means of the Brenke-type polynomials. Suppose that

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \quad B(t) = \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0 \quad (r \geq 0) \tag{1.6}$$

are analytic functions. The Brenke-type polynomials [5] have generating functions of the form

$$A(t) B(xt) = \sum_{k=0}^{\infty} p_k(x) t^k \tag{1.7}$$

from which the explicit form of $p_k(x)$ is as follows:

$$p_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \quad k = 0, 1, 2, \ldots. \tag{1.8}$$

Under the assumptions

$$\begin{aligned}
&\text{(i)}\quad A(1)\neq 0,\ \frac{a_{k-r}b_r}{A(1)}\geq 0,\quad 0\leq r\leq k,\ k=0,1,2,\ldots,\\
&\text{(ii)}\quad B:[0,\infty)\longrightarrow(0,\infty),\\
&\text{(iii)}\quad \text{(1.6) and (1.7) converge for } |t|<R,\quad (R>1),
\end{aligned} \tag{1.9}$$

Varma et al. introduced the linear positive operators $L_n(f;x)$ via

$$L_n(f;x) := \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.10}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

The aim of this paper is to present a Kantorovich type of the operators given by (1.10) and to give their some approximation properties. We consider the Kantorovich version of the operators (1.10) under the assumptions (1.9) as follows:

$$K_n(f;x) := \frac{n}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\int_{k/n}^{(k+1)/n} f(t)dt, \tag{1.11}$$

where $n \in \mathbb{N}$, $x \geq 0$, and $f \in C[0,\infty)$. It is easy to see that K_n defined by (1.11) is linear and positive.

In the case of $B(t) = e^t$ and $A(t) = 1$, with the help of (1.7), it follows that $p_k(x) = x^k/k!$, so the operators (1.11) reduce to the Szasz-Mirakyan-Kantorovich operators defined by [6]

$$K_n(f;x) := ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{k/n}^{(k+1)/n} f(t)dt. \tag{1.12}$$

Various approximation properties of the Szasz-Mirakyan-Kantorovich operators and their iterates may be found in [7–13].

The case of $B(t) = e^t$ gives the Kantorovich version of the operators (1.3).

The structure of the paper is as follows. In Section 2, the convergence of the operators (1.11) is given by means of Korovkin's theorem. The order of approximation is obtained with the help of a classical approach, the second modulus of continuity, and Peetre's K-functional in Section 3. Finally, as an example, we present a Kantorovich type of the operators including Gould-Hopper polynomials and then we give a Voronovskaya-type theorem for the operators including Gould-Hopper polynomials.

2. Approximation Properties of K_n Operators

In this section, we give our main theorem with the help of Korovkin theorem. We begin with the following lemma which is necessary to prove the main result.

Lemma 2.1. *For all $x \in [0,\infty)$, the operators K_n defined by* (1.11) *verify*

$$K_n(1;x) = 1, \tag{2.1}$$

$$K_n(s;x) = \frac{B'(nx)}{B(nx)}x + \frac{2A'(1)+A(1)}{2nA(1)}, \tag{2.2}$$

$$K_n\left(s^2;x\right) = \frac{B''(nx)}{B(nx)}x^2 + \frac{2B'(nx)[A'(1)+A(1)]}{nA(1)B(nx)}x + \frac{1}{n^2A(1)}\left\{A''(1)+2A'(1)+\frac{A(1)}{3}\right\}. \tag{2.3}$$

Proof. Using the generating function of the Brenke-typepolynomials given by (1.7), we can write

$$\begin{gathered}\sum_{k=0}^{\infty} p_k(nx) = A(1)B(nx),\\ \sum_{k=0}^{\infty} kp_k(nx) = A'(1)B(nx) + nxA(1)B'(nx),\\ \sum_{k=0}^{\infty} k^2p_k(nx) = n^2x^2A(1)B''(nx) + nxB'(nx)\{2A'(1)+A(1)\} + B(nx)\{A''(1)+A'(1)\}.\end{gathered} \tag{2.4}$$

From these equalities, the assertions of the lemma are obtained. □

Lemma 2.2. *For $x \in [0,\infty)$, one has*

$$\begin{aligned}K_n\left((s-x)^2;x\right) = {}&\left\{\frac{B''(nx)-2B'(nx)+B(nx)}{B(nx)}\right\}x^2\\ &+\left\{\frac{2A'(1)[B'(nx)-B(nx)]+A(1)[2B'(nx)-B(nx)]}{nA(1)B(nx)}\right\}x\\ &+\frac{1}{n^2A(1)}\left\{A''(1)+2A'(1)+\frac{A(1)}{3}\right\}.\end{aligned} \tag{2.5}$$

Proof. From the linearity of K_n, we get

$$K_n\left((s-x)^2;x\right) = K_n\left(s^2;x\right) - 2xK_n(s;x) + x^2K_n(1;x). \tag{2.6}$$

Next, we apply Lemma 2.1. □

Theorem 2.3. *Let*

$$E := \left\{f : x\in[0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \longrightarrow \infty\right\}, \tag{2.7}$$

$$\lim_{y\to\infty}\frac{B'(y)}{B(y)} = 1, \qquad \lim_{y\to\infty}\frac{B''(y)}{B(y)} = 1. \tag{2.8}$$

If $f \in C[0,\infty) \cap E$, *then*

$$\lim_{n\to\infty} K_n(f;x) = f(x), \tag{2.9}$$

and the operators K_n *converge uniformly in each compact subset of* $[0,\infty)$.

Proof. Using Lemma 2.1 and taking into account the equality (2.8) we get

$$\lim_{n\to\infty} K_n\left(s^i;x\right) = x^i, \quad i = 0,1,2. \tag{2.10}$$

The above convergence is satisfied uniformly in each compact subset of $[0,\infty)$. We can then apply the universal Korovkin-type property (vi) of Theorem 4.1.4 in [14] to obtain the desired result. □

3. The Order of Approximation

In this section, we deal with the rates of convergence of the $K_n(f)$ to f by means of a classical approach, the second modulus of continuity, and Peetre's K-functional.

Let $f \in \widetilde{C}[0,\infty)$. If $\delta > 0$, the modulus of continuity of f is defined by

$$w(f;\delta) := \sup_{\substack{x,y\in[0,\infty) \\ |x-y|\le\delta}} |f(x) - f(y)|, \tag{3.1}$$

where $\widetilde{C}[0,\infty)$ denotes the space of uniformly continuous functions on $[0,\infty)$. It is also well known that, for any $\delta > 0$ and each $x \in [0,\infty)$,

$$|f(x) - f(y)| \le w(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{3.2}$$

The next result gives the rate of convergence of the sequence $K_n(f)$ to f by means of the modulus of continuity.

Theorem 3.1. *Let* $f \in \widetilde{C}[0,\infty) \cap E$. *The* K_n *operators satisfy the following inequality:*

$$|K_n(f;x) - f(x)| \le 2w\left(f;\sqrt{\lambda_n(x)}\right), \tag{3.3}$$

where

$$\lambda := \lambda_n(x) = K_n\Big((s-x)^2; x\Big) = \left\{ \frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)} \right\} x^2$$
$$+ \left\{ \frac{2A'(1)[B'(nx) - B(nx)] + A(1)[2B'(nx) - B(nx)]}{nA(1)B(nx)} \right\} x \tag{3.4}$$
$$+ \frac{1}{n^2 A(1)} \left\{ A''(1) + 2A'(1) + \frac{A(1)}{3} \right\}.$$

Proof. Using (2.1), (3.2), and the linearity property of K_n operators, we can write

$$\begin{aligned}
\left|K_n(f;x) - f(x)\right| &\le \frac{n}{A(1)B(nx)} \sum_{k=0}^{\infty} P_k(nx) \int_{k/n}^{(k+1)/n} \left|f(s) - f(x)\right| ds \\
&\le \frac{n}{A(1)B(nx)} \sum_{k=0}^{\infty} P_k(nx) \int_{k/n}^{(k+1)/n} \left(\frac{|s-x|}{\delta} + 1 \right) w(f;\delta) ds \\
&\le \left\{ 1 + \frac{n}{A(1)B(nx)\delta} \sum_{k=0}^{\infty} P_k(nx) \int_{k/n}^{(k+1)/n} |s-x| ds \right\} w(f;\delta).
\end{aligned} \tag{3.5}$$

By using the Cauchy-Schwarz inequality for integration, we get

$$\int_{k/n}^{(k+1)/n} |s-x| ds \le \frac{1}{\sqrt{n}} \left(\int_{k/n}^{(k+1)/n} |s-x|^2 ds \right)^{1/2} \tag{3.6}$$

which holds that

$$\sum_{k=0}^{\infty} P_k(nx) \int_{k/n}^{(k+1)/n} |s-x| ds \le \frac{1}{\sqrt{n}} \sum_{k=0}^{\infty} P_k(nx) \left(\int_{k/n}^{(k+1)/n} |s-x|^2 ds \right)^{1/2}. \tag{3.7}$$

By applying the Cauchy-Schwarz inequality for summation on the right-hand side of (3.7), we have

$$\begin{aligned}
\sum_{k=0}^{\infty} P_k(nx) \int_{k/n}^{(k+1)/n} |s-x| ds &\le \frac{\sqrt{A(1)B(nx)}}{\sqrt{n}} \left(\frac{A(1)B(nx)}{n} K_n\Big((s-x)^2; x\Big) \right)^{1/2} \\
&= \frac{A(1)B(nx)}{n} \left(K_n\Big((s-x)^2; x\Big) \right)^{1/2} \\
&= \frac{A(1)B(nx)}{n} (\lambda_n(x))^{1/2},
\end{aligned} \tag{3.8}$$

where $\lambda_n(x)$ is given by (3.4). If we use this in (3.5), we obtain

$$\left|K_n(f;x) - f(x)\right| \leq \left\{1 + \frac{1}{\delta}\sqrt{\lambda_n(x)}\right\} w(f;\delta). \tag{3.9}$$

On choosing $\delta = \sqrt{\lambda_n(x)}$, we arrive at the desired result. □

Recall that the second modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$w_2(f;\delta) := \sup_{0<t\leq\delta} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C_B}, \tag{3.10}$$

where $C_B[0,\infty)$ is the class of real valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)}|f(x)|$.

Peetre's K-functional of the function $f \in C_B[0,\infty)$ is defined by

$$K(f;\delta) := \inf_{g\in C_B^2[0,\infty)} \left\{\|f - g\|_{C_B} + \delta\|g\|_{C_B^2}\right\}, \tag{3.11}$$

where

$$C_B^2[0,\infty) := \left\{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\right\}, \tag{3.12}$$

and the norm $\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$ (see [15]). It is clear that the following inequality:

$$K(f;\delta) \leq M\left\{w_2\left(f;\sqrt{\delta}\right) + \min(1,\delta)\|f\|_{C_B}\right\}, \tag{3.13}$$

holds for all $\delta > 0$. The constant M is independent of f and δ.

Theorem 3.2. *Let $f \in C_B^2[0,\infty)$. The following*

$$\left|K_n(f;x) - f(x)\right| \leq \zeta\|f\|_{C_B^2} \tag{3.14}$$

holds, where

$$\begin{aligned}
\zeta &:= \zeta_n(x) \\
&= \left\{\frac{B''(nx) - 2B'(nx) + B(nx)}{2B(nx)}\right\}x^2 \\
&\quad + \left\{\frac{2A'(1)[B'(nx) - B(nx)] + A(1)[2(n+1)B'(nx) - (2n+1)B(nx)]}{2nA(1)B(nx)}\right\}x \\
&\quad + \frac{1}{2n^2A(1)}\left\{A''(1) + 2A'(1) + \frac{A(1)}{3}\right\} + \frac{2A'(1) + A(1)}{2nA(1)}.
\end{aligned} \tag{3.15}$$

Proof. From the Taylor expansion of f, the linearity of the operators K_n and (2.1), we have

$$K_n(f;x) - f(x) = f'(x)K_n(s-x;x) + \frac{1}{2}f''(\eta)K_n\left((s-x)^2;x\right), \quad \eta \in (x,s). \tag{3.16}$$

Since

$$K_n(s-x;x) = \left\{\frac{B'(nx)-B(nx)}{B(nx)}\right\}x + \frac{2A'(1)+A(1)}{2nA(1)} \geq 0 \tag{3.17}$$

for $s \geq x$, by considering Lemmas 2.1 and 2.2 in (3.16), we can write that

$$\begin{aligned}
\left|K_n(f;x) - f(x)\right| &\leq \left\{\left(\frac{B'(nx)-B(nx)}{B(nx)}\right)x + \frac{2A'(1)+A(1)}{2nA(1)}\right\}\|f'\|_{C_B} \\
&\quad + \frac{1}{2}\left[\left\{\frac{B''(nx)-2B'(nx)+B(nx)}{B(nx)}\right\}x^2\right. \\
&\qquad + \left\{\frac{2A'(1)[B'(nx)-B(nx)]+A(1)[2B'(nx)-B(nx)]}{nA(1)B(nx)}\right\}x \\
&\qquad \left. + \frac{1}{n^2A(1)}\left\{A''(1)+2A'(1)+\frac{A(1)}{3}\right\}\right]\|f''\|_{C_B} \\
&\leq \left[\left\{\frac{B''(nx)-2B'(nx)+B(nx)}{2B(nx)}\right\}x^2\right. \\
&\quad + \left\{\frac{2A'(1)[B'(nx)-B(nx)]+A(1)[2(n+1)B'(nx)-(2n+1)B(nx)]}{2nA(1)B(nx)}\right\}x \\
&\quad \left. + \frac{1}{2n^2A(1)}\left\{A''(1)+2A'(1)+\frac{A(1)}{3}\right\} + \frac{2A'(1)+A(1)}{2nA(1)}\right]\|f\|_{C_B^2}
\end{aligned} \tag{3.18}$$

which completes the proof. □

Theorem 3.3. *Let $f \in C_B[0,\infty)$. Then*

$$\left|K_n(f;x) - f(x)\right| \leq 2M\left\{w_2\left(f;\sqrt{\delta}\right) + \min(1,\delta)\|f\|_{C_B}\right\}, \tag{3.19}$$

where

$$\delta := \delta_n(x) = \frac{1}{2}\zeta_n(x) \tag{3.20}$$

and $M > 0$ is a constant which is independent of the functions f and δ. Also, $\zeta_n(x)$ is the same as in Theorem 3.2.

Proof. Suppose that $g \in C_B^2[0,\infty)$. From Theorem 3.2, we can write

$$\begin{aligned} \left|K_n(f;x) - f(x)\right| &\le \left|K_n(f-g;x)\right| + \left|K_n(g;x) - g(x)\right| + \left|g(x) - f(x)\right| \\ &\le 2\|f-g\|_{C_B} + \zeta\|g\|_{C_B^2} \\ &= 2\left[\|f-g\|_{C_B} + \delta\|g\|_{C_B^2}\right]. \end{aligned} \tag{3.21}$$

The left-hand side of inequality (3.21) does not depend on the function $g \in C_B^2[0,\infty)$, so

$$\left|K_n(f;x) - f(x)\right| \le 2K(f;\delta), \tag{3.22}$$

where $K(f;\delta)$ is Peetre's K-functional defined by (3.11). By the relation between Peetre's K-functional and the second modulus of smoothness given by (3.13), inequality (3.21) becomes

$$\left|K_n(f;x) - f(x)\right| \le 2M\left\{w_2\left(f;\sqrt{\delta}\right) + \min(1,\delta)\|f\|_{C_B}\right\} \tag{3.23}$$

whence we have the result. □

Remark 3.4. Note that when $n \to \infty$, then λ_n, ζ_n, and δ_n tend to zero in Theorems 3.1–3.3 under the assumption (2.8).

4. Special Cases and Further Properties

Gould-Hopper polynomials $g_k^{d+1}(x,h)$ [16], which are d-orthogonal polynomial sets of Hermite type [17], are generated by

$$e^{ht^{d+1}} \exp(xt) = \sum_{k=0}^{\infty} g_k^{d+1}(x,h)\frac{t^k}{k!} \tag{4.1}$$

from which it follows that

$$g_k^{d+1}(x,h) = \sum_{m=0}^{[k/(d+1)]} \frac{k!}{m!(k-(d+1)m)!} h^m x^{k-(d+1)m}, \tag{4.2}$$

where, as usual, $[\cdot]$ denotes the integer part.

In [4], the authors showed that the Gould-Hopper polynomials are Brenke-type polynomials with $A(t) = e^{ht^{d+1}}$ and $B(t) = e^t$, and the restrictions (1.9) and condition (2.8) for the operators given by (1.10) are satisfied under the assumption $h \ge 0$. These operators including the Gould-Hopper polynomials are as follows:

$$L_n^*(f;x) := e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,h)}{k!} f\left(\frac{k}{n}\right), \tag{4.3}$$

where $x \in [0,\infty)$.

The special case $A(t) = e^{ht^{d+1}}$ and $B(t) = e^t$ of (1.11) gives the following Kantorovich version of $K_n(f;x)$ including the Gould-Hopper polynomials:

$$K_n^*(f;x) := ne^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\int_{k/n}^{(k+1)/n} f(t)dt \tag{4.4}$$

under the assumption $h \geq 0$.

Remark 4.1. For $h = 0$, we find $g_k^{d+1}(nx,0) = (nx)^k$ and the operators given by (4.4) reduce to the Szasz-Mirakyan-Kantorovich operators given by (1.12).

Now, we give a Voronovskaya-type theorem for the operators (4.4). In order to prove this theorem, we need the following lemmas.

Lemma 4.2. *For the operators K_n^*, one has*

$$\begin{aligned}
K_n^*(1;x) &= 1,\\
K_n^*(s;x) &= x + \frac{h(d+1)}{n} + \frac{1}{2n},\\
K_n^*\left(s^2;x\right) &= x^2 + \frac{2}{n}(h(d+1)+1)x\\
&\quad + \frac{1}{n^2}\left[h(d+1)\{h(d+1)+d+2\} + \frac{1}{3}\right],\\
K_n^*\left(s^3;x\right) &= x^3 + \frac{3x^2}{n}\left\{h(d+1)+\frac{3}{2}\right\} + \frac{x}{n^2}\left\{3h^2(d+1)^2 + 3h(d+1)(d+3) + \frac{7}{2}\right\}\\
&\quad + \frac{1}{n^3}\left\{h^2(h+3)(d+1)^3 + h(d+1)^3 + \frac{3}{2}h(h+1)(d+1)^2 + h(d+1) + \frac{1}{4}\right\},\\
K_n^*\left(s^4;x\right) &= x^4 + \frac{4x^3}{n}\{h(d+1)+2\} + \frac{3x^2}{n^2}\left\{2h(h+1)(d+1)^2 + 6h(d+1) + 5\right\}\\
&\quad + \frac{x}{n^3}\left\{12h^2(d+1)^2(d+2) + 4h^3(d+1)^3 + 2h(d+1)\left(2d^2+10d+15\right) + 6\right\}\\
&\quad + \frac{1}{n^4}\left\{h^3(h+6)(d+1)^4 + 8h(1+3h)(d+1)^3 - 9h(h+1)(d+1)^2\right.\\
&\qquad + 7h(d+1) + 2h^3(d+1)^3 + 4d(d-1)(d+1)^2h^2\\
&\qquad \left. + 3h^2d^2(d+1)^2 + (d-2)(d-1)d(d+1)h + \frac{1}{5}\right\}.
\end{aligned} \tag{4.5}$$

Proof. From the generating function (4.1) for the Gould-Hopper polynomials, one can easily find the above equalities. □

Lemma 4.3. *For $x \in [0, \infty)$, one has*

$$
\begin{aligned}
K_n^*\left((s-x)^2; x\right) &= \frac{x}{n} + \frac{1}{n^2}\left[h(d+1)\{h(d+1)+d+2\} + \frac{1}{3}\right] \\
K_n^*\left((s-x)^4; x\right) &= \frac{3x^2}{n^2} + \frac{x}{n^3}\left\{6h(h+2)(d+1)^2 + 2h(d+1)(-3d+2) + 5\right\} \\
&\quad + \frac{1}{n^4}\left\{h^3(h+6)(d+1)^4 + 2h\left(h^2+12h+4\right)(d+1)^3\right. \\
&\qquad + 7h(d+1) - 9h(h+1)(d+1)^2 + 4d(d-1)(d+1)^2h^2 \\
&\qquad \left. + 3h^2d^2(d+1)^2 + (d-2)(d-1)d(d+1)h + \frac{1}{5}\right\}.
\end{aligned}
\tag{4.6}
$$

Proof. It is enough to use Lemma 4.2 to obtain above equalities. □

Theorem 4.4. *Let $f \in C^2[0, a]$. Then one has*

$$
\lim_{n\to\infty} n\left[K_n^*(f;x) - f(x)\right] = f'(x)\left\{h(d+1) + \frac{1}{2}\right\} + \frac{x f''(x)}{2!}. \tag{4.7}
$$

Proof. By Taylor's theorem, we get

$$
f(s) = f(x) + (s-x)f'(x) + \frac{(s-x)^2}{2!}f''(x) + (s-x)^2\eta(s;x), \tag{4.8}
$$

where $\eta(s;x) \in C[0,a]$ and $\lim_{s\to x}\eta(s;x) = 0$. If we apply the operator K_n^* to the both sides of (4.8), we obtain

$$
\begin{aligned}
K_n^*(f;x) &= f(x) + f'(x)K_n^*(s-x;x) \\
&\quad + \frac{f''(x)}{2!}K_n^*\left((s-x)^2;x\right) + K_n^*\left((s-x)^2\eta(s;x);x\right).
\end{aligned}
\tag{4.9}
$$

In view of Lemmas 4.2 and 4.3, the equality (4.9) can be written in the form

$$
\begin{aligned}
n\left[K_n^*(f;x) - f(x)\right] &= n\left\{\frac{h(d+1)}{n} + \frac{1}{2n}\right\}f'(x) \\
&\quad + n\left\{\frac{x}{n} + \frac{1}{n^2}\left[h(d+1)\{h(d+1)+d+2\} + \frac{1}{3}\right]\right\}\frac{f''(x)}{2!} \\
&\quad + nK_n^*\left((s-x)^2\eta(s;x);x\right),
\end{aligned}
\tag{4.10}
$$

where

$$K_n^*\left((s-x)^2\eta(s;x);x\right) = ne^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\int_{k/n}^{(k+1)/n}(s-x)^2\eta(s;x)ds. \tag{4.11}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} &nK_n^*\left((s-x)^2\eta(s;x);x\right) \\ &\quad\le n^2e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\left(\int_{k/n}^{(k+1)/n}(s-x)^4ds\right)^{1/2}\left(\int_{k/n}^{(k+1)/n}\eta^2(s;x)ds\right)^{1/2}. \end{aligned} \tag{4.12}$$

If we use Cauchy-Schwarz inequality again on the right-hand side of the inequality above, then we conclude that

$$\begin{aligned} nK_n^*\left((s-x)^2\eta(s;x);x\right) &\le \left(n^3e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\int_{k/n}^{(k+1)/n}(s-x)^4ds\right)^{1/2} \\ &\quad\cdot\left(ne^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k^{d+1}(nx,h)}{k!}\int_{k/n}^{(k+1)/n}\eta^2(s;x)ds\right)^{1/2} \\ &= \sqrt{n^2K_n^*\left((s-x)^4;x\right)}\sqrt{K_n^*\left(\eta^2(s;x);x\right)}. \end{aligned} \tag{4.13}$$

In view of Lemma 4.3,

$$\lim_{n\to\infty} n^2K_n^*\left((s-x)^4;x\right) = 3x^2 \tag{4.14}$$

holds. On the other hand, since $\eta(s;x)\in C[0,a]$ and $\lim_{s\to x}\eta(s;x)=0$, then it follows from Theorem 2.3 that

$$\lim_{n\to\infty} K_n^*\left(\eta^2(s;x);x\right) = \eta^2(x;x) = 0. \tag{4.15}$$

Considering (4.13), (4.14), and (4.15), we immediately see that

$$\lim_{n\to\infty} nK_n^*\left((s-x)^2\eta(s;x);x\right) = 0. \tag{4.16}$$

Then, taking limit as $n\to\infty$ in (4.10) and using (4.16), we have

$$\lim_{n\to\infty} n\left[K_n^*(f;x)-f(x)\right] = f'(x)\left\{h(d+1)+\frac{1}{2}\right\}+\frac{xf''(x)}{2!} \tag{4.17}$$

which completes the proof. □

Remark 4.5. Getting $h = 0$ in Theorem 4.4 gives a Voronovskaya-type result for the Szasz-Mirakyan-Kantorovich operators given by (1.12).

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