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Research Article

Generalized Proximal ψ-Contraction Mappings and Best Proximity Points

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We generalized the notion of proximal contractions of the first and the second kinds and established the best proximity point theorems for these classes. Our results improve and extend recent result of Sadiq Basha (2011) and some authors.

1. Introduction

The significance of fixed point theory stems from the fact that it furnishes a unified treatment and is a vital tool for solving equations of form $Tx = x$ where T is a self-mapping defined on a subset of a metric space, a normed linear space, topological vector space or some suitable space. Some applications of fixed point theory can be found in [1–12]. However, almost all such results dilate upon the existence of a fixed point for self-mappings. Nevertheless, if T is a non-self-mapping, then it is probable that the equation $Tx = x$ has no solution, in which case best approximation theorems explore the existence of an approximate solution whereas best proximity point theorems analyze the existence of an approximate solution that is optimal. A classical best approximation theorem was introduced by Fan [13]; that is, if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [14], Reich [15], Sehgal, and Singh [16, 17], have derived extensions of Fan's theorem in many directions. Other works of the existence of a best proximity point for contractions can be seen in [18–21]. In 2005, Eldred et al. [22] have obtained best proximity point theorems for relatively nonexpansive mappings. Best proximity point theorems for several types of contractions have been established in [23–36].

Recently, Sadiq Basha in [37] gave necessary and sufficient to claimed that the existence of best proximity point for proximal contraction of first kind and the second kind which are non-self mapping analogues of contraction self-mappings and also established some best proximity and convergence theorem as follow.

Theorem 1.1 (see [37, Theorem 3.1]). *Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions.*

(a) *S and T are proximal contractions of first kind.*

(b) *g is an isometry.*

(c) *The pair (S, T) is a proximal cyclic contraction.*

(d) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$.*

(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{1.1}$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{1.2}$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A, B), \tag{1.3}$$

converges to the element y.

On the other hand, a sequence $\{u_n\}$ in A converges to x if there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n \to \infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{1.4}$$

where $z_{n+1} \in A$ satisfies the condition that $d(z_{n+1}, Su_n) = d(A, B)$.

Theorem 1.2 (see [37, Theorem 3.4]). *Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) *S is proximal contractions of first and second kinds.*

(b) *g is an isometry.*

(c) *S preserves isometric distance with respect to g.*

(d) $S(A_0) \subseteq B_0$.

(e) $A_0 \subseteq g(A_0)$.

Then, there exists a unique point $x \in A$ *such that*

$$d(gx, Sx) = d(A, B). \tag{1.5}$$

Moreover, for any fixed $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{1.6}$$

converges to the element x.

On the other hand, a sequence $\{u_n\}$ *in* A *converges to* x *if there is a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\longrightarrow\infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{1.7}$$

where $z_{n+1} \in A$ *satisfies the condition that* $d(z_{n+1}, Su_n) = d(A, B)$.

The aim of this paper is to introduce the new classes of proximal contractions which are more general than class of proximal contraction of first and second kinds, by giving the necessary condition to have best proximity points and we also give some illustrative examples of our main results. The results of this paper are extension and generalizations of main result of Sadiq Basha in [37] and some results in the literature.

2. Preliminaries

Given nonvoid subsets A and B of a metric space (X, d), we recall the following notations and notions that will be used in what follows:

$$\begin{aligned} d(A, B) &:= \inf\{d(x, y) : x \in A,\ \ y \in B\}, \\ A_0 &:= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned} \tag{2.1}$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty. Further, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B, respectively, provided A and B are closed subsets of a normed linear space such that $d(A, B) > 0$ (see [31]).

Definition 2.1 ([37, Definition 2.2]). A mapping $S : A \to B$ is said to be a *proximal contraction of the first kind* if there exists $\alpha \in [0, 1)$ such that

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(u, v) \leq \alpha d(x, y) \tag{2.2}$$

for all $u, v, x, y \in A$.

It is easy to see that a self-mapping that is a proximal contraction of the first kind is precisely a contraction. However, a non-self-proximal contraction is not necessarily a contraction.

Definition 2.2 (see [37, Definition 2.3]). A mapping $S : A \to B$ is said to be a *proximal contraction of the second kind* if there exists $\alpha \in [0, 1)$ such that

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(Su, Sv) \leq \alpha d(Sx, Sy) \tag{2.3}$$

for all $u, v, x, y \in A$.

Definition 2.3. Let $S : A \to B$ and $T : B \to A$. The pair (S, T) is said to be a *proximal cyclic contraction pair* if there exists a nonnegative number $\alpha < 1$ such that

$$d(a, Sx) = d(b, Ty) = d(A, B) \Longrightarrow d(a, b) \leq \alpha d(x, y) + (1 - \alpha) d(A, B) \tag{2.4}$$

for all $a, x \in A$ and $b, y \in B$.

Definition 2.4. Leting $S : A \to B$ and an isometry $g : A \to A$, the mapping S is said to preserve isometric distance with respect to g if

$$d(Sgx, Sgy) = d(Sx, Sy) \tag{2.5}$$

for all $x, y \in A$.

Definition 2.5. A point $x \in A$ is said to be a *best proximity point* of the mapping $S : A \to B$ if it satisfies the condition that

$$d(x, Sx) = d(A, B). \tag{2.6}$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.6. A is said to be *approximatively compact with respect to* B if every sequence $\{x_n\}$ in A satisfies the condition that $d(y, x_n) \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

We observe that every set is approximatively compact with respect to itself and that every compact set is approximatively compact. Moreover, A_0 and B_0 are nonempty set if A is compact and B is approximatively compact with respect to A.

3. Main Results

Definition 3.1. A mapping $S : A \to B$ is said to be *a generalized proximal ψ-contraction of the first kind*, if for all $u, v, x, y \in A$ satisfies

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(u, v) \leq \psi(d(x, y)), \tag{3.1}$$

where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$.

Definition 3.2. A mapping $S : A \to B$ is said to be *a generalized proximal ψ-contraction of the second kind*, if for all $u, v, x, y \in A$ satisfies

$$d(u, Sx) = d(v, Sy) = d(A, B) \Longrightarrow d(Su, Sv) \leq \psi(d(Sx, Sy)), \tag{3.2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous from the right such that $\psi(t) < t$ for all $t > 0$.

It is easy to see that if we take $\psi(t) = \alpha t$, where $\alpha \in [0, 1)$, then a generalized proximal ψ-contraction of the first kind and generalized proximal ψ-contraction of the second kind reduce to a proximal contraction of the first kind Definition 2.1 and a proximal contraction of the second kind Definition 2.2, respectively. Moreover, it is easy to see that a self-mapping generalized proximal ψ-contraction of the first kind and the second kind reduces to the condition of Boy and Wong' s fixed point theorem [3].

Next, we extend the result of Sadiq Basha [37] and the Banach's contraction principle to the case of non-self-mappings which satisfy generalized proximal ψ-contraction condition.

Theorem 3.3. *Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$, and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(a) *S and T are generalized proximal ψ-contraction of the first kind;*

(b) *g is an isometry;*

(c) *The pair (S, T) is a proximal cyclic contraction;*

(d) *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*

(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.3}$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.4}$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A, B), \tag{3.5}$$

converges to the element y.

On the other hand, a sequence $\{u_n\}$ in A converges to x if there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n \to \infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{3.6}$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.

Proof. Let x_0 be a fixed element in A_0. In view of the fact that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it is ascertained that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A, B). \tag{3.7}$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A, B). \tag{3.8}$$

By similar fashion, we can find x_n in A_0. Having chosen x_n, one can determine an element $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B). \tag{3.9}$$

Because of the facts that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, by a generalized proximal ψ-contraction of the first kind of S, g is an isometry and property of ψ, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(gx_{n+1}, gx_n) \\ &\leq \psi(d(x_n, x_{n-1})) \\ &\leq d(x_n, x_{n-1}). \end{aligned} \tag{3.10}$$

This means that the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{3.11}$$

If $r > 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} d(x_{n+1}, x_n) \\ &\leq \lim_{n\to\infty} \psi(d(x_n, x_{n-1})) \\ &= \psi(r) \\ &< r, \end{aligned} \tag{3.12}$$

which is a contradiction unless $r = 0$. Therefore,

$$\alpha_n := \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.13}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad d(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{3.14}$$

for $k \in \{1,2,3,\ldots\}$. Thus

$$\begin{aligned} \varepsilon \le r_k &\le d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1}. \end{aligned} \tag{3.15}$$

It follows from (3.13) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.16}$$

On the other hand, by constructing the sequence $\{x_n\}$, we have

$$d(gx_{m_k+1}, Sx_{m_k}) = d(A,B), \qquad d(gx_{n_k+1}, Sx_{n_k}) = d(A,B). \tag{3.17}$$

Sine S is a generalized proximal ψ-contraction of the first kind and g is an isometry, we have

$$d(x_{m_k+1}, x_{n_k+1}) = d(gx_{m_k+1}, gx_{n_k+1}) \le \psi(d(x_{m_k}, x_{n_k})). \tag{3.18}$$

Notice also that

$$\begin{aligned} \varepsilon \le r_k &\le d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &\le \alpha_{m_k} + \alpha_{n_k} + \psi(d(x_{m_k}, x_{n_k})). \end{aligned} \tag{3.19}$$

Taking $k \to \infty$ in above inequality, by (3.13), (3.16), and property of ψ, we get $\varepsilon \le \psi(\varepsilon)$. Therefore, $\varepsilon = 0$, which is a contradiction. So we obtain the claim and hence converge to some element $x \in A$. Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that there is a sequence $\{y_n\}$ such that $d(gy_{n+1}, Sy_n) = d(A,B)$ and converge to some element $y \in B$. Since the pair (S,T) is a proximal cyclic contraction and g is an isometry, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) \le \alpha d(x_n, y_n) + (1-\alpha) d(A,B). \tag{3.20}$$

We take limit in (3.20) as $n \to \infty$; it follows that

$$d(x,y) = d(A,B), \tag{3.21}$$

so, we concluded that $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there is $u \in A$ and $v \in B$ such that

$$d(u, Sx) = d(A,B) \tag{3.22}$$

$$d(v, Ty) = d(A,B). \tag{3.23}$$

From (3.9), (3.22), and the notion of generalized proximal ψ-contraction of first kind of S, we get

$$d(u, gx_{n+1}) \leq \psi(d(x, x_n)). \tag{3.24}$$

Letting $n \to \infty$, we get $d(u, gx) \leq \psi(0) = 0$ and thus $u = gx$. Therefore

$$d(gx, Sx) = d(A, B). \tag{3.25}$$

Similarly, we can show that $v = gy$ and then

$$d(gy, Ty) = d(\mathrm{A}, B). \tag{3.26}$$

From (3.21), (3.25), and (3.26), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B). \tag{3.27}$$

Next, to prove the uniqueness, let us suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*, y \neq y^*$ such that

$$\begin{aligned} d(gx^*, Sx^*) &= d(A, B), \\ d(gy^*, Ty^*) &= d(A, B). \end{aligned} \tag{3.28}$$

Since g is an isometry, S and T are generalized proximal ψ-contractions of the first kind and the property of ψ; it follows that

$$\begin{aligned} d(x, x^*) &= d(gx, gx^*) \leq \psi(d(x, x^*)) < d(x, x^*), \\ d(y, y^*) &= d(gy, gy^*) \leq \psi(d(y, y^*)) < d(y, y^*), \end{aligned} \tag{3.29}$$

which is a contradiction, so we have $x = x^*$ and $y = y^*$. On the other hand, let $\{u_n\}$ be a sequence in A and let $\{\epsilon_n\}$ be a sequence of positive real numbers such that

$$\lim_{n \to \infty} \epsilon_n = 0, \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{3.30}$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$. Since S is a generalized proximal ψ-contraction of first kind and g is an isometry, we have

$$d(x_{n+1}, z_{n+1}) \leq \psi(d(x_n, u_n)). \tag{3.31}$$

Given $\epsilon > 0$, we choose a positive integer N such that $\epsilon_n \leq \epsilon$ for all $n \geq N$; we obtain that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\leq d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\leq \psi(d(x_n, u_n)) + \epsilon_n. \end{aligned} \tag{3.32}$$

Therefore, we get

$$\begin{aligned} d(u_{n+1}, x) &\leq d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x) \\ &\leq \psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x). \end{aligned} \tag{3.33}$$

We claim that $d(u_n, x) \rightarrow 0$ as $n \rightarrow \infty$; supposing the contrary, by inequality (3.33) and property of ψ, we get

$$\begin{aligned} \lim_{n\to\infty} d(u_{n+1}, x) &\leq \lim_{n\to\infty} (d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x)) \\ &\leq \lim_{n\to\infty} \left(\psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x)\right) \\ &= \psi\left(\lim_{n\to\infty} d(x_n, u_n)\right) \\ &< \lim_{n\to\infty} d(x_n, u_n) \\ &\leq \lim_{n\to\infty} (d(x_n, x) + d(x, u_n)) \\ &= \lim_{n\to\infty} d(x, u_n), \end{aligned} \tag{3.34}$$

which is a contradiction, so we have $\{u_n\}$ is convergent and it converges to x. This completes the proof of the theorem. □

If g is assumed to be the identity mapping, then by Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \rightarrow B, T : B \rightarrow A$ and $g : A \cup B \rightarrow A \cup B$ satisfy the following conditions:*

(a) *S and T are generalized proximal ψ-contraction of the first kind;*

(b) *$S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$;*

(c) *the pair (S, T) is a proximal cyclic contraction.*

Then, there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.35}$$

If we take $\psi(t) = \alpha t$, where $0 \leq \alpha < 1$, we obtain following corollary.

Corollary 3.5 (see [37, Theorem 3.1]). *Let (X, d) be a complete metric space and A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $S : A \rightarrow B$, $T : B \rightarrow A$ and $g : A \cup B \rightarrow A \cup B$ satisfy the following conditions:*

(a) *S and T are proximal contractions of first kind;*

(b) *g is an isometry;*

(c) *the pair* (S, T) *is a proximal cyclic contraction;*

(d) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$*;*

(e) $A_0 \subseteq g(A_0)$ *and* $B_0 \subseteq g(B_0)$*.*

Then, there exists a unique point $x \in A$ *and there exists a unique point* $y \in B$ *such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.36}$$

Moreover, for any fixed $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.37}$$

converges to the element x*. For any fixed* $y_0 \in B_0$*, the sequence* $\{y_n\}$*, defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B), \tag{3.38}$$

converges to the element y*.*

If g is assumed to be the identity mapping in Corollary 3.5, we obtain the following corollary.

Corollary 3.6. *Let* (X, d) *be a complete metric space and let* A *and* B *be nonempty, closed subsets of* X*. Further, suppose that* A_0 *and* B_0 *are nonempty. Let* $S : A \to B$*,* $T : B \to A$*, and* $g : A \cup B \to A \cup B$ *satisfy the following conditions:*

(a) S *and* T *are proximal contractions of first kind;*

(b) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$*;*

(c) *the pair* (S, T) *is a proximal cyclic contraction.*

Then, there exists a unique point $x \in A$ *and there exists a unique point* $y \in B$ *such that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B). \tag{3.39}$$

For a self-mapping, Theorem 3.3 includes the Boy and Wong' s fixed point theorem [3] as follows.

Corollary 3.7. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a mapping that satisfies* $d(Tx, Ty) \leq \psi(d(x, y))$ *for all* $x, y \in X$*, where* $\psi : [0, \infty) \to [0, \infty)$ *is an upper semicontinuous function from the right such that* $\psi(t) < t$ *for all* $t > 0$*. Then* T *has a unique fixed point* $v \in X$*. Moreover, for each* $x \in X$*,* $\{T^n x\}$ *converges to* v*.*

Next, we give an example to show that Definition 3.1 is different form Definition 2.1; moreover we give an example which supports Theorem 3.3.

Example 3.8. Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|, \tag{3.40}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Let

$$A = \{(0, y) : 0 \leq y \leq 1\}, \quad B = \{(1, y) : 0 \leq y \leq 1\}. \tag{3.41}$$

Then $d(A, B) = 1$. Define the mappings $S : A \to B$ as follows:

$$S((0, y)) = \left(1, y - \frac{y^2}{2}\right). \tag{3.42}$$

First, we show that S is generalized proximal ψ-contraction of the first kind with the function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} t - \dfrac{t^2}{2}, & 0 \leq t \leq 1, \\ t - 1, & t > 1. \end{cases} \tag{3.43}$$

Let $(0, x_1), (0, x_2), (0, a_1)$ and $(0, a_2)$ be elements in A satisfying

$$d((0, x_1), S(0, a_1)) = d(A, B) = 1, \quad d((0, x_2), S(0, a_2)) = d(A, B) = 1. \tag{3.44}$$

It follows that

$$x_i = a_i - \frac{a_i^2}{2} \quad \text{for } i = 1, 2. \tag{3.45}$$

Without loss of generality, we may assume that $a_1 - a_2 > 0$, so we have

$$\begin{aligned} d((0, x_1), (0, x_2)) &= d\left(\left(0, a_1 - \frac{a_1^2}{2}\right), \left(0, a_2 - \frac{a_2^2}{2}\right)\right) \\ &= \left|\left(a_1 - \frac{a_1^2}{2}\right) - \left(a_2 - \frac{a_2^2}{2}\right)\right| \\ &= (a_1 - a_2) - \left(\frac{a_1^2}{2} - \frac{a_2^2}{2}\right) \\ &\leq (a_1 - a_2) - \frac{1}{2}(a_1 - a_2)^2 \\ &= \psi(d((0, a_1), (0, a_2))). \end{aligned} \tag{3.46}$$

Thus S is a generalized proximal ψ-contraction of the first kind.

Next, we prove that S is not a proximal contraction. Suppose S is proximal contraction then for each $(0, x), (0, y), (0, a), (0, b) \in A$ satisfying

$$d((0, x), S(0, a)) = d(A, B) = 1, \qquad d((0, y), S(0, b)) = d(A, B) = 1, \tag{3.47}$$

there exists $k \in [0,1)$ such that

$$d\big((0,x),(0,y)\big) \leq kd((0,a),(0,b)). \tag{3.48}$$

From (3.47), we get

$$x = a - \frac{a^2}{2}, \qquad y = b - \frac{b^2}{2}, \tag{3.49}$$

and thus

$$\begin{aligned} \left|\left(a - \frac{a^2}{2}\right) - \left(b - \frac{b^2}{2}\right)\right| &= d\big((0,x),(0,y)\big) \\ &\leq kd((0,a),(0,b)) \\ &= k|a-b|. \end{aligned} \tag{3.50}$$

Letting $b = 0$ with $a \neq 0$, we get

$$1 = \lim_{a \to 0^+}\left(1 - \frac{a}{2}\right) \leq k < 1, \tag{3.51}$$

which is a contradiction. Therefore S is not a proximal contraction and Definition 3.1 is different form Definition 2.1.

Example 3.9. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$\begin{aligned} A &= \{(0,y) : y \in \mathbb{R}\}, \\ B &= \{(1,y) : y \in \mathbb{R}\}. \end{aligned} \tag{3.52}$$

Define two mappings $S : A \to B, T : B \to A$ and $g : A \cup B \to A \cup B$ as follows:

$$S\big((0,y)\big) = \left(1, \frac{y}{4}\right), \quad T\big((1,y)\big) = \left(0, \frac{y}{4}\right), \quad g\big((x,y)\big) = (x,-y). \tag{3.53}$$

Then it is easy to see that $d(A,B) = 1$, $A_0 = A$, $B_0 = B$ and the mapping g is an isometry.

Next, we claim that S and T are generalized proximal ψ-contractions of the first kind. Consider a function $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = t/2$ for all $t \geq 0$. If $(0,y_1),(0,y_2) \in A$ such that

$$d\big(a, S(0,y_1)\big) = d(A,B) = 1, \qquad d\big(b, S(0,y_2)\big) = d(A,B) = 1 \tag{3.54}$$

for all $a, b \in A$, then we have

$$a = \left(0, \frac{y_1}{4}\right), \qquad b = \left(0, \frac{y_2}{4}\right). \tag{3.55}$$

Because,

$$\begin{aligned}
d(a,b) &= d\left(\left(0,\frac{y_1}{4}\right),\left(0,\frac{y_2}{4}\right)\right)\\
&= \left|\frac{y_1}{4}-\frac{y_2}{4}\right|\\
&= \frac{1}{4}|y_1-y_2|\\
&\le \frac{1}{2}|y_1-y_2|\\
&= \frac{1}{2}d((0,y_1),(0,y_2))\\
&= \psi(d((0,y_1),(0,y_2))).
\end{aligned} \tag{3.56}$$

Hence S is a generalized proximal ψ-contraction of the first kind. If $(1,y_1),(1,y_2) \in B$ such that

$$d(a,T(1,y_1)) = d(A,B) = 1, \qquad d(b,T(1,y_2)) = d(A,B) = 1 \tag{3.57}$$

for all $a,b \in B$, then we get

$$a = \left(1,\frac{y_1}{4}\right), \qquad b = \left(1,\frac{y_2}{4}\right). \tag{3.58}$$

In the same way, we can see that T is a generalized proximal ψ-contraction of the first kind. Moreover, the pair (S,T) forms a proximal cyclic contraction and other hypotheses of Theorem 3.3 are also satisfied. Further, it is easy to see that the unique element $(0,0) \in A$ and $(1,0) \in B$ such that

$$d(g(0,0),S(0,0)) = d(g(1,0),T(1,0)) = d((0,0),(1,0)) = d(A,B). \tag{3.59}$$

Next, we establish a best proximity point theorem for non-self-mappings which are generalized proximal ψ-contractions of the first kind and the second kind.

Theorem 3.10. *Let (X,d) be a complete metric space and let A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *S is a generalized proximal ψ-contraction of first and second kinds;*

(b) *g is an isometry;*

(c) *S preserves isometric distance with respect to g;*

(d) *$S(A_0) \subseteq B_0$;*

(e) *$A_0 \subseteq g(A_0)$.*

Then, there exists a unique point $x \in A$ such that

$$d(gx, Sx) = d(A, B). \tag{3.60}$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.61}$$

converges to the element x.

On the other hand, a sequence $\{u_n\}$ in A converges to x if there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0, \qquad d(u_{n+1}, z_{n+1}) \leq \epsilon_n, \tag{3.62}$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.

Proof. Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, similarly in the proof of Theorem 3.3, we can construct the sequence $\{x_n\}$ of element in A_0 such that

$$d(gx_{n+1}, Sx_n) = d(A, B) \tag{3.63}$$

for nonnegative number n. It follows from g that is an isometry and the virtue of a generalized proximal ψ-contraction of the first kind of S; we see that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \psi(d(x_n, x_{n-1})) \tag{3.64}$$

for all $n \in \mathbb{N}$. Similarly to the proof of Theorem 3.3, we can conclude that the sequence $\{x_n\}$ is a Cauchy sequence and converges to some $x \in A$. Since S is a generalized proximal ψ-contraction of the second kind and preserves isometric distance with respect to g that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sgx_n, Sgx_{n+1}) \\ &\leq \psi(d(Sx_{n-1}, Sx_n)) \\ &\leq d(Sx_{n-1}, Sx_n), \end{aligned} \tag{3.65}$$

this means that the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is nonincreasing and bounded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r. \tag{3.66}$$

If $r > 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} d(Sx_{n+1}, Sx_n) \\ &\le \lim_{n\to\infty} \psi(d(Sx_{n-1}, Sx_n)) \\ &= \psi(r) \\ &< r, \end{aligned} \tag{3.67}$$

which is a contradiction, unless $r = 0$. Therefore

$$\beta_n := \lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{3.68}$$

We claim that $\{Sx_n\}$ is a Cauchy sequence. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ such that $n_k > m_k \ge k$ with

$$r_k := d(Sx_{m_k}, Sx_{n_k}) \ge \varepsilon, \qquad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon \tag{3.69}$$

for $k \in \{1, 2, 3, \ldots\}$. Thus

$$\begin{aligned} \varepsilon \le r_k &\le d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + \beta_{n_k-1}, \end{aligned} \tag{3.70}$$

it follows from (3.68) that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.71}$$

Notice also that

$$\begin{aligned} \varepsilon \le r_k &\le d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &= \beta_{m_k} + \beta_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\le \beta_{m_k} + \beta_{n_k} + \psi(d(Sx_{m_k}, Sx_{n_k})). \end{aligned} \tag{3.72}$$

Taking $k \to \infty$ in previous inequality, by (3.68), (3.71), and property of ψ, we get $\varepsilon \le \psi(\varepsilon)$. Hence, $\varepsilon = 0$, which is a contradiction. So we obtain the claim and then it converges to some $y \in B$. Therefore, we can conclude that

$$d(gx, y) = \lim_{n\to\infty} d(gx_{n+1}, Sx_n) = d(A, B). \tag{3.73}$$

That is $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in A_0$ and then $d(gx, gz) = 0$. By the fact that g is an isometry, we have $d(x, z) = d(gx, gz) = 0$. Hence $x = z$ and so x becomes to a point in A_0. As $S(A_0) \subseteq B_0$ that

$$d(u, Sx) = d(A, B) \tag{3.74}$$

for some $u \in A$. It follows from (3.63) and (3.74) that S is a generalized proximal ψ-contraction of the first kind that

$$d(u, gx_{n+1}) \leq \psi(d(x, x_n)) \tag{3.75}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$, we get the sequence $\{gx_n\}$ converging to a point u. By the fact that g is continuous, we have

$$gx_n \longrightarrow gx \quad \text{as } n \longrightarrow \infty. \tag{3.76}$$

By the uniqueness of limit of the sequence, we conclude that $u = gx$. Therefore, it results that $d(gx, Sx) = d(u, Sx) = d(A, B)$. The uniqueness and the remaining part of the proof follow as in Theorem 3.3. This completes the proof of the theorem. □

If g is assumed to be the identity mapping, then by Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $S : A \to B$ satisfy the following conditions:*

(a) *S is a generalized proximal ψ-contraction of first and second kinds;*

(b) *$S(A_0) \subseteq B_0$.*

Then, there exists a unique point $x \in A$ such that

$$d(x, Sx) = d(A, B). \tag{3.77}$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Sx_n) = d(A, B), \tag{3.78}$$

converges to the best proximity point x of S.

If we take $\psi(t) = \alpha t$, where $0 \leq \alpha < 1$ in Theorem 3.10, we obtain following corollary.

Corollary 3.12 (see [37, Theorem 3.4]). *Let (X, d) be a complete metric space and let A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *S is a proximal contraction of first and second kinds;*

(b) *g is an isometry;*

(c) *S preserves isometric distance with respect to g;*

(d) $S(A_0) \subseteq B_0$;

(e) $A_0 \subseteq g(A_0)$.

Then, there exists a unique point $x \in A$ such that

$$d(gx, Sx) = d(A, B). \tag{3.79}$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.80}$$

converges to the element x.

If g is assumed to be the identity mapping in Corollary 3.12, we obtain the following corollary.

Corollary 3.13. *Let (X, d) be a complete metric space and let A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $S : A \to B$ satisfy the following conditions:*

(a) *S is a proximal contraction of first and second kinds;*

(b) $S(A_0) \subseteq B_0$.

Then, there exists a unique point $x \in A$ such that

$$d(x, Sx) = d(A, B). \tag{3.81}$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Sx_n) = d(A, B), \tag{3.82}$$

converges to the best proximity point x of S.

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