Research Article

Bifurcation of a Cohen-Grossberg Neural Network with Discrete Delays

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A simple Cohen-Grossberg neural network with discrete delays is investigated in this paper. The existence of local Hopf bifurcations is first considered by choosing the appropriate bifurcation parameter, and then explicit formulas are given to determine the direction of Hopf bifurcation and stability of the periodic solutions. Moreover, a set of sufficient conditions are given to guarantee the global Hopf bifurcation. Numerical simulations are given to illustrate the obtained results.

1. Introduction

In 1983, Cohen and Grossberg [1] proposed a kind of neural networks, which are now called Cohen-Grossberg neural networks. The networks have been successfully applied to signal processing, pattern recognition, optimization, and associative memories.

It is well known that the analysis of the dynamical behaviors is a necessary step for practical design of neural networks since their applications heavily depend on the dynamical behaviors. The research on dynamical behaviors of neural networks involves not only the dynamic analysis of equilibrium but also that of periodic solution, bifurcation, and chaos, especially, the periodic oscillatory behavior of the neural networks is of great interest in many applications [2, 3]. Since periodic oscillatory can arise through the Hopf bifurcation in different system with or without time delays, it is very important to discuss the Hopf bifurcation of neural networks. Up to now, to the best of the author's knowledge, bifurcation of Hopfield neural networks has been discussed by many researchers [4–12], but only a few results on the bifurcation of Cohen-Grossberg neural networks have been obtained. Zhao discussed the bifurcation of a discrete-time Cohen-Grossberg neural network in [13]

and the bifurcation of a continuous-time Cohen-Grossberg neural network with distributed delays in which kernel function is $\alpha e^{-\alpha s}$ in [14]; Liu discussed the local Hopf bifurcation of the following Cohen-Grossberg neural network with discrete delays by regarding time delay as the bifurcation parameter in [15]:

$$
\begin{aligned}
\dot{x}_1(t) &= -a_1(x_1(t))\left[b_1(x_1(t)) - d_1 f_2(x_2(t - \tau_2))\right], \\
\dot{x}_2(t) &= -a_2(x_2(t))\left[b_2(x_2(t)) - d_2 f_1(x_1(t - \tau_1))\right],
\end{aligned}
\tag{1.1}
$$

where $x_i(t)$ denotes the state variable associated with the ith neuron; $a_i(\cdot)$ represents amplification function; $f_i(\cdot)$ denotes the signal function of the ith neuron at time t; $b_i(\cdot)$ is appropriately behaved function; $d_i(\cdot)$ is connection weight of the neural network; discrete delay τ_i corresponds to the finite speed of the axonal signal transmission at time t, $i = 1,\ 2$.

On the other hand, realistic modelling of neural networks inevitably depend on careful design and variation of the connection weight; in [16], Liu discussed the Neimark-Sacker bifurcation of a discrete-time version of the neural network (1.1) by regarding the connection weight as the bifurcation parameter. The objective of this paper is to discuss Hopf bifurcation of continuous-time system (1.1) by regarding the connection weight as the bifurcation parameter while time delays are fixed. The rest of this paper is organized as follows. Stability property, existence of Hopf bifurcation, and the stability of bifurcating periodic solutions for system (1.1) are obtained in Section 2. Global existence of Hopf bifurcation for system (1.1) is obtained in Section 3. An example is given in Section 4 to demonstrate the main results.

2. Stability Analysis and Existence of Local Hopf Bifurcation

Throughout this paper, we assume that

(H_1) $b_1(0) = b_2(0) = 0$, $f_1(0) = f_2(0) = 0$;

(H_2) there exist constants $\underline{a}_i$, $\overline{a}_i$ such that $0 < \underline{a}_i \le a_i(\cdot) \le \overline{a}_i$ for $i = 1, 2$;

(H_3) there exist constants $\underline{b}_i$ such that $b_i'(\cdot) \ge \underline{b}_i > 0$ for $i = 1, 2$.

Let $u_1(t) = x_1(t - \tau_1)$ and $u_2(t) = x_2(t)$; we transform system (1.1) into the following system

$$
\begin{aligned}
\dot{u}_1(t) &= -a_1(u_1(t))\left[b_1(u_1(t)) - d_1 f_2(u_2(t - \tau))\right], \\
\dot{u}_2(t) &= -a_2(u_2(t))\left[b_2(u_2(t)) - d_2 f_1(u_1(t))\right],
\end{aligned}
\tag{2.1}
$$

in which $\tau = \tau_1 + \tau_2$.

Obviously, system (2.1) has an equilibrium $(0,0)$ under conditions (H_1) and (H_2). The linearized system of system (2.1) at $(0,0)$ is

$$
\begin{aligned}
\dot{u}_1(t) &= -a_1(0)b_1'(0)u_1(t) + a_1(0)d_1 f_2'(0)u_2(t - \tau), \\
\dot{u}_2(t) &= -a_2(0)b_2'(0)u_2(t) + a_2(0)d_2 f_1'(0)u_1(t).
\end{aligned}
\tag{2.2}
$$

The associated characteristic equation of system (2.2) is

$$
\lambda^2 + a\lambda + b\left(c - de^{-\lambda\tau}\right) = 0,
\tag{2.3}
$$

in which

$$a = a_1(0)b_1'(0) + a_2(0)b_2'(0), \qquad b = a_1(0)a_2(0), \qquad c = b_1'(0)b_2'(0), \qquad d = d_1d_2f_1'(0)f_2'(0). \tag{2.4}$$

If $\tau = 0$, it is easy to know that (2.3) has two roots with negative real parts when $d < c$. Equation (2.10) has at least one positive root when $d > c$.

If $\tau \neq 0$, let us assume that $\lambda = i\omega$ $(\omega > 0)$ is a root of (2.3); we have

$$bc - \omega^2 = bd \cos \omega\tau, \qquad a\omega = -bd \sin \omega\tau. \tag{2.5}$$

Adding up the squares of the corresponding sides of the above equation (2.5), we obtain

$$\omega^4 + k\omega^2 + b^2\left(c^2 - d^2\right) = 0, \tag{2.6}$$

where $k = (a_1(0)b_1'(0))^2 + (a_2(0)b_2'(0))^2$.

Denote $z = \omega^2$; then (2.6) becomes

$$z^2 + kz + b^2\left(c^2 - d^2\right) = 0. \tag{2.7}$$

Since $k > 0$, (2.7) has no positive real roots when $|d| \leq c$. Equation (2.7) has a positive real roots when $|d| > c$, which means (2.3) has a pair of purely imaginary root for every $|d| > c$.

We have from (2.5) that

$$\frac{a\omega}{\omega^2 - bc} = \tan \omega\tau. \tag{2.8}$$

It has roots as follows [4, 12]:

$$\omega_j^+ \in \left[\frac{2(j-1)\pi}{\tau}, \frac{(2j-1)\pi}{\tau}\right], \qquad \omega_j^- \in \left[\frac{(2j-1)\pi}{\tau}, \frac{(4j-1)\pi}{2\tau}\right], \quad j = 1, 2, \ldots. \tag{2.9}$$

From (2.5) and the discussion of (2.7), we obtain that

$$d_j^- = -\frac{a\omega_j^+}{b \sin \omega_j^+\tau} < -b_1'(0)b_2'(0) < 0, \qquad d_j^+ = -\frac{a\omega_j^-}{b \sin \omega_j^-\tau} > b_1'(0)b_2'(0) > 0, \tag{2.10}$$

that is, (2.3) has a pair of purely imaginary roots $\pm i\omega_0$ when $d = d_j^\pm$.

On the other hand, we have

$$\lambda'(d)^{-1} = \frac{2\lambda + a}{be^{-\lambda\tau}} + \tau d = \left(\frac{2\lambda + a}{\lambda^2 + a\lambda + bc} + \tau\right)d, \tag{2.11}$$

thus we have

$$\mathrm{Re}\left(\lambda'(d)^{-1}\right)\Big|_{d=d_j^\pm} = \left(\frac{abc + a\left(\omega_j^\pm\right)^2}{\left(bc - \omega_j^\pm\right)^2 + a^2\left(\omega_j^\pm\right)^2} + \tau\right)d_j^\pm; \qquad (2.12)$$

which means

$$\mathrm{Re}\left(\lambda'(d)^{-1}\right)\Big|_{d=d_j^+} > 0, \qquad \mathrm{Re}\left(\lambda'(d)^{-1}\right)\Big|_{d=d_j^-} < 0, \qquad (2.13)$$

and note that at $d = b_1(0)b_2(0) = c$, (2.3) has a root $\lambda(c) = 0$, and $\lambda'(c)^{-1} = (a/b + \tau)c > 0$ according to (2.11).

From the above discusses and Hopf bifurcation theorem in [17] for functional differential equations, we have the following results.

Theorem 2.1. *Under assumptions (H_1)–(H_3), we have*

(1) *if $d_1 d_2 f_1'(0) f_2'(0) \in (d^-, b_1'(0)b_2'(0))$, the equilibrium $(0,0)$ of system (1.1) is asymptotically stable;*

(2) *if $d_1 d_2 f_1'(0) f_2'(0) \notin [d^-, b_1'(0)b_2'(0)]$, the equilibrium $(0,0)$ of system (1.1) is unstable;*

(3) *$d_j^\pm (j = 0,1,2,\ldots)$ are Hopf bifurcation values for system (1.1),*

where $d_j^\pm$ is given by (2.10) and $d^- = \max_{j\geq 1}\{d_j^-\}$.

From discussion above, we obtain some conditions under which system (1.1) undergoes local Hopf bifurcation near $d = d_j^\pm (j = 1,2,\ldots)$. In addition, we can obtain explicit formulas for determining the properties of the Hopf bifurcation at critical $d = d_j^\pm$.

Based on the normal form theory and the center manifold theorem [18], similar to discussion in [15], we can compute the following quantities:

$$\begin{aligned}
C_1(0) &= \frac{i}{2\omega_j^\mp \tau}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{\left|g_{20}\right|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\left\{\lambda'\left(d_j^\pm\right)\right\}}, \\
\beta_2 &= 2\,\mathrm{Re}\{C_1(0)\}, \\
T_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2\,\mathrm{Im}\left\{\lambda'\left(d_j^\pm\right)\right\}}{\tau\omega_j^\mp},
\end{aligned} \qquad (2.14)$$

where the explicit formulas for g_{ij} in (2.14) can be obtained when we replace τ^j and ω_0 in g_{ij} by τ and $\omega_j^\mp$ in [15], respectively.

It is well known that μ_2 determines the direction of the Hopf bifurcation and β_2 determines the stability of the bifurcating periodic solutions [18]. Since $\mathrm{Re}\{\lambda'(d_j^+)\} > 0$,

we know if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical(subcritical), the bifurcating periodic solutions exist for $d > d_j^+$ ($d < d_j^+$), and the bifurcating periodic solutions are stable(unstable), and due to $\mathrm{Re}\{\lambda'(d_j^-)\} < 0$, we know if $\mu_2 < 0$ ($\mu_2 > 0$), then the Hopf bifurcation is supercritical(subcritical), the bifurcating periodic solutions exist for $d < d_j^-$ ($d > d_j^-$), and the bifurcating periodic solutions are stable(unstable). T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

Remark 2.2. The main results above coincide with the main results in [16].

3. Global Existence of Hopf Bifurcation for the Model

In this section, we mainly prove that the local Hopf bifurcation of (1.1) can be extended for large values by applying the global Hopf bifurcation theorem in [19].

(H$_4$) There exist constants $M_i > 0$ such that $|f_i(\cdot)| \leq M_i$ and $f'(\cdot) \geq 0$, $u_i f''(u_i) < 0$ for $i = 1, 2$.

Remark 3.1. Assumption (H$_4$) is suitable, for example, $f_i(u_i) = \tanh u_i$ and $f_i(u_i) = (1 - e^{-u_i})/(1 + e^{-u_i})$ satisfy (H$_4$).

Theorem 3.2. *Under assumptions (H$_1$)–(H$_4$) and either $(d_1, f_1(0), f_2(0))$ or $(d_2, f_1(0), f_2(0))$ is fixed, system (1.1) has at least a periodic solution for $d < d_j^-$.*

Proof. First, we can prove that system (1.1) has a unique equilibrium $(0, 0)$ under conditions (H$_1$)–(H$_4$) when $d < d^-$.

The equilibria of system (1.1) satisfy the following equation according to condition (H$_2$):

$$
\begin{aligned}
b_1(u_1) - d_1 f_2(u_2) &= 0, \\
b_2(u_2) - d_2 f_1(u_1) &= 0.
\end{aligned}
\tag{3.1}
$$

From the first equation of (3.1), we have $u_1 = b_1^-(d_1 f_2(u_2))$; substitute it into the second equation of (3.1); we obtain

$$
b_2(u_2) - d_2 f_1(b_1^-(d_1 f_2(u_2))) = 0.
\tag{3.2}
$$

Denote

$$
g(u_2) = b_2(u_2) - d_2 f_1(b_1^-(d_1 f_2(u_2))),
\tag{3.3}
$$

and we have

$$
g'(u_2) = b_2'(u_2)\left[1 - d_1 d_2 \frac{f_1'(b_1^-(d_1 f_2(u_2))) f_2'(u_2)}{b_2'(u_2) b_1'(d_1 f_2(u_2))}\right].
\tag{3.4}
$$

When $d_1 d_2 < 0$, obviously, $g'(u_2) > 0$ due to $f_i'(\cdot) > 0$ according to (H$_4$). When $d_1 d_2 > 0$, we obtain that $f_i'(0) = \max_{u_i \in R} f_i'(u_i)$, $i = 1, 2$, due to $u_i f_i''(u_i) < 0$ according to (H$_4$), combined

with $d = d_1 d_2 f_1'(0) f_2'(0) < d^- < 0$ and condition (H$_3$); we obtain $d_1 d_2 f_1'(b_1^-(d_1 f_2(u_2))) f_2'(u_2) < d_1^- < 0$; consequently, $g'(u_2) > 0$. Hence $g(u_2)$ is an increasing bijective function, since $g_2(0) = 0$, $u_2 = 0$ is a unique solution of (3.3), and (3.1) has a unique solution $(0,0)$, that is, $(0,0)$ is a unique equilibrium of system (1.1) when $d < d^-$.

Second, let us prove the global existence of Hopf bifurcation of system (1.1). Following the work of Wu [19], we make the following definitions.

Denote the equivalent system (2.1) of system (1.1) as follows:

$$\dot{u} = F(u_t(\theta), \tau) \tag{3.5}$$

in which $u = (u_1, u_2)$, $u_t(\theta) = u(t + \theta) \in C([-\tau, 0], R^2)$.

And denote

$$X = C\left([-\tau, 0], R^2\right), \qquad \Sigma = \mathrm{cl}\{(u(t), d, p) \in X \times R \times R^+\}, \tag{3.6}$$

where $u(t)$ is a p-periodic solution of (2.1). Let $\ell(0, d_j^-, 2\pi/\omega_j^+)$ denote the connected component through $(0, d_j^-, 2\pi/\omega_j^+)$ in Σ, where d_j^- and ω_j^+ are given by (2.9) and (2.10), respectively. From Theorem 2.1, we know that $(0, d_j^-, 2\pi/\omega_j^+)$ is nonempty, and all centers are isolated center. At fact, sets of centers are

$$\left\{\left(0, d_j^-, \frac{2\pi}{\omega_j^+}\right), \ j = 1, 2, \dots\right\}. \tag{3.7}$$

The characteristic function

$$\Delta(0, d, p)(\lambda) = \lambda^2 + a\lambda + b\left(c - de^{-\lambda\tau}\right) \tag{3.8}$$

is continuous in $(d, p, \lambda) \in R \times R \times R \times C$. Theorem 2.1 above ensures that there exist $\varepsilon > 0$, $\epsilon > 0$ and a smooth curve $\lambda : (d_j^- - \delta, d_j^- + \delta) \to C$ such that $\Delta(\lambda(d)) = 0$, $|\lambda(d) - i\omega_j^+| < \epsilon$ for all $d \in [d_j^- - \delta, d_j^- + \delta]$, and $\lambda(d_j^-) = i\omega_j^+$, $\mathrm{Re}(\lambda'(d)^{-1})|_{d=d_j^-} < 0$. Define $\Omega_\epsilon = \{(u, p) : 0 < u < \epsilon, \ |p - 2\pi/\omega_j^+| < \varepsilon$. It is not difficult to show that $\Delta(0, d, p)(u + i2\pi/p) = 0$ on $[d_j^- - \delta, d_j^- + \delta] \times \partial\Omega_\epsilon$ if and only if $d = d_j^-, u = 0, p = 2\pi/\omega_j^+$.

Moreover, if we define

$$H^\pm\left(0, d_j^-, \frac{2\pi}{\omega_j^+}\right)(u, p) = \Delta\left(0, d_j^- \pm \delta, p\right)\left(u + i\frac{2\pi}{p}\right), \tag{3.9}$$

and we can compute the crossing number of every isolated center $(0, d_j^-, 2\pi/(\omega_j^+))$ as follows:

$$\gamma\left(0, d_j^-, \frac{2\pi}{\omega_j^+}\right) = \deg_B\left(H^-\left(0, d_j^-, \frac{2\pi}{\omega_j^+}\right), \Omega_\varepsilon\right) - \deg_B\left(H^+\left(0, d_j^-, \frac{2\pi}{\omega_j^+}\right), \Omega_\varepsilon\right) = -1, \tag{3.10}$$

hence we can obtain from Theorem 3.3 in [19] that the connected component $\ell(0, d_j^-, 2\pi/(\omega_j^+))$ in Σ is unbounded.

On the other hand, we can prove that all nontrivial periodic solutions of system (2.1) are uniformly bounded.

Let $V(t) = \sqrt{u_1^2 + u_2^2}$, we have

$$
\begin{aligned}
\dot{V}(t) &= \frac{1}{V(t)} \big[-a_1(u_1(t))b_1(u_1(t))u_1(t) + d_1 f_2(u_2(t-\tau))u_1(t) \\
&\qquad\quad -a_2(u_2(t))b_2(u_2(t))u_2(t) + d_2 f_1(u_1(t)u_2(t)) \big] \\
&\leq \frac{1}{V(t)} \big[-\underline{a}_1 b_1 u_1(t) + |d_1| M_2 u_1(t) - \underline{a}_2 b_2 u_2(t) + |d_2| M_1 u_2(t) \big] \\
&\leq \frac{1}{V(t)} \big[-k_1 V^2(t) + \sqrt{2} k_2 V(t) \big] \\
&\leq -k_1 V(t) + \sqrt{2} k_2,
\end{aligned}
\tag{3.11}
$$

where $k_1 = \max\{\underline{a}_1 b_1, \underline{a}_2 b_2\}$ and $k_2 = \max\{|d_1| M_1, |d_2| M_2\}$. It follows that

$$
V(t) \leq k + V(0)e^{-kt} \xrightarrow{t \to +\infty} k, \quad k = \frac{\sqrt{2}k_2}{k_1}.
\tag{3.12}
$$

Hence, all nontrivial periodic solutions of system (2.1) are uniformly bounded.

At the same time, we can prove that the period p of a periodic solution of system (2.1) with $d < d^-$ on $\ell(0, d_j^-, 2\pi/\omega_j^+)$ is also uniformly bounded. In fact, system has no nontrivial τ-periodic solution. For a contradiction, suppose that system (2.1) has a nontrivial τ-periodic solution, then the following differential equation also has a nontrivial τ-periodic solution

$$
\begin{aligned}
\dot{u}_1(t) &= -a_1(u_1(t))\big[b_1(u_1(t)) - d_1 f_2(u_2(t)) \big] = P(u_1, u_2), \\
\dot{u}_2(t) &= -a_2(u_2(t))\big[b_2(u_2(t)) - d_2 f_1(u_1(t)) \big] = Q(u_1, u_2).
\end{aligned}
\tag{3.13}
$$

On the other hand, let

$$
B(u_1, u_2) = \frac{1}{a_1(u_1)a_2(u_2)},
$$

$$
\frac{\partial(BP)}{\partial u_1} + \frac{\partial(BQ)}{\partial u_2} = -\left[\frac{b_1'(u_1)}{a_2(u_2)} + \frac{b_2'(u_2)}{a_1(u_1)} \right] \leq -\left[\frac{b_1}{\overline{a}_2} + \frac{b_2}{\overline{a}_2} \right] < 0.
\tag{3.14}
$$

Thus, we conclude that system (2.6) has no τ-periodic solution according to Bendixson-Dulac criterion. Consequently, system (2.1) has no τ/n-periodic solution for positive integer $n \geq 1$. From definition of ω_j^+, we have

$$
\frac{\tau}{m} < p < \frac{2\pi}{\omega_1^+},
\tag{3.15}
$$

where m is a positive integer. Therefore, the projection of $\ell(0, d_j^-, 2\pi/\omega_j^+)$ on p-space is bounded.

Hence, we obtain that the projection $\ell(0, d_j^-, 2\pi/\omega_j^+)$ on d-space must be unbounded. As a result, for any $d < d_j^-$, system (2.1) has at least a periodic solution with a period in $(\tau/m, 2\pi/\omega_1^+)$ and the proof of Theorem 3.2 is complete. $\square$

Remark 3.3. The discussions of bifurcations of system (1.1) at $d = b_1'(0)b_2'(0)$ are a very cockamamie case. It is difficult to obtain good conditions to guarantee the global existence of periodic solutions of system (1.1), but if $a_1(x_1(t)) = a_2(x_2(t)) = 1$, $b_1(t, x_1(t)) = \mu_1 x_1(t)$, and $b_2(t, x_2(t)) = \mu_2(t)x_2(t)$, system (1.1) reduces to the following Hopfield neural network:

$$\dot{x}_1(t) = -\mu_1 x_1(t) + d_1 f_2(x_2(t - \tau_2)),$$
$$\dot{x}_2(t) = -\mu_1 x_1(t) + d_2 f_1(x_1(t - \tau_1)). \tag{3.16}$$

A supercritical pitchfork of system (3.16) can occur at $d = \mu_1 \mu_2$ under assumptions (H_1)–(H_4) [11], and the periodic solutions of system (2.2) are also globally existent [12]. Of course, Theorems 2.2 and Theorem 3.1 for (3.16) in [12] are special cases of Theorems 2.1 and 3.2 in this paper.

4. A Numerical Example

Consider the following Cohen-Grossberg neural network with discrete delays:

$$\dot{x}_1(t) = -(6 + \cos(x_1(t)))[x_1(t) - d_1 \tanh(x_2(t - 2))],$$
$$\dot{x}_2(t) = -4[2x_2(t) + \tanh(x_1(t - 1))]. \tag{4.1}$$

Since $f'(0) = 1$, $b_1'(0) = 1$, $b_2'(0) = 2$, $a_1(0) = 7$, and $a_2(0) = 4$, $d_2 = 1$, using MATLAB 7.0, we can obtain that

$$b_1'(0)b_2'(0) = 2,$$
$$d = d_1 d_2 f_1'(0) f_2'(0) = d_1,$$
$$\omega_1^+ \approx 0.9618, \qquad d_1^- \approx -2.0334,$$
$$\omega_2^+ \approx 2.8952, \qquad d_2^- \approx -2.3023,$$
$$\omega_3^+ \approx 4.8522, \qquad d_3^- \approx -2.8461, \ldots, \tag{4.2}$$
$$d^- = d_1^- \approx 2.0334.$$

Moreover, due to $f''(0) = 0$, $f'''(0) = 2$, $a_1'(0) = a_2'(0) = a_2''(0) = 0$, $a_1''(0) = 1$, and $b_1'(0) = b_2'(0) = 0$, it is easy to know that $g_{20} = g_{11} = g_{02} = 0$ and

$$g_{21} = 3\overline{D}\left[14 d_1 \overline{\rho} \rho^2 e^{-i3\omega_j^+} + 8\right]. \tag{4.3}$$

Case 1. Let $d_1 = -1.8$. Note that $d = -1.8 \in (-2.0334, 2)$, system (4.1) satisfies Theorem 2.1, and the equilibrium $(0,0)$ of system (4.1) is asymptotically stable. Figure 1 shows the dynamic behaviors of system (4.1) with initial condition $(0.1, 0.2)$.
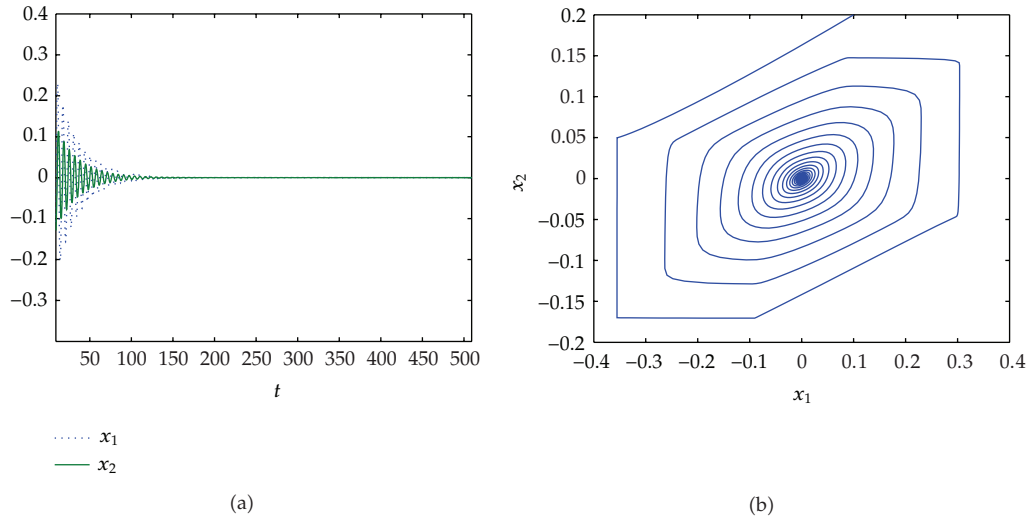
Figure 1: (a) Phase plot in space (x_1, x_2) for system (4.1) with $d_1 = -1.8$. (b) Phase plot in space (x_1, x_2) for system (4.1) with $d_1 = -1.8$.
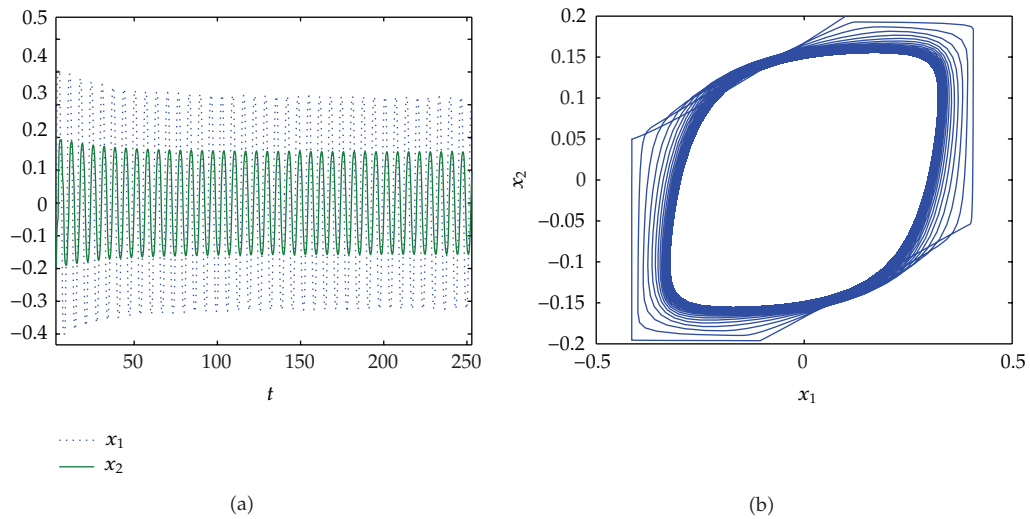

Figure 2: (a) Phase plot in space (x_1, x_2) for system (4.1) with $d_1 = -2.1$. (b) Phase plot in space (x_1, x_2) for system (4.1) with $d_1 = -2.1$.

Case 2. Let $d_1 = -2.1$ and -3.0, respectively. Note that $d = -2.1, -3.0 < d^- = -2.0334$, the equilibrium $(0, 0)$ loses its stability, and Hopf bifurcations occurs, that is, periodic solutions bifurcate from $(0, 0)$ according to Theorem 3.2. Furthermore, we can from (4.3) obtain that $g_{21}(d_1^-) = -70.7796 - 1.8445i$, $g_{21}(d_2^-) = -9.7399 - 1.1614i$, $g_{21}(d_3^-) = -5.6912 - 0.9305i, \ldots$; so, $\mu_2 < 0$ for d_j^-, $j = 1, 2, 3$. Hence, the periodic solutions are stable for $d_1 = -2.1, -3.0$. Figures 2 and 3 show the dynamic behaviors of system (4.1) with initial condition $(0.1, 0.2)$.
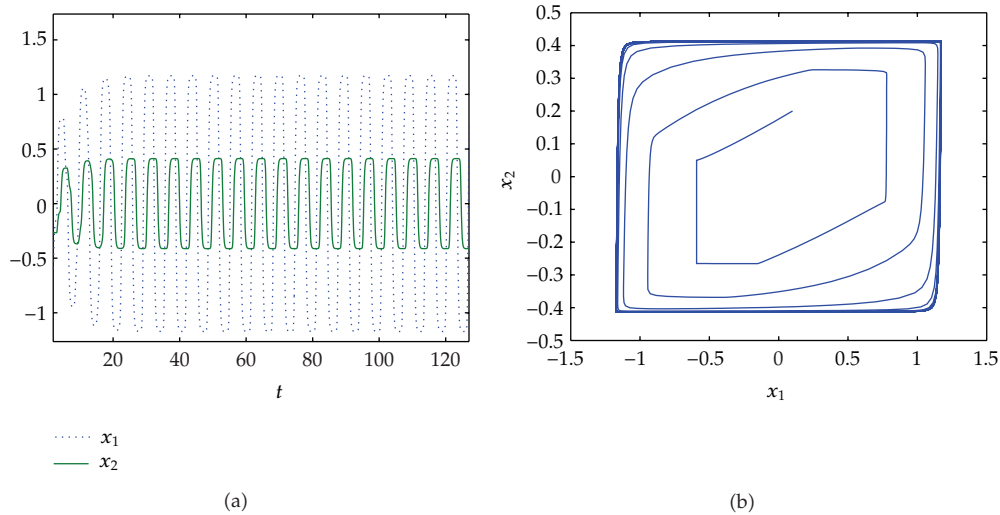
(a)   (b)

Figure 3: (a) Phase plot in space (x_1, x_2) for system (4.1) with $d_1 = -3.0$. (b) Phase plot in space (x_1, x_2) for system (4.1) with $d_1 = -3.0$.

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